Independent arithmetic progressions in clique-free graphs on the natural numbers

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Abstract

We show that if G is a K_r-free graph on $\mathbb{N}$, there are independent sets in G which contain an arbitrarily long arithmetic progression together with its difference. This is a common generalization of theorems of Schur, van der Waerden, and Ramsey. We also discuss various related questions regarding (m, p, c)-sets and parameter words.

1 Introduction

We use the notation $\mathbb{N} = \{1, 2, \ldots\}$, $\mathbb{Z} = \{\ldots, -2, -1, 0, 1, 2, \ldots\}$, $[a, b] = \{c \in \mathbb{Z} : a \leq c \leq b\}$; we may abbreviate $[1, n]$ by simply $[n]$. For a set S and cardinal κ, let $[S]^\kappa = \{K \subseteq S : |K| = \kappa\}$.

We are interested in (simple) graphs $G = (\mathbb{N}, E)$ on vertex set $\mathbb{N}$ with edge set $E = E(G) \subseteq [\mathbb{N}]^2$. A set $Y \subset V(G)$ is called *independent* in G if $[Y]^2 \cap E(G) = \emptyset$. When $E(G) = [V(G)]^2$, we say that G is complete, and the complete graph on n vertices is denoted by K_n. A graph is r-partite if its vertex set can be partitioned into r sets, each set containing no edges. The graph $K_{m,n}$ is the complete bipartite graph on disjoint vertex sets of sizes m and n.

Given a set $\{x_i\}_{i \in I}$ of distinct positive integers, let

$$\mathrm{FS}(\{x_i\}_{i\in I}) = \left\{ \sum_{j \in J} x_j : \emptyset \neq J \subseteq I, |J| < \infty \right\}.$$

denote a *Folkman set*, the finite sums from the set $\{x_i\}_{i\in I}$. If I is infinite, we say that $\mathrm{FS}(\{x_i\}_{i\in I})$ is a *Hindman set.*

Investigations considered in this paper were in part inspired by Hajnal asking the following question (see [4]) in 1995.

Question 1.1 *If G is a triangle-free graph on $\mathbb{N}$, does there always exist an Hindman set independent in G?*

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A negative answer to Question 1.1 was found by Deuber, Gunderson, Hindman and Strauss in [2], yet variants of the question have been shown to indeed have a positive answer, for example, if the condition "triangle-free" is replaced by "$K_{k,k}$-free" (see [2] and [9]).

Before a solution was known to Question 1.1, Erdős [3] "retaliated" with a finite version:

Question 1.2 *If G is a triangle-free graph on $\mathbb{N}$, does there always exist an independent Schur triple, that is, does there exist x, y, $x \neq y$ so that $FS(x, y) = \{x, y, x + y\}$ is independent in G?*

Using an application of the Milliken-Taylor theorem, (cf. [10]) Łuczak, Rödl, and Schoen answered Question 1.2 in the affirmative with a strong statement:

Theorem 1.3 ([9]) *Fix r and d. If G is a K_r-free graph on $\mathbb{N}$, then there exist distinct integers $a_1, a_2, ..., a_d$, so that $FS(\{a_1, \ldots, a_d\})$ is an independent set in G.*

Since $(r-1)$-partite graphs are K_r-free, and an $(r-1)$-partite graph on $\mathbb{N}$ determines an $(r-1)$-colouring of $\mathbb{N}$, Theorem 1.3 implies, for example, Schur's theorem.

Theorem 1.4 (Schur [16]) *For any positive integer k, there exists a least n so that for every colouring $\Delta : [n] \to k$ there exist distinct $x, y \in [n]$ so that $\Delta(x) = \Delta(y) = \Delta(x + y)$.*

2 Results

One of the goals in this paper is to strengthen van der Waerden's theorem in the same way that Theorem 1.3 extends Schur's theorem.

Theorem 2.1 (van der Waerden [17]) *For positive integers r, ℓ, there exists a least n so that for any coloring $\Delta : [n] \to r$ there is a monochromatic ℓ-term arithmetic progression.*

In Section 4 we attained this goal:

Theorem 2.2 *For each $k \geq 3$, and each $\ell \geq 3$, in any K_k-free graph G on $\mathbb{N}$ there exists an independent set in G which contains an arithmetic progression of length ℓ.*

In Section 4, Theorem 2.2 follows fairly easily from a lemma yielding independent lines in Hales-Jewett cubes on vertices (0-parameter words) of a K_k-free graph.

For integers s and ℓ, an *s-fold arithmetic progression of length ℓ* is a set of the form $\{a_0 + \lambda_1 a_1 + \cdots + \lambda_s a_s : \lambda_1, \ldots, \lambda_s \in [0, \ell - 1]\}$. In Theorem 4.3 the result corresponding to Theorem 2.2 for s-fold arithmetic progressions is given. This is derived from Corollary 4.2, guaranteeing that every K_k-free graph G defined on the vertices of a Hales-Jewett cube always contains an m-space which spans an independent set in G. With two trivial exceptions (Corollaries 3.1 and 3.2), attempts to generalize these results to graphs on general parameter words fails; counterexamples are delayed until Section 6.

Considering Theorems 1.3 and 2.2, a natural question might be to ask what other kinds of arithmetic structures can we find in independent sets in K_k-free graphs (for every k). A system $A\boldsymbol{x} = \boldsymbol{0}$ of linear equations is called *partition regular* if for every partition of $\mathbb{Z}$ into finitely many classes there exists a solution completely contained in one class. The equation $x + y - z = 0$ describes Schur triples, and so is partition regular; the equation $x + y - 2z = 0$

describes 3-term arithmetic progressions and so is also partition regular. Similarly, systems of equations describing any longer arithmetic progressions, s-fold arithmetic progressions or Folkman sets form partition regular systems. Partition regular equations were first completely characterized by Rado [14]. An example of a simple system which is not partition regular is $x + y = 3z$. (See, for example, [5] for a more detailed discussion.)

Conjecture 2.3 *For any $k \geq 2$ and any K_k-free graph on $\mathbb{N}$ one can always solve any partition regular system in an independent set?*

Notice that if for every k and any K_k-free graph on $\mathbb{N}$ one can solve a particular linear system of equations in an independent set, then this system must be partition regular since to each $(k-1)$-partition corresponds a K_k-free graph. So a "yes" answer to Conjecture 2.3 would be, in some sense, optimal in that it would strengthen results of Rado *et al.*

One of the simplest sets described by partition regular equations which is neither a Folkman set nor an arithmetic progression is an arithmetic progression together with its difference. In Section 5 we accomplish a step in answering Conjecture 2.3 by extending Theorem 2.2 as follows.

Theorem 2.4 *For any $k \geq 3$ and $\ell \geq 3$, in any K_k-free graph on $\mathbb{N}$, there exists an ℓ-term arithmetic progression together with its difference, all contained in an independent set.*

The proof of Theorem 2.4 is less straightforward than the proof of Theorem 2.2, and answering Conjecture 2.3 in general turns out to be even more technical. [Very recently, we found a positive answer to Conjecture 2.3, however due to its length and technical nature, the proof will appear in a separate subsequent paper.]

On the other hand, when one replaces "K_k-free" in Conjecture 2.3 with "$K_{k,k}$-free," the problem becomes much simpler and we present an easy averaging argument providing a positive answer in Section 7 (Theorem 7.3). Similar questions have been considered in [2] and [9].

Note that one can view Theorem 2.2 (and its generalizations) as a common generalization of Ramsey's theorem [15] and van der Waerden's theorem. To explain, we use the notation $[n] \to (a,b)^2$ to indicate that under any red-blue colouring of the pairs $[n]^2$, there is either $A \in [n]^a$ so that all pairs $[A]^2$ are red, or $B \in [n]^b$ so that pairs $[B]^2$ are blue. For any a and b, Ramsey's theorem (for 2-colouring pairs) guarantees a least n satisfying $[n] \to (a,b)^2$. With a slight abuse of this notation, Theorem 2.2 says that for any k and ℓ, $\mathbb{N} \to (k, \mathrm{AP}_\ell)^2$; Theorem 1.3 might similarly say that for any r, d, $\mathbb{N} \to (r, \mathrm{FS}_d)^2$. In this sense, most theorems in this paper are "one-sided" generalizations of Ramsey's theorem for colouring pairs of integers. Are there similar "two-sided" generalizations? Unfortunately not; counterexamples appear in [2] and [9] showing $\mathbb{N} \not\to (\mathrm{AP}_3, \mathrm{AP}_3)$ and $\mathbb{N} \not\to (\mathrm{FS}_2, \mathrm{FS}_2)$.

We start in Section 3 with a brief discussion of facts about parameter words.

3 Parameter words

See [12] for a survey of results, applications, and notation for parameter words. Here we use fairly standard notation.

Let A be a finite alphabet and $\xi_1, \xi_2, \ldots, \xi_m$ be symbols not in A, called *parameters*. As usual, we use $A^n = \{f : n \to A\}$. For $0 \leq m \leq n$, define the set of *m-parameter words* of length

n over A by

$$[A]\binom{n}{m} = \left\{ \begin{array}{rl} f : n \to (A \cup \{\xi_1, \dots, \xi_m\}) : & \forall j \le m, f^{-1}(\xi_j) \ne \emptyset, \text{ and,} \\ & \forall i < j, \quad \min f^{-1}(\xi_i) < \min f^{-1}(\xi_j) \end{array} \right\}.$$

So $[A]\binom{n}{m}$ can be viewed as a set of ordered n-tuples containing each of $\xi_1, \dots, \xi_m$ at least once and if $i < j$, the first occurrence of ξ_i must precede the first occurrence of ξ_j. We make the trivial observation that $A^n = [A]\binom{n}{0}$. For $f \in [A]\binom{n}{m}$ and $g \in [A]\binom{m}{k}$ we define the composition $f \circ g \in [A]\binom{n}{k}$ by

$$f \circ g(i) = \begin{cases} f(i) & \text{if } f(i) \in A, \\ g(j) & \text{if } f(i) = \xi_j. \end{cases}$$

It is straightforward to check that composition of parameter words is associative. The shorthand notation $f \circ [A]\binom{m}{k} = \{f \circ g : g \in [A]\binom{m}{k}\}$ is often useful.

For $f \in [A]\binom{n}{m}$, define the *space* of f, $\mathrm{sp}(f) = f \circ [A]\binom{m}{0}$, to be the set of (0-parameter) words from $[A]\binom{n}{0}$ which are formed by faithfully replacing parameters in f with elements from A (that is, the same letter replaces all occurrences of one parameter). The space of a parameter word is often referred to as a *parameter set*. An *m-dimensional (combinatorial) subspace* of A^n (or simply, m-space) is the space of some word in $[A]\binom{n}{m}$. If $f \in [A]\binom{n}{1}$ then we say $\mathrm{sp}(f)$ is a *combinatorial line* in A^n.

Extending these notions, for $f \in [A]\binom{n}{m}$ define $\mathrm{sp}_k(f) = f \circ [A]\binom{m}{k}$ to be the set of k-parameter words which are formed by replacing parameters in f with elements from $[A]\binom{m}{k}$.

The independence result for finite sums (Theorem 1.3) may be expressed in terms of parameter words in two ways. One way is to use the bijection between $[1]\binom{n}{1}$ and $\mathcal{P}(n)\backslash\{\emptyset\}$ (the occurrences of ξ_1 are interpreted as the characteristic function of sets).

Corollary 3.1 *Given k and m, there is a least n such that for every K_k-free graph $G = \big([1]\binom{n}{1}, E(G)\big)$ there exists $h \in [1]\binom{n}{m}$ so that $sp_1(h)$ is an independent set in G.*

Secondly, using the bijection between $[0]\binom{n+1}{2}$ and $\mathcal{P}(n)\backslash\{\emptyset\}$ (interpreting the occurrences of ξ_2 as the characteristic function of sets) we obtain a corresponding result for 2-parameter words over the empty alphabet.

Corollary 3.2 *Given k and m, there is a least n such that for every K_k-free graph $G = \big([0]\binom{n}{2}, E(G)\big)$ there exists $h \in [0]\binom{n}{m}$ so that $sp_2(h)$ is an independent set in G.*

We now state two of the major theorems regarding parameter sets.

Theorem 3.3 (Hales-Jewett [8]) *Let $m \ge 0$, $r \ge 1$ and a finite alphabet A be given. Then there exists a smallest integer $n = HJ(|A|, m, r)$ so that for every r-colouring $\Delta : A^n \longrightarrow r$, there exists $f \in [A]\binom{n}{m}$ so that $sp(f) = f \circ [A]\binom{m}{0}$ is monochromatic.*

Note that the Hales-Jewett theorem immediately implies van der Waerden's theorem by setting $A = \{0, 1, \dots, \ell - 1\}$ and defining $\psi : [A]\binom{n}{0} \to \ell^n$ by $\psi(f) = \sum_{i=1}^{n} f(i)\ell^i$; then the space of each $f \in [A]\binom{n}{1}$ determines a ℓ-term arithmetic progression under the mapping ψ.

The proof of Theorem 2.2 relies heavily on the following generalization of the Hales-Jewett theorem from 0-parameter spaces to k-parameter spaces.

Theorem 3.4 (Graham-Rothschild [6]) *Let $m \geq k \geq 0$, $r \geq 1$ and a finite alphabet A be given. Then there exists a smallest integer $n = GR(|A|, k, m, r)$ so that for every r-colouring $\Delta : [A]\binom{n}{k} \longrightarrow r$, there exists $f \in [A]\binom{n}{m}$ so that $sp_k(f) = f \circ [A]\binom{m}{k}$ is monochromatic.*

In Section 5, we use a result by Gallai (see [14]) and Witt [18]. A now standard proof of their result which uses the Hales-Jewett theorem (very similar to the proof alluded to above for van der Waerden's theorem; see, for example, [11] or [7] for details) enables us to state a special case of the Gallai-Witt theorem in the following simple form.

Theorem 3.5 (Gallai-Witt) *For every finite $X \subset \mathbb{N} \times \mathbb{N}$ and number of colours ρ there exists $n = GW(X, \rho)$ such that for any ρ-colouring $\chi : [0, n] \times [1, n] \rightarrow \rho$ there exist integers u, v, and c so that $\{(u, v) + c(s, t) : (s, t) \in X\} \subset [n] \times [n]$ and is monochromatic.*

4 Independent arithmetic progressions

We start with a lemma guaranteeing independent lines in a Hales-Jewett cube on vertices of a K_r-free graph.

Lemma 4.1 *Given r and alphabet $A = \{a_1, a_2, \ldots, a_\ell\}$ with $\ell \geq 2$ letters, there exists n so that for every K_r-free graph $G = (A^n, E(G))$, there exists $h \in [A]\binom{n}{1}$ so that $sp(h)$ is independent in G.*

Proof: Let $m \geq r - 1$, put $n = \mathrm{GR}(|A|, 1, m, \binom{\ell}{2} + 1)$, and let $G = (A^n, E(G))$ be a graph on vertex set $A^n = [A]\binom{n}{0}$ which is K_r-free.

Define a colouring $\Delta : [A]\binom{n}{1} \longrightarrow \binom{\ell}{2} + 1$ as follows: for each $h \in [A]\binom{n}{1}$, if $\{h \circ (a_i), h \circ (a_j)\} \in E(G)$ and (i, j) is least (in some lexicographic order, say) so that this is so, then set $\Delta(h) = \{i, j\}$; if $E(G) \cap [\mathrm{sp}(h)]^2 = \emptyset$, that is, if no edge occurs in the graph induced by $\mathrm{sp}(h)$, then put $\Delta(h) = 0$. Under this colouring, by the choice of n, there exists a monochromatic $f \in [A]\binom{n}{m}$ all of whose lines (1-spaces) receive the same colour.

First suppose that this colour is not 0, and so let $\{\alpha, \beta\}$ be so that $\Delta |_{\{f \circ h : h \in [A]\binom{m}{1}\}} = \{\alpha, \beta\}$. Examine the $m + 1$ vertices

$$\begin{aligned} f_0 &= f \circ (a_\alpha, a_\alpha, \ldots, a_\alpha), \\ f_1 &= f \circ (a_\beta, a_\alpha, \ldots, a_\alpha), \\ &\vdots \\ f_m &= f \circ (a_\beta, a_\beta, \ldots, a_\beta). \end{aligned}$$

For $0 \leq i < j \leq m$, both f_i and f_j are in the same 1-space (the line $f \circ h$ where h is the word of length m of the form $h = (a_\beta, \ldots, a_\beta, \xi, \ldots, \xi, a_\alpha, \ldots, a_\alpha)$, the first i symbols being a_β's, the next $j - i$ being ξ's, and the rest a_α's). So for each such i and j, there is an edge between f_i and f_j, producing a K_{m+1}, a contradiction when $m + 1 \geq r$. Therefore we must have $\Delta |_{\{f \circ h : h \in [A]\binom{m}{1}\}} = 0$. In this case, every 1-subspace of f is independent, namely, for every $h \in [A]\binom{m}{1}$, the set of vertices in A^n given by $\mathrm{sp}(f \circ h) = \{f \circ h \circ (a_1), f \circ h \circ (a_2), \ldots, f \circ h \circ (a_l)\}$ is an independent l-set. □

We are now ready to give a simple proof of Theorem 2.2 (in any K_r-free graph on $\mathbb{N}$ there is an independent ℓ-term arithmetic progression).

Proof of Theorem 2.2: Let $A = \{0, 1, 2, \ldots, \ell - 1\}$ and define a map $\psi : [A]\binom{n}{0} \longrightarrow [\ell^n]$ by $\psi(f) = 1 + \sum_{i=1}^{n} f(i)\ell^i$. Observe that ψ is one to one on A^n. For an element $f \in [A]\binom{n}{1}$, sp(f) determines an arithmetic progression of ℓ terms under the mapping ψ. Now by Lemma 4.1, we are done. □

Applying Lemma 4.1 to the alphabet $B = A^m$, we obtain also an m-space $h \in [A]\binom{n}{m}$ so that sp(h) is an independent set. More precisely, we get the following corollary:

Corollary 4.2 *Given k, m and A, there exists $n = IS(|A|, k, m)$ such that for every K_k-free graph $G = \big([A]\binom{n}{0}, E(G)\big)$ there exists $h \in [A]\binom{n}{m}$ so that sp(h) is an independent set in G.*

Theorem 4.3 *For each $k \geq 2$, $s \geq 1$, and $\ell \geq 2$ and any K_k-free graph G on $\mathbb{N}$, there is an s-fold arithmetic progression of length ℓ which is independent in G.*

Proof: Apply Corollary 4.2 in the same manner as Lemma 4.1 was applied in the proof of Theorem 2.2. □

5 Independent arithmetic progression plus difference

A simple example of a partition regular set which is neither a Folkman set nor is contained in any arithmetic progression is an arithmetic progression together with its difference. For example, $\{a, a+d, a+2d, d\}$ is such a set.

In this section, it will be convenient to abbreviate "arithmetic progression of length ℓ" by "AP$_\ell$" and "arithmetic progression of length ℓ together with its difference" by "AP$_\ell$D". For each $a \geq 1$ and $d \geq 1$, identify a specific AP$_\ell$D by

$$\mathrm{AP}_\ell\mathrm{D}(a, d) = \{a, a+d, a+2d, \ldots, a+(\ell-1)d, d\}.$$

The main goal of this section is to prove that for any k and ℓ, any K_k-free graph on $\mathbb{N}$ contains independent AP$_\ell$D. We first address the case $k = 3$.

Theorem 5.1 *For each $\ell \geq 2$ and any K_3-free graph G on $\mathbb{N}$, there exists an $AP_\ell D$ which is an independent set in G.*

Proof: Fix $\ell \geq 2$ and let G be a graph on $\mathbb{N}$. We will show that if each $AP_\ell D(a, d)$ contains an edge then G contains a triangle.

Assume that for each $a \in \mathbb{N}$ and $d \in \mathbb{N}$ $AP_\ell D(a, d)$ induces an edge, say $\eta(a, d) \in [AP_\ell D(a, d)]^2 \cap E(G)$. Define

$$\chi(a, d) = \begin{cases} \kappa & \text{if } \eta(a, d) = \{a + \kappa d, d\}; \\ \{\lambda, \mu\} & \text{if } \eta(a, d) = \{a + \lambda d, a + \mu d\}, \end{cases}$$

a colouring of $\mathbb{N} \times \mathbb{N}$ with $\ell + \binom{\ell}{2} = \binom{\ell+1}{2}$ colours according to the position of the edge in each $AP_\ell D(a, d)$. Applying the Gallai-Witt theorem (Theorem 3.5) with $X = [0, 2(\ell-1)^3] \times [0, 2(\ell-1)]$ there is (p, q) and a constant c so that

$$X^* = (p, q) + cX = \{(p + cs, q + ct) : (s, t) \in X\}$$

is monochromatic with respect to χ.

CLAIM: There exist two disjoint AP_ℓ's A and B with the same difference so that for every $a \in A$ and $b \in B$, $\{a, b\} \in E(G)$.

To prove the claim, we examine two cases, according to the kind of colour of X^*.

Case 1: $\chi\mid_{X^*} = \kappa$; then for each $s \in [0, \ell(\ell-1)]$ and $t \in [0, \ell-1]$,

$$\eta(p + cs, q + ct) = \{p + cs + \kappa(q + ct), q + ct\} \in E(G). \tag{1}$$

Examine the following two arithmetic progressions, both with difference c:

$$\begin{aligned} A^* &= \{p + \kappa q + ic : i = 0, 1, 2, ...\}, \\ B^* &= \{q + jc : j = 0, 1, 2, \ldots\}. \end{aligned}$$

For some particular i and j, to show that $p + \kappa q + ic \in A^*$ is connected to $q + jc \in B^*$, by (1) it suffices to find appropriate s and t so that

$$\begin{aligned} i &= s + \kappa t, \\ j &= t. \end{aligned}$$

Solving this system for s and t yields

$$\begin{aligned} s &= i - \kappa j, \\ t &= j. \end{aligned}$$

If $i \in [(\ell-1)^2, \ell(\ell-1)]$ and $j \in [0, \ell-1]$, then $s \in [0, \ell(\ell-1)]$ and $t \in [0, \ell-1]$, and so the arithmetic progressions

$$\begin{aligned} A &= \{p + \kappa q + ic : i \in [(\ell-1)^2, \ell(\ell-1)]\}, \\ B &= \{q + jc : j \in [0, \ell-1]\} \end{aligned}$$

satisfy the claim. Observe that $\min A = p + \kappa q + (\ell-1)^2 c > q + (\ell-1)c = \max B$ and thus A and B are disjoint.

Case 2: $\chi_{|X^*} = \{\lambda, \mu\}$, where $0 \le \lambda < \mu \le \ell - 1$. For each $s \in [0, 2(\ell-1)^3]$, $t \in [0, 2\ell-2]$,

$$\{p + cs + \lambda(q + ct), p + cs + \mu(q + ct)\} \in E(G). \tag{2}$$

Examine the following two arithmetic progressions with common difference $(\mu - \lambda)c$:

$$\begin{aligned} A^* &= \{p + \lambda q + i(\mu-\lambda)c : i = 0, 1, 2, \ldots\}, \\ B^* &= \{p + \mu q + j(\mu-\lambda)c : j = 0, 1, 2, \ldots\}. \end{aligned}$$

For a particular choice of i and j, to see that $p + \lambda q + i(\mu-\lambda)c \in A^*$ is connected to $p + \mu q + j(\mu-\lambda)c \in B^*$, by (2) it suffices to find appropriate s and t so that

$$\begin{aligned} i(\mu-\lambda) &= s + \lambda t, \\ j(\mu-\lambda) &= s + \mu t. \end{aligned}$$

Solving this system for s and t yields

$$
\begin{aligned}
s &= i\mu - j\lambda, \\
t &= j - i.
\end{aligned}
$$

If i_0 and j_0 are the smallest i and j so that both s and t are non-negative, then we must have $j_0 \geq i_0 + \ell - 1$, and and for every λ and μ satisfying $0 \leq \lambda < \mu \leq \ell - 1$ we require $i_0 \geq 2\lambda(\ell-1)/(\mu - \lambda)$; since this last expression is minimized for $\lambda = \ell - 2$ and $\mu = \ell - 1$, the conditions $i_0 \geq 2(\ell-1)(\ell-2)$ and $j_0 \geq 2(\ell-1)(\ell-2) + \ell - 1 = (\ell-1)(2\ell-3)$ are necessary. Choosing these lower bounds for i_0 and j_0 yield (after a short calculation) $i_0 \leq s \leq (\ell-1)^2(2\ell-3) < 2(\ell-1)^3$ and $0 \leq t \leq 2(\ell-1)$ as required. Hence

$$
\begin{aligned}
A &= \{p + \lambda q + i(\mu - \lambda)c : i \in [2(\ell-1)(\ell-2), (2\ell-3)(\ell-1)]\}, \\
B &= \{p + \mu q + j(\mu - \lambda)c : j \in [(2\ell-3)(\ell-1), 2(\ell-1)^2]\}
\end{aligned}
$$

satisfy the claim. Using the facts that $\mu > \lambda$ and $j \geq i$ one can verify that $\max A < \min B$.

So we have proved the claim in both cases, producing disjoint AP_ℓ's A and B with the same difference which span a complete bipartite graph. If either A or B induces an edge, then many triangles exist in G. If both A and B are independent sets, then since each $\mathrm{AP}_\ell\mathrm{D}$ contains an edge, there is an edge from the difference (c in Case 1, or $c(\mu - \lambda)$ in Case 2) to a point in A and to a point in B, again yielding a triangle. □

In each case of the above proof, the two arithmetic progressions A and B could have been chosen arbitrarily long (by letting s and t vary over larger intervals) and so we have the following consequence.

Corollary 5.2 *Let G be a graph on $\mathbb{N}$ and fix $w \geq \ell \geq 2$. If each $AP_\ell D$ induces an edge in G, then there exist two AP_w's, A and B, which have the same difference, satisfy* $\max A < \min B$, *and form a complete bipartite graph in G.*

We now extend Theorem 5.1 to Theorem 2.4; for convenience, we repeat the statement (in a slightly modified but equivalent form).

Theorem 2.4 *Let $m \geq 3$, $\ell \geq 2$, and let G be a graph on $\mathbb{N}$. If every $AP_\ell D$ in $\mathbb{N}$ induces an edge in G, then G contains a K_m.*

Proof: The case $\ell = 2$ is trivial, so fix $\ell \geq 3$ and let G be a graph on $\mathbb{N}$ where each $\mathrm{AP}_\ell D(a,d)$ contains an edge of G. To see that G contains a K_m we instead prove the following much stronger claim, from which it trivially follows that G contains a K_m.

CLAIM: For any $z \geq \ell$ and $m \geq 2$, G contains m different AP_z's, $A_1, \ldots, A_m$ with a common difference so that for every $\alpha \neq \beta$, each element of A_α is connected to each element of A_β.

Proof of the claim is by induction on m; the base case $m = 2$ is Corollary 5.2.

Fix $z \geq \ell$ and $m \geq 3$. Using $X = [0, 2(z-1)^3] \times [0, 2(z-1)]$ and $\rho = \binom{\ell+1}{2}^{m-1}$, let $n = \mathrm{GW}(X, \rho)$ be as in Theorem 3.5. Set $w = \ell n$. For the induction hypothesis, assume that there exist distinct arithmetic progressions

$$\begin{aligned}
A_1^* &= \{x_1 + id : i = 0, 1, \ldots, w\},\\
A_2^* &= \{x_2 + id : i = 0, 1, \ldots, w\},\\
\vdots \quad & \quad \vdots\\
A_{m-1}^* &= \{x_{m-1} + id : i = 0, 1, \ldots, w\},
\end{aligned}$$

are so that all pairs of points from different A_α^*'s are connected.

For each $\alpha = 1, 2, \ldots, m-1$, and $(q, r) \in [0, n] \times [1, n]$, observe that $x_\alpha + qd + \ell rd \leq x_\alpha + wd$ and consequently $\mathrm{AP}_\ell\mathrm{D}(x_\alpha + qd, rd) \subset A_\alpha^*$.

For each $(q, r) \in [0, n] \times [1, n]$, select an edge $\eta(x_\alpha + qd, rd)$ from $\mathrm{AP}_\ell\mathrm{D}(x_\alpha + qd, rd)$. Define an $\binom{\ell+1}{2}^{m-1}$-colouring of the pairs $(q, r) \in [0, n] \times [1, n]$ as follows. First, for $\alpha = 1, \ldots, m-1$, define

$$\chi_\alpha(q, r) = \begin{cases} \kappa & \text{if } \eta(x_\alpha + qd, rd) = \{x_\alpha + qd + \kappa rd, rd\};\\ \{\lambda, \mu\} & \text{if } \eta(x_\alpha + qd, rd) = \{x_\alpha + qd + \lambda rd, x_\alpha + qd + \mu rd\}.\end{cases}$$

Finally, put

$$\chi(q, r) = (\chi_1(q, r), \chi_2(q, r), \ldots, \chi_{m-1}(q, r)),$$

an $(m-1)$-tuple indicating edge positions in each of $\mathrm{AP}_\ell\mathrm{D}(x_1+qd, rd), \ldots, \mathrm{AP}_\ell\mathrm{D}(x_{m-1}+qd, rd)$, respectively. By the choice of n, there exists $(u, v) \in [n] \times [n]$ and a constant c so that χ is monochromatic on $\{(u, v) + c(s, t) : s \in [0, 2(z-1)^3], t \in [0, 2(z-1)]\}$.

We divide the proof of the inductive step of the claim into cases according to the kind of colour.

Case 1: For every α there is a κ_α so that for each s and t,

$$\chi_\alpha(u + cs, v + ct) = \kappa_\alpha,$$

indicating an edge from the common difference $(v + ct)d$ to each of the arithmetic progressions. In other words, for each s and t, and for each $\alpha = 1, \ldots, m-1$,

$$\begin{aligned}
\{x_\alpha + (u + cs)d + \kappa_\alpha(v + ct)d, (v + ct)d\} &=\\
\{x_\alpha + ud + \kappa_\alpha vd + (s + \kappa_\alpha t)cd, vd + tcd\} &\in E(G). \qquad (3)
\end{aligned}$$

Examine the $m-1$ arithmetic progressions

$$\begin{aligned}
A'_1 &= \{x_1 + ud + \kappa_1 vd + icd : i = 0, 1, 2, \ldots, z(z-1)\},\\
A'_2 &= \{x_2 + ud + \kappa_2 vd + icd : i = 0, 1, 2, \ldots, z(z-1)\},\\
\vdots &= \vdots\\
A'_{m-2} &= \{x_{m-2} + ud + \kappa_{m-2} vd + icd : i = 0, 1, \ldots, z(z-1)\},\\
A_{m-1} &= \{vd + kcd : j = 0, 1, 2, \ldots, z-1\}.
\end{aligned}$$

For each $\alpha = 1, \ldots, m-1$, $A'_\alpha \subset A_\alpha^*$, so by the induction hypothesis, for each $1 \leq \alpha < \beta \leq m-1$, every $a \in A_\alpha$ is connected to every $b \in A_\beta$.

For each α and some i and j, to show that an element $vd + jcd \in A_m$ is connected to $x_\alpha + ud + \kappa_\alpha vd + icd \in A'_\alpha$, by (3) it suffices to produce appropriate s and t so that

$$\begin{aligned} i &= s + \kappa_\alpha t, \\ j &= t. \end{aligned}$$

The values $t = j$ and $s = i - \kappa_\alpha j$ satisfy these equations, and if $i \in [(z-1)^2, z(z-1)]$ and $j \in [0, z-1]$ then $s \in [0, z(z-1)]$ and $t \in [0, z-1]$ are as required. Thus, the arithmetic progressions

$$\begin{aligned} A_1 &= \{x_1 + ud + \kappa_1 vd + icd : i \in [(z-1)^2, z(z-1)]\}, \\ A_2 &= \{x_2 + ud + \kappa_2 vd + icd : i \in [(z-1)^2, z(z-1)]\}, \\ \vdots &= \vdots \\ A_{m-1} &= \{x_{m-1} + ud + \kappa_{m-1} vd + icd : i \in [(z-1)^2, z(z-1)]\}, \\ A_m &= \{vd + jcd : j \in [0, z-1]\}, \end{aligned}$$

satisfy the claim.

Case 2: At least one of the χ_α's indicates an edge inside all the specified AP_ℓ's. To fix ideas, suppose that for every s and t, $\chi_1(u + cs, v + ct) = \{\lambda, \mu\}$ (where $\lambda < \mu$), that is,

$$\begin{aligned} \{x_1 + (u+cs)d + \lambda(v+ct)d, x_1 + (u+cs)d + \mu(v+ct)d\} &= \\ \{x_1 + ud + \lambda vd + (s + \lambda t)cd, x_1 + ud + (s + \mu t)cd\} &\in E(G). \end{aligned} \tag{4}$$

Examine the m arithmetic progressions

$$\begin{aligned} B^* &= \{x_1 + ud + \lambda vd + i(\mu - \lambda)cd : i = 0, 1, ...\} \subset A_1^*, \\ C^* &= \{x_1 + ud + \mu vd + j(\mu - \lambda)cd : j = 0, 1, ...\} \subset A_1^*, \\ A_2 &= \{x_2 + i(\mu - \lambda)cd : i \in [0, z-1]\} \subset A_2^*, \\ A_3 &= \{x_3 + i(\mu - \lambda)cd : i \in [0, z-1]\} \subset A_3^*, \\ &\vdots \\ A_{m-1} &= \{x_{m-1} + i(\mu - \lambda) : i \in [0, z-1]\} \subset A_{m-1}^*. \end{aligned}$$

By the induction hypothesis, for each $\alpha = 2, \ldots, m-1$, every element of B^* and every element of C^* is connected to each element of A_α, and for $2 \le \alpha < \beta \le m-1$, every vertex of A_α is connected to every vertex of A_β. So to prove the claim in this case, it suffices to find two AP_z's, $B \subset B^*$ and $C \subset C^*$, which are totally connected. For a given i and j, to show that $x_1 + ud + \lambda vd + i(\mu - \lambda) \in B$ is connected to $x_1 + ud + \mu vd + j(\mu - \lambda) \in C$, by (4) it suffices to exhibit suitable s and t so that

$$\begin{aligned} i(\mu - \lambda) &= s + \lambda t, \\ j(\mu - \lambda) &= s + \mu t. \end{aligned}$$

Solving for s and t yields

$$\begin{aligned} s &= i\mu - j\lambda, \\ t &= j - i. \end{aligned}$$

Identical calculations to those in the proof of Theorem 5.1 show that (with z instead of ℓ) for a particular i and j satisfying $2(z-1)(z-2) \leq i \leq (2z-3)(z-1)$ and $(2z-3)(z-1) \leq j \leq 2(z-1)^2$, then $s \in [0, 2(z-1)^3]$ and $t \in [0, 2(z-1)]$ as required. Hence

$$\begin{aligned} B &= \{x_1 + ud\lambda vd + i(\mu - \lambda)cd : i \in [2(z-1)(z-2), (2z-3)(z-1)]\}, \\ C &= \{x_1 + ud + \mu vd + j(\mu - \lambda)cd : j \in [(2z-3)(z-1), 2(z-1)^2]\}. \end{aligned}$$

are as required, proving the claim in this case (with AP_z's B, C, $A_2, \ldots, A_{m-1}$).

So the claim is proved in both cases, finishing the proof of the theorem. □

6 Clique-free graphs on parameter words

A natural approach to extend results on independent arithmetic progressions to independent (m, p, c)-sets might be to first to extend the corresponding result for 0-parameter words to general k-parameter words. As we will show in this section, this task fails completely—with the two minor exceptions of Corollary 3.1 (for $|A| = 1$ and K_k-free graphs defined on 1-spaces) and Corollary 3.2 (for $|A| = 0$ and K_k-free graphs defined on 2-spaces).

Proposition 6.1 *Let A be such that $|A| \geq 2$. Then for every $n \geq 2$ there exists a K_3-free graph $G = \left([A]\binom{n}{1}, E(G)\right)$ such that for every $h \in [A]\binom{n}{2}$ the set $sp_1(h)$ contains at least one edge.*

Proof: Let $n \geq 2$ and $f, g \in [A]\binom{n}{1}$ be such that $\min f^{-1}(\xi_1) < \min g^{-1}(\xi_1)$. We define f, g to be an edge if and only if the following three conditions are fulfilled:

(1) If $f(i) = \xi_1$ for some i, then $g(i) = 0$.

(2) If $g(j) = \xi_1$ for some j, then $f(j) = 1$.

(3) In all positions where neither f nor g has ξ_1 as value, their values coincide.

First we observe that G does not contain any triangle. Assume to the contrary that $h_0, h_1, h_2 \in [A]\binom{n}{1}$ do form a triangle. Without loss of generality, we can assume that

$$\min h_0^{-1}(\xi_1) < \min h_1^{-1}(\xi_1) < \min h_2^{-1}(\xi_1).$$

Let $i = \min h_1^{-1}(\xi_1)$. Then, by (1), $h_2(i) = 0$ and, by (2), $h_0(i) = 1$; this implies by (3) that $\{h_0, h_2\} \notin E(G)$.

Every $h \in [A]\binom{n}{2}$ the set $sp_1(h)$ contains an edge since if $f = (\xi_1, 1)$ and $g = (0, \xi_1)$ then $h \circ f, h \circ g \in [A]\binom{n}{1}$ satisfy conditions (1)–(3). □

Proposition 6.2 *For every $n \geq 3$ there exists a K_3-free graph $G = \left([1]\binom{n}{2}, E(G)\right)$ such that for every $h \in [1]\binom{n}{3}$ the set $sp_2(h)$ contains at least one edge.*

Proof: Let $n \geq 3$ and $f, g \in [1]\binom{n}{2}$ be such that $\min f^{-1}(\xi_1) < \min g^{-1}(\xi_1)$. We say that $\{f, g\}$ is an edge if and only if $\min f^{-1}(\xi_2) = \min g^{-1}(\xi_1)$.

Assume that $h_0, h_1, h_2 \in [1]\binom{n}{2}$ form a triangle. Then the fact that $\{h_0, h_1\}$ and $\{h_1, h_2\}$ are edges implies that

$$\min h_0^{-1}(\xi_2) = \min h_1^{-1}(\xi_1) < \min h_1^{-1}(\xi_2) = \min h_2^{-1}(\xi_1),$$

contradicting the fact that $\{h_0, h_2\}$ is an edge.

Every $h \in [1]\binom{n}{3}$ the set $sp_2(h)$ contains an edge, for if $f = (\xi_1, \xi_2, 0)$ and $g = (0, \xi_1, \xi_2)$ then $h \circ f, h \circ g \in [1]\binom{n}{1}$ satisfy the required condition. □

Proposition 6.3 *For every $n \geq 4$ there exists a K_3-free graph $G = \left([0]\binom{n}{3}, E(G)\right)$ such that for every $h \in [0]\binom{n}{4}$ the set $sp(h)$ contains at least one edge.*

Proof: The proof mimics the proof of Proposition 6.2 just letting ξ_1 play the role of the letter 0. So let $n \geq 4$ and $f, g \in [0]\binom{n}{3}$ be such that $\min f^{-1}(\xi_2) < \min g^{-1}(\xi_2)$. We define $\{f, g\}$ to be an edge if and only if $\min f^{-1}(\xi_3) = \min g^{-1}(\xi_2)$. This graph does not contain any triangle and for every $h \in [0]\binom{n}{4}$ the set $sp_3(h)$ contains an edge. □

The ideas leading to these counterexamples can be ramified and extended to show that there exist K_3-free graphs on, say, every (n, q, d)-set such that every (m, p, c)-subset of this (n, q, d)-set which spans an independent set in this graph can be forced to be of some very special structure.

7 Independent (m, p, c)-sets

Arithmetic progressions and finite sum sets are both solutions to partition regular systems of equations. As we will soon see, all solutions to a particular partition regular system are, in a sense, constructed from arithmetic progressions and finite sum sets.

A characterization of partition regular systems of equations was first given by Rado [13]. Deuber [1] later gave another characterization of partition regular systems using structures called (m, p, c)-sets which we now define.

Definition 7.1 *A set of integers M is an (m, p, c)-set if $M \subset \mathbb{N}$ and there exist positive integers $x_0, x_1, \ldots, x_m$ so that M is a union $M = R_0(M) \cup R_1(M) \cup \cdots \cup R_m(M)$, where*

$$\begin{array}{lcl} R_0(M) & = & \{cx_0 + \lambda_1 x_1 + \lambda_2 x_2 + \ldots + \lambda_m x_m : \ \lambda_1, \ldots, \lambda_m \in [-p, p]\}, \\ R_1(M) & = & \{cx_1 + \lambda_2 x_2 + \ldots + \lambda_m x_m : \ \lambda_2, \ldots, \lambda_m \in [-p, p]\}, \\ \vdots & & \vdots \\ R_{m-1}(M) & = & \{cx_{m-1} + \lambda_m x_m : \ \lambda_m \in [-p, p]\}, \\ R_m(M) & = & \{cx_m\}. \end{array}$$

Deuber proved that a linear system $A\boldsymbol{x} = \boldsymbol{0}$ is partition regular if and only if there exist positive integers m, p, c such that every (m, p, c)-set contains a solution of $A\boldsymbol{x} = \boldsymbol{0}$. So solving Conjecture 2.3 is tantamount to answering the following (perhaps first due to Deuber).

Conjecture 7.2 *Given k, m, p, c, and any K_k-free graph G on $\mathbb{N}$, one always find an (m, p, c)-set which is independent in G.*

Recently we have found a proof of this conjecture, but due to the length and complicated nature of the argument, it will appear in a subsequent paper.

As a related problem, if one considers $K_{k,k}$-free graphs instead of K_k-free graphs on $\mathbb{N}$ we indeed can expect to find independent (m, p, c)-sets with a fairly easy averaging argument.

Theorem 7.3 *For every k, m, p and c there exists an integer n such that every $K_{k,k}$-free graph G on vertex set $[n]$ contains an (m, p, c)-set which is independent in G.*

Proof: Assume that every (m, p, c)-set in $\{1, 2, \ldots, n\}$ contains an edge of G. Since each (m, p, c)-set is determined by the choice of $x_0, x_1, \ldots, x_m \in \{1, 2, \ldots, n\}$, observe that there exists $\alpha \geq \frac{1}{mp}$ such that there are at least αn^{m+1} many (m, p, c)-sets in $\{1, 2, \ldots, n\}$. The cardinality of each (m, p, c)-set is at most

$$l =: (2p+1)^m + (2p+1)^{m-1} + \ \ldots \ + (2p+1) + 1$$

and since each edge of G can be in at most $\binom{l}{2} n^{m-1}$ (m, p, c)-sets (having determined two elements of an (m, p, c)-set we have $(m-1)$ "degrees of freedom" to choose the rest) we infer that

$$|E(G)| \geq \frac{\alpha n^{m+1}}{\binom{l}{2} n^{m-1}} \geq \frac{2\alpha n^2}{l^2}.$$

Let $d_1 \geq d_2 \geq \cdots \geq d_n$ be the degree sequence of vertices of G. Then by Jensen's inequality (provided $n > \frac{1}{2^k} k l^{2k} m^k p^k$) we have

$$\frac{1}{\binom{n}{k}} \sum_{i=1}^{n} \binom{d_i}{k} \ \geq \ \frac{n}{\binom{n}{k}} \binom{\frac{2\alpha}{l^2} n}{k} \ \sim \ n \left(\frac{2\alpha}{l^2} \right)^k \ \geq \ k,$$

and hence, there are distinct vertices $y_1, \ldots, y_k$ completely joined to some k-set, say $\{t_1, \ldots, t_k\}$. $\square$

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