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A Canonical Partition Theorem for Equivalence Relations on Z^t

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A canonical version of the multidimensional version of van der Waerden's theorem on arithmetic progressions is proved.

1. INTRODUCTION

The starting point for our investigations is Theorem A which is due to Erdös and Graham [1].

THEOREM A. *For every positive integer m there exists a positive integer n such that for every coloring Δ: $\{0,..., n-1\} \to \omega$ there exists an arithmetic progression a, $a+d$,..., $a+(m-1)d$ of length m such that the restriction of Δ to $\{a, a+d,..., a+(m-1)d\}$ is either a constant or a one-to-one mapping.*

This theorem is the so called canonical version of van der Waerden's theorem on arithmetic progressions [7]. Originally, the consideration of canonical partition theorems goes back to Erdös and Rado [2] who proved this generalization of Ransey's theorem known as "Erdös–Rado canonization theorem."

THEOREM B. *Let k and m be positive integers. Then there exists a positive integer n satisfying,*

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(CAN) *For every coloring* $\Delta\colon [n]^k \to \omega$ *of the k-element subsets of n with arbitrarily many colors, there exists an m-element subset* $X \in [n]^m$ *and a (possibly empty) subset* $\mathbf{K} \subseteq \{0,..., k-1\}$ *such that for every two k-element subsets* $\{\alpha_0,..., \alpha_{k-1}\}$, $\{\beta_0,..., \beta_{k-1}\}$ *of X, (where* $\alpha_0 < \alpha_1 < \cdots < \alpha_{k-1}$ *and* $\beta_0 < \beta_1 < \cdots < \beta_{k-1}$), $\Delta(\{\alpha_0,..., \alpha_{k-1}\}) = \Delta(\{\beta_0,..., \beta_{k-1}\})$ *iff* $\alpha_i = \beta_i$ *for every* $i \in \mathbf{K}$.

In contrast to Theorem A, here one does not only have the alternative "either constant or one-to-one." None of the 2^k many types of colorings given in Theorem B, however, may be omitted without violating the property (CAN). For a general definition and further examples of canonical partition theorems, see [4].

The multidimensional version of van der Waerden's theorem on arithmetic progressions is due independently to Gallai (see [5]) and Witt [8].

THEOREM C. *Let t be a positive integer. Then for every positive integer M there exists a positive integer N such that for every coloring* $\Delta\colon N^t \to 2$, *where* $N^t = \{(a_0,..., a_{t-1}) \in \mathbf{Z}^t\colon\ 0 \leqslant a_0,..., a_{t-1} < N\}$ *there exists a monochromatic homothetic copy of* M^t, *i.e., there exist a positive integer d and a vector* $\mathbf{a} = (a_0,..., a_{t-1}) \in \mathbf{Z}^t$ *such that* $\phi_d^{\mathbf{a}}(M^t) = \{\mathbf{a} + d \cdot (\lambda_0,..., \lambda_{t-1})\colon (\lambda_0,..., \lambda_{t-1}) \in M^t\}$ *is a subset of* N^t *and* $\Delta \mid \phi_d^{\mathbf{a}}(M^t)$, *i.e., the restriction of* Δ *to* $\phi_d^{\mathbf{a}}(M^t)$, *is a constant mapping.*

The main result of this paper is a canonical version of this theorem. For convenience, we shall use equivalence relations rather than colorings with arbitrarily many colors. The main theorem may be stated in the following way. Think of M^t as being embedded into $\mathbf{Q}^t$, the t-dimensional vector space over the rationals. A vector $\mathbf{v} \neq 0$ is a *cross direction* for M^t iff there exists a line $\mathbf{a} + \langle \mathbf{v} \rangle$ which intersects M^t in at least two points. Let $\mathbf{U}(M, t)$ be the set of subspaces U of $\mathbf{Q}^t$ possessing a basis of crossing directions. Furthermore, add the null space $\{0\}$ to $\mathbf{U}(M, t)$. Suppose $U \in \mathbf{U}(M, t)$. An equivalence relation π on M^t is said to be of type $\pi(U)$ iff it is the coset equivalence relation on M^t modulo U which is defined by

$$\mathbf{x} \approx \mathbf{y} \bmod \pi \quad \text{iff} \quad \mathbf{x} + U = \mathbf{y} + U, \quad \text{where} \quad \mathbf{x}, \mathbf{y} \in M^t.$$

These equivalence relations are defined in a rather natural way and if $\mathbf{Z}^t$ is partitioned according to $\pi(U)$, then each homothetic copy of M^t is partitioned according to $\pi(U)$ as well. Thus, a canonical version of Theorem C has to consider at least the equivalence relations $\pi(U)$, $U \in \mathbf{U}(M, t)$. Our main result is that it suffices to consider just these equivalence relations.

MAIN THEOREM. *Let t be a positive integer. Then, for every positive integer M, there exists a positive integer* N^* *which satisfies:*

(CAN*) *For every equivalence relation* π *on the points of* N^t, *where* $N \geqslant N^*$, *there exists a homothetic copy* $\phi_d^{\mathbf{a}}(M^t) \subseteq N^t$ *of* M^t *and there exists a linear subspace* $U \in \mathbf{U}(M, t)$ *such that* $\pi \mid \phi_d^{\mathbf{a}}(M^t) = \pi(U)$, *i.e., the restriction of* π *to* $\phi_d^{\mathbf{a}}(M^t)$, *is of type* $\pi(U)$. *Moreover, the set* $\{\pi(U)\colon U \in \mathbf{U}(M, t)\}$ *is the unique minimal set of equivalence relations satisfying* (CAN*).

Remark (i). For $t = 1$ this is only Theorem A, since $\pi(\{0\})$ is the identity and $\pi(\mathbf{Q})$ is the one-block equivalence relation (corresponding to the constant coloring in Theorem A).

Remark (ii). All presently known proofs of Theorem A make use of the density version of van der Waerden's theorem on arithmetic progressions, i.e., Szemerédi's theorem [6]. Analogously, our proof of the main theorem uses Fürstenberg and Katznelson's density version of Theorem C [3]. Since the only available proofs of this density result involve heavy ergodic theoretic tools it still remains to find an elementary proof of our theorem.

Let us state Fürstenberg and Katznelson's result explicitly:

THEOREM D [3]. *Let* t *be a positive integer. Then for every positive integer* M *and positive rational* $\varepsilon > 0$, *there exists a positive integer* N^* *such that if* $\mathbf{S} \subseteq N^t$, *where* $N \geqslant N^*$, *is a set of points of* N^t *with* $|\mathbf{S}| \geqslant \varepsilon\, |N^t|$, *then there exists a homothetic copy* $\phi_d^{\mathbf{a}}(M^t)$ *of* M^t *contained in* $\mathbf{S}$, *i.e.,* $\phi_d^{\mathbf{a}}(M^t) \subseteq \mathbf{S}$.

Remark. The case $t = 1$ is Szemerédi's density result on arithmetic progressions [6].

Basically, the proof of the main theorem proceeds by induction on t, the case $t = 1$ being the canonical version of van der Waerden's theorem, i.e., Theorem A. Each step involves a little counting. In Section 2, we prepare the tools for doing this. In Section 3, the case $t = 1$ is proved explicitly and Section 4 contains the inductive step from t to $t + 1$.

2. SOME PRELIMINARIES

Let G be a 1-dimensional linear subspace of $\mathbf{Q}^{t+1}$, i.e., $G = \langle (g_0, \ldots, g_t) \rangle$ for some vector $(g_0, \ldots, g_t) \neq \mathbf{0}$. One easily verifies that $(g_0, \ldots, g_t)$ can always be chosen in such a way that:

(α) if g_i is the first nonzero entry, i.e., $g_i \neq 0$ and $g_0 = \cdots = g_{i-1} = 0$, then $g_i > 0$,

(β) $g_0, \ldots, g_t \in \mathbf{Z}$,

(γ) the numbers $g_0, \ldots, g_t$ are relatively prime, i.e., g.c.d. $(g_0, \ldots, g_t) = 1$. Throughout this paper we shall tacitly assume that $(g_0, \ldots, g_t)$ satisfies (α), (β), (γ), and $G = \langle (g_0, \ldots, g_t) \rangle$.

Let $\mathbf{x} \in \mathbf{Q}^{t+1}$. We say that the coset $\mathbf{x}+G$ is "a coset of G in M^{t+1}", where M is a positive integer, iff $(\mathbf{x}+G) \cap M^{t+1} \neq \varnothing$.

Fact 2.1

Let G be a fixed 1-dimensional subspace of $\mathbf{Q}^{t+1}$. Then the number of cosets of G in M^{t+1} is $O(M^t)$ as $M \to \infty$.

Proof. Let g_{i_0} be the first nonzero entry in $g=(g_0,..., g_t)$ and let $g_j^* = g_j/g_{i_0}$, $j=0,..., t$. Thus, $(g_0^*,..., g_t^*)=(0,..., 0, 1, g_{i_0+1}^*,..., g_t^*)$. The vectors $(x_0,..., x_t) \in \mathbf{Q}^{t+1}$ with $x_{i_0}=0$ form a system of representatives for the cosets of G. Thus, we have to count the number of vectors $\mathbf{x}=(x_0,..., x_t) \in \mathbf{Q}^t$ with $x_{i_0}=0$ and $(\mathbf{x}+\langle \mathbf{g}\rangle) \cap M^{t+1} \neq \varnothing$. If $g_j^* \geqslant 0$, then $x_i \in [-(M-1)g_j^*, M-1] \cap \mathbf{Z}$, and if $g_j^* < 0$, then $x_i \in [0, (M-1)(1-g_j^*)] \cap \mathbf{Z}$, where $j=0,..., t$, $j \neq i_0$. This immediately gives the desired result. ∎

Fact 2.2

Let $G=\langle \mathbf{g}\rangle$ be a fixed 1-dimensional subspace of $\mathbf{Q}^{t+1}$ and let $g=\operatorname{Max}\{|g_i|: i=0,..., t\}$. Then there exists a constant c (depending on G) such that the number of cosets of G in M^{t+1} which contain precisely l points of M^{t+1}, where $l < \lfloor M/g \rfloor$, is less than cM^{t-1}.

Proof. Let us call a point $(x_0,..., x_t) \in M^{t+1}$ a "right-hand corner" iff there exists an $i \in \{0,..., t\}$ such that $x_i+g_i \geqslant M$ or $x_i+g_i<0$. Analogously, a point $(x_0,..., x_t) \in M^{t+1}$ is a "left-hand corner" iff there exists an $i \in \{0,..., t\}$ such that $x_i-g_i \geqslant M$ or $x_i-g_i<0$. Now let $\mathbf{x}$ be a right-hand corner. Then, $|(\mathbf{x}+G) \cap M^{t+1}|=l$ iff $\mathbf{x}-(l-1)\mathbf{g}$ is a left-hand corner. Thus, there exists a $j \in \{0,..., t\}$ with $x_j+g_j \geqslant M$ or $x_j+g_j<0$ and there exists a $j^* \in \{0,..., t\}$ with $x_{j^*}-lg_{j^*} \geqslant M$ or $x_{j^*}-lg_{j^*}<0$, but with $\mathbf{x}-(l-1)\mathbf{g} \in M^{t+1}$.

We claim that $j \neq j^*$. Assume to the contrary that $j=j^*$.

Case 1. $(x_j+g_j \geqslant M)$. In particular it then follows that $x_j-lg_j<0$. But $x_j-(l-1)g_j \geqslant 0$. This implies that $M=(l-1)g_j+R$ for some R, $0<R<2g_j$, contradicting the choice of l as being strictly smaller than $\lfloor M/g \rfloor$.

Case 2. $(x_j+g_j<0)$. In particular, $x_j+lg_j \geqslant M$. However, $x_j+(l-1)g_j<M$ which contradicts the choice of l.

Next, observe that there exist at most $\binom{t+1}{2}$ possible choices for j and j^*. Also, $x_j \in ([0, g_j) \cup [M-g_j, M)) \cap \mathbf{Z}$ and similarly, $x_{j^*}-(l-1)g_{j^*} \in ([0, g_{j^*}) \cup [M+g_{j^*}, M)) \cap \mathbf{Z}$. Thus, it suffices to take $c=4g^2\binom{t+1}{2}$. ∎

Fact 2.3

Let M be a fixed positive integer. The number of homothetic copies of M^t in N^t is at least cN^{t+1} for a positive constant c depending only on t.

Proof. Recall that each homothetic copy of M^t in N^t is given by $\phi_d^{\mathbf{a}}(M^t) = \{\mathbf{a} + d \cdot \mathbf{x} : \mathbf{x} \in M^t\}$.

3. Proof of Theorem A

Notation. For sets X, we denote by $\Pi(X)$ the set of equivalence relations on X. For $x \in X$ and $\pi \in \Pi(X)$, let $\pi(x)$ denote the block (equivalence class) of π containing x. Let N be a positive integer and $\varepsilon > 0$ be a rational. An equivalence relation $\pi \in \Pi(N)$ is called *ε-injective* iff $|\pi(x) \cap N| < \varepsilon N$ for every $x \in N$, i.e., π is ε-injective iff every block of π consists of less than an εth part of $\{0,..., N-1\}$. Thus, there exist at least $1/\varepsilon$ different equivalence classes. The name "ε-injective" is justified by

Lemma 3.1. *Let M be a positive integer. Then there exists a rational $\varepsilon > 0$ and a positive integer N^* such that for every positive integer $N > N^*$ and every ε-injective equivalence relation $\pi \in \Pi(N)$ there exists an arithmetic progression $a, a+d,..., a+(M-1)d$ of length M in N such that $\pi \mid \{a,..., a+(M-1)d\}$ is the identity, i.e., $a + id \not\approx a + jd (\operatorname{mod} \pi)$ for every $0 \leqslant i < j < M$.*

Proof. Choose ε sufficiently small and N sufficiently large (to be specified later) and let $\pi \in \Pi(N)$ be an ε-injective equivalence relation. We claim that there exists an arithmetic progression of length M satisfying the properties of Lemma 3.1. Assume to the contrary that for each arithmetic progression in N of length M there exist at least two members x and y with $x \approx y (\operatorname{mod} \pi)$.

(*) By Fact 2.3 there exist at least cN^2 two-element subsets $\{x, y\} \subseteq N$ with $x \approx y$ $(\operatorname{mod} \pi)$.

However, π is ε-injective. Thus, each block of π provides at most $\varepsilon^2 O(N^2)$ such two-element subsets.

One easily observes that the number of two-element subsets $\{x, y\} \subseteq N$ with $x \approx y$ $(\operatorname{mod} \pi)$ is maximized if the cardinalities of all blocks of π are as large as possible. Hence, there exist at most $\varepsilon^2 O(N^2)\, \varepsilon^{-1} = \varepsilon O(N^2)$ such two-element subsets. This, however, is a contradiction, provided $\varepsilon > 0$ is sufficiently small and N is sufficiently large in order to violate (*). ∎

Theorem A can now be proved as follows. Let M be a positive integer. Let $\varepsilon > 0$ and N^* be chosen according to Lemma 3.1. Also let N^{**} be such that for every $N \geqslant N^{**}$ and every $S \subseteq N$ with $|S| \geqslant \varepsilon N$, there exists an M-element arithmetic progression contained in S. The existence of such an N^{**} is guaranteed by Szemerédi's theorem, i.e., Theorem D with $t = 1$.

We claim that every $N \geqslant \max(N^*, N^{**})$ satisfies the requirements of Theorem A. For if $\pi \in \Pi(N)$ is ε-injective, the assertion follows by Lemma 3.1. If π is not ε-injective, then $|\pi(X)| \geqslant \varepsilon N$ for some $x \in N$ and the assertion follows by choice of N^{**}.

4. The Inductive Step from t to $t+1$

In this section, we assume that the main theorem is valid for some positive integer t. Under this assumption, we shall prove it for $t+1$. In particular, the positive integer t will be fixed throughout this section.

Notation. For positive integers M, let $G(M)$ be the set of those 1-dimensional linear subspaces of G of $\mathbf{Q}^{t+1}$ which are crossing directions for M^{t+1}, i.e., for which there exists $\mathbf{a} \in \mathbf{Q}^{t+1}$ such that $|(\mathbf{a} + G) \cap M^{t+1}| \geqslant 2$.

Definition. Let M and N be positive integers and $\varepsilon > 0$ be rational. An equivalence relation $\pi \in \Pi(N^{t+1})$ is called *ε-injective with respect to M* iff for every $G \in \mathbf{G}(M)$ the following inequality is valid:

$$\begin{aligned} &|\{y + G \colon y \in \mathbf{Q}^{t+1}, (y + \mathbf{G}) \cap N^{t+1} \neq \varnothing, |\pi(x) \cap (y + G) \cap N^{t+1}| \\ &\quad < \varepsilon\, |(y + G) \cap N^{t+1}| \text{ for every } x \in (y + G) \cap N^{t+1}\}| \\ &\quad \geqslant (1 - \varepsilon)|\, \{(y + G) \colon y \in \mathbf{Q}^{t+1},\ (y + G) \cap N^{t+1} \neq \varnothing\}|. \end{aligned}$$

Remark. $\pi \in \Pi(N^{t+1})$ is ε-injective with respect to M iff for every $G \in \mathbf{G}(M)$ the equivalence relation π acts ε injectively (in the sense of Section 3) on at least an $(1 - \varepsilon)$th of the cosets of G in N^{t+1}. Similar to Section 3, we have Lemma 4.1 (which justifies the name "ε-injective").

Lemma 4.1. *Let M be a positive integer. Then, there exists a rational $\varepsilon > 0$ and a positive integer N^* such that for every equivalence relation $\pi \in \Pi(N^{t+1})$, where $N \geqslant N^*$, which is ε-injective with respect to M, there exists a homothetic copy $\phi_d^{\mathbf{a}}(M^{t+1}) \subseteq N^{t+1}$ such that $\pi | \phi_d^{\mathbf{a}}(M^{t+1})$ is the identity, i.e., $\pi | \phi_d^{\mathbf{a}}(M^{t+1}) = \pi(\{0\})$.*

Proof. Choose $\varepsilon > 0$ sufficiently small and N sufficiently large (to be specified later) and let $\pi \in \Pi(N^{t+1})$ be ε-injective with respect to M. Assume that the conclusion of Lemma 4.1 fails to be true, i.e., for each homothetic copy $\phi_d^{\mathbf{a}}(M^{t+1}) \subseteq N^{t+1}$ there exists at least one two-element subset $\{x, y\} \subseteq \phi_d^{\mathbf{a}}(M^{t+1})$ with $x \approx y (\operatorname{mod} \pi)$ and $x + G = y + G$ for some $G \in \mathbf{G}(M)$. By Fact 2.3 there exist at least cN^{t+2} such two-element subsets in

N^{t+1}, i.e., for some fixed $G \in \mathbf{G}(M)$ there exist at least cN^{t+2} two-element subsets $\{x, y\} \in N^{t+1}$ with $x + G = y + G$ and $x \approx y \pmod{\pi}$.

Since π is ε-injective, then we can find an upper bound for such two element subsets using the ε-injectivity of π. The cosets $x + G$ of G in N^{t+1} can be split into two disjoint sets:

$$A = \{y + G\colon y \in \mathbf{Q}^{t+1}, (y+G) \cap N^{t+1} \neq \varnothing \text{ and } |\pi(x) \cap (y+G) \cap N^{t+1}| < \varepsilon\, |(y+G) \cap N^{t+1}| \text{ for every } x \in (y+G) \cap N^{t+1}\},$$

$$B = \{y + G\colon y \in \mathbf{Q}^{t+1}, (y+G) \cap N^{t+1} \neq \varnothing \text{ and } |\pi(x) \cap (y+G) \cap N^{t+1}| \geqslant \varepsilon\, |(y+G) \cap N^{t+1}| \text{ for some } x \in (y+G) \cap N^{t+1}\}.$$

By Fact 2.1 we have $|A \cup B| = O(N^t)$ and since π is ε-injective, it follows that $|B| \leqslant \varepsilon\, |A \cup B| \leqslant \varepsilon O(N^t)$. Consider first those cosets $x + G$ which belong to B. Assume that the worst possible case occurs, i.e., that $\pi \mid (x + G)$ is the constant (one-block) equivalence relation for every coset $x + G \in B$. Since $|(x + G) \cap N^{t+1}| \leqslant N$, this yields at most $\varepsilon O(N^t)\, O(N^2) = \varepsilon O(N^{t+2})$ two-element subsets $\{x, y\} \in N^{t+1}$ with $x \approx y\, (\mathrm{mod})\pi$ and $x + G = y + G \in B$.

Next, consider those cosets $x + G$ which belong to A. Each block of $\pi \mid (x + G)$ contains at most εN elements, and thus, provides at most $\varepsilon^2 O(N^2)$ two-element sets $\{y, z\} \subseteq (x + G)$ with $y \approx z \pmod{\pi}$. Again, the number of such subsets is maximized iff the cardinalities of blocks of $\pi \mid (x + G)$ are as large as possible. Since $|A| \leqslant O(N^t)$ by Fact 2.1, then there exist at most $O(N^t)\, \varepsilon^2 O(N^2)\, \varepsilon^{-1}$ two-element subsets $\{x, y\} \subseteq N^{t+1}$ with $x \approx y \pmod{\pi}$ and $x + G = y + G$.

Putting the preceding remarks together we see that there exist at most $\varepsilon O(N^{t+2}) + \varepsilon O(N^{t+2}) = \varepsilon O(N^{t+2})$ two-element subsets $\{x, y\} \in N^{t+1}$ with $x \approx y \pmod{\pi}$ and $x + G = y + G$. This, however, is a contradiction to the remark made at the top of the page, provided that $\varepsilon > 0$ is sufficiently small and N is sufficiently large. ∎

LEMMA 4.2. *Let M be a positive integer and $\varepsilon > 0$ be rational. Then, there exists a positive integer N^* such that for every positive integer $N \geqslant N^*$ and every equivalence relation $\pi \in \Pi(N^{t+1})$ which is not ε-injective with respect to M the following is valid*:

There exists a $G \in \mathbf{G}(M)$ and there exists a homothetic copy $\phi_d^{\mathbf{a}}(M^{t+1}) \subseteq N^{t+1}$ such that $x \approx y \pmod{\pi}$ for all $x, y \in \phi_d^{\mathbf{a}}(M^{t+1})$ with $x + G = y + G$.

Proof. Let N be sufficiently large and let $\pi \in \Pi(N^{t+1})$ be an equivalence relation which is not ε injective with respect to M. Hence, there exists $G \in \mathbf{G}(M)$ such that for a suitable $c_0 > 0$, $|B| > \varepsilon c_0 N^t$, where A and B are defined as in the proof of Lemma 4.1. For each coset $x + G \in B$ let

$S_x \subseteq (x+G) \cap N^{t+1}$ be such that $y \approx z \pmod{\pi}$ for every $y, z \in S_x$ and such that $|S_x| \geqslant \varepsilon\,|(x+G) \cap N^{t+1}|$. Let $\mathbf{S}$ be the union of all sets S_x, $x+G \in B$. In what follows, $c_1, c_2, \ldots$, will denote suitably chosen positive constants.

We claim that $|S| \geqslant \varepsilon^3 c_1 N^{t+1}$. One easily observes that $|\bigcup B| \geqslant \varepsilon^2 c_2 N^{t+1}$, for by Fact 2.2 it follows that $|(x+G) \cap N^{t+1}| = l$ for at most $c_3 N^{t-1}$ cosets $x+G \in B$, where $l < \lfloor N/g \rfloor$ and g is defined as in Fact 2.2. Set $p = (1/c_3 N^{t-1})\, \varepsilon c_4 N^t$. Since $|B| > \varepsilon c_0 N^t$ it follows that

$$\left|\bigcup B\right| \geqslant c_5 N^{t-1} \sum_{l=1}^{p} l = \varepsilon^2 c_6 N^{t+1}.$$

Hence, $|S| \geqslant \varepsilon\varepsilon^2 c_7 N^{t+1} = \varepsilon^3 c_7 N^{t+1}$. Now N may be chosen to be sufficiently large so that the Fürstenberg–Katznelson theorem (Theorem D) applies. Thus, there exists a homothetic copy $\phi_d^{\mathbf{a}}(M^{t+1})$ of M^{t+1} which is contained in $\mathbf{S}$. By construction $\phi_d^{\mathbf{a}}(M^{t+1})$ has the desired properties. ∎

LEMMA 4.3. *Let M be a positive integer. Then there exists a positive integer N^* such that for every positive integer $N \geqslant N^*$ and every equivalence relation $\pi \in \Pi(N^{t+1})$ such that for some $G \in \mathbf{G}(M)$ it follows that $x \approx y \pmod{\pi}$ for all $x, y \in N^{t+1}$ with $x+G = y+G$, the following is true:*

There exists a homothetic copy $\phi_d^{\mathbf{a}}(M^{t+1})$ of M^{t+1} in N^{t+1} and there exists a subspace $U \in \mathbf{U}(M, t+1)$ such that $\pi \mid \phi_d^{\mathbf{a}}(M^{t+1}) = \pi(U)$.

Proof. Let N be sufficiently large and $\pi \in \Pi(N^{t+1})$ and $G \in \mathbf{G}(M)$ be as above. More precisely, let $G = \langle \mathbf{g} \rangle$, where $(g_0, \ldots, g_t) \in \mathbf{Z}^{t+1}$ and the entries $g_0, \ldots, g_t$ are relatively prime. Without loss of generality assume that $g_0 = \max\{|g_i| : i = 0, \ldots, t\}$—all other cases can be handled analogously. Put $M^* = g_0(3M-2)+1$. N should be large enough for the inductive hypothesis on M^* and t to hold. Since $\mathbf{G}(M)$ is finite, we easily can find an N which is sufficiently large with respect to each $G \in \mathbf{G}(M)$.

Now, consider the equivalence relation $\hat{\pi} \in \Pi(N^t)$ which is given by $(x_1, \ldots, x_t) \approx (y_1, \ldots, y_t) \pmod{\hat{\pi}}$ iff $(0, x_1, \ldots, x_t) \approx (0, y_1, \ldots, y_t) \pmod{\pi}$. Using the inductive hypothesis, there exists a subspace $U^* \in \mathbf{U}(M^*, t)$ and a homothetic copy $\phi_{a^*}^{\mathbf{a}^*}(M^{*t}) \subseteq N^t$ such that

$$x \approx y \pmod{\hat{\pi}} \qquad \text{iff} \quad \mathbf{x} + U^* = \mathbf{y} + U^*, \qquad \text{where} \quad \mathbf{x}, \mathbf{y} \in \phi_{a^*}^{\mathbf{a}^*}(M^{*t}).$$

Write

$$\mathbf{a}^* = (a_1^*, \ldots, a_t^*)$$

and let

$$\mathbf{a} = (0, a_1^* + Mg_0 d^*, \ldots, a_t^* + Mg_0 d^*) \qquad \text{and} \qquad d = g_0 d^*.$$

Finally, let U' be the subspace of $\mathbf{Q}^{t+1}$ which is generated by $G \cup \{(0, x_1, ..., x_t): (x_1, ..., x_t) \in U^*\}$. We claim that

$$\mathbf{x} \approx \mathbf{y} \pmod{\pi} \quad \text{iff } \mathbf{x} + U' = \mathbf{y} + U', \quad \text{where } \mathbf{x}, \mathbf{y} \in \phi_d^{\mathbf{a}}(M^{t+1}) \subseteq N^{t+1}.$$

The inclusion $\phi_d^{\mathbf{a}}(M^{t+1}) \subseteq N^{t+1}$ is obvious. Let $\mathbf{x} \in \phi_d^{\mathbf{a}}(M^{t+1})$ be arbitrary, say $\mathbf{x} = (\lambda_0 g_0 d, a_1^* + Mg_0 d + \lambda_1 g_0 d, ..., a_t^* + Mg_0 d + \lambda_t g_0 d)$, for some $(\lambda_0, ..., \lambda_t) \in M^{t+1}$. Consider $\hat{x} = \mathbf{x} - \lambda_0 d(g_0, ..., g_t) = (0, a_1^* + d(Mg_0 + \lambda_1 g_0 - \lambda_0 g_1), ..., a_t^* + d(Mg_0 + \lambda_t g_0 - \lambda_0 g_t))$. By the choice of π, it follows that $\hat{x} = \mathbf{x} (\text{mod } \pi)$ and since,

$$(a_1^* + d(Mg_0 + \lambda_1 g_0 - \lambda_0 g_1), ..., a_t^* + d(Mg_0 + \lambda_t g_0 - \lambda_0 g_t)) \in \phi_d^{\mathbf{a}^*}(M^{*t}),$$

it follows that $\mathbf{x} + U'$ belongs to the block of π containing $\mathbf{x}$, i.e., $(\mathbf{x} + U') \cap \phi_d^{\mathbf{a}}(M^{t+1}) = \pi \mid \phi_d^{\mathbf{a}}(M^{t+1})(\mathbf{x})$. Thus, $\pi \mid \phi_d^{\mathbf{a}}(M^{t+1}) = \pi(U')$. However, $U' \in \mathbf{U}(M^*, t+1)$ and possibly $U' \notin \mathbf{U}(M, t+1)$. In this case, though, it is easy to find a subspace $U \subseteq U'$ such that $U \in \mathbf{U}(M, t+1)$ and $\pi \mid \phi_d^{\mathbf{a}}(M^{t+1}) = \pi(U)$. It only suffices to consider any maximal independent subset $\{v_1, ..., v_r\} \subseteq U'$ such that $\langle v_i \rangle \in \mathbf{G}(M)$, $i = 1, ..., r$, and set $U = \langle v_1, ..., v_r \rangle$. This proves the lemma and the proof of the main theorem is complete. ∎

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