ON MONOCHROMATIC SUBSETS OF A RECTANGULAR GRID

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ABSTRACT. For $n \in \mathbb{N}$, let $[n]$ denote the integer set $\{0, 1, \ldots, n-1\}$. For any subset $V \subset \mathbb{Z}^2$, let $Hom(V) = \{cV + \mathbf{b} : c \in \mathbb{N}, \mathbf{b} \in \mathbb{Z}^2\}$. For $k \in \mathbb{N}$, let $R_k(V)$ denote the least integer N_0 such that for any $N \geq N_0$ and for any k-coloring of $[N]^2$, there is a monochromatic subset $U \in Hom(V)$. The argument of Gallai ensures that $R_k(V)$ is finite. We investigate bounds on $R_k(V)$ when V is a three or four-point configuration in general position. In particular, we prove that $R_2(S) \leq VW(8)$, where VW is the classical van der Waerden number for arithmetic progressions and S is a square $S = \{(0,0), (0,1), (1,0), (1,1)\}$.

1. Introduction

Let, for a positive integer n, $[n] = \{0, 1, \ldots, n-1\}$. The classical Theorem of van der Waerden [16] claims that for any $n, k \in \mathbb{N}$, there is $N_0 \in \mathbb{N}$ such that for all $N \geq N_0$ and any k-coloring $\chi : [N] \to [k]$, there is a monochromatic arithmetic progression of length n (n-AP). Define $VW(k, n)$ to be the least such integer guaranteed by van der Waerden's Theorem. The number $VW(n) = VW(2, n)$ is usually referred to as the classical van der Waerden number. The best known bounds are

$$(n-1)2^{n-1} \leq VW(n) \leq 2^{2^{2^{2^{2^{n+9}}}}},$$

with the lower bound valid for values of $n-1$ which are prime. Here, the upper and lower bounds are due to Gowers [4], and Berlekamp [1], respectively; see also [5]. The only known exact values for VW are $VW(3) = 9$, $VW(4) = 35$, and $VW(5) = 178$; the first two are due to Chvátal [2], while the third is due to Stevens and Shantaram [14]. Kouril proved in [10] that $VW(6) \geq 1132$, and conjectured that equality holds; his proof of this conjecture is featured in a paper which is unavailable at the time of this writing.

The density version of van der Waerden's Theorem (see the celebrated result of Szemerédi [15]) asserts that an arithmetic progression of a fixed length is always present in dense subsets of integers, thus implying the classical van der Waerden Theorem. For the improved bounds, see the results of Gowers [4] and Shkredov [13].

In search for better bounds and better understanding of van der Waerden numbers, some connections between higher-dimensional problems and the original problem have been established by Graham [5]. In this note, we continue this effort by studying a problem of independent interest when instead of arithmetic progressions in $[n]$, configurations in $[n]^2$ are being considered.

For a set $V \subseteq \mathbb{Z}^2$ and $\mathbf{b} \in \mathbb{Z}^2$, define $V + \mathbf{b} = \{\mathbf{v} + \mathbf{b} : \mathbf{v} \in V\}$. We say that a subset U of the grid is homothetic to a set V in the grid if $U = cV + \mathbf{b}$, for some constants $c \in \mathbb{R}$, $c \neq 0$, and $\mathbf{b} \in \mathbb{Z}^2$. In particular, we consider the set of all squares, i.e., sets homothetic to $S = \{(0,0),(0,1),(1,0),(1,1)\}$, and the set of L-sets homothetic to $L = \{(0,0),(0,1),(1,0)\}$. In this note we consider a stronger notion, when the coefficient c above is a natural number. Let

$$Hom(V) = \{cV + \mathbf{b} : c \in \mathbb{N}, \mathbf{b} \in \mathbb{Z}^2\}.$$

Given $k \in \mathbb{N}$, let

$$R_k(V) = \min\{n : \text{any } k\text{-coloring of } [n]^2 \text{ contains a monochromatic set from } Hom(V)\}.$$

The argument of Gallai, see for example [6], implies that $R_k(V)$ exists for finite V. Gallai's Theorem together with results of Shelah found in [12] immediately give the upper bound in terms of Hales-Jewett numbers, HJ,

$$R_2(S) \leq 2^{2^{2^{\cdot^{\cdot^{\cdot^{2}}}}}} 2^{HJ(3,4)};$$

where the height of the tower is 24, see Appendix A for details. Here, we improve this bound to $R_2(S) \leq \min\{VW(8), 5 \cdot 2^{2^{40}}\}$. One of the results we use is the bound by Graham and Solymosi [7]:

$$R_k(L) \leq 2^{2^k}. \tag{1}$$

Recall that a collection of points in the plane in *general position* means that no three of them are collinear. Note than an immediate lower bound on $R_k(V)$ for any V in general position with $|V| \geq 3$ is $R_k(V) \geq k$; this can be seen by coloring the i^{th} row of $[k]^2$ with color i. Since each row has its own color and no three points of any $X \in Hom(V)$ can lie on one row, we avoid a monochromatic homothetic copy of V.

In this manuscript, we study mostly $R_2(V)$, when V is a 3 or 4-element set in general position. Theorem 1 gives an argument using forbidden configuration for squares. Theorem 2 provides bounds for arbitrary 3 and 4-element sets in a general position in terms of $R_k(L)$ using a reduction argument (independent of Theorem 1) by looking at a smaller grid but using more colors; see also presentations of Bill Gasarch [3] on the topic.

Theorem 1. $13 \leq R_2(S) \leq VW(8)$.

For a set $A \subseteq [n]^2$, let the *square-size* of A be $s_A = \min\{\ell : \ell \in \mathbb{N}, \exists X \subseteq [\ell]^2 \text{ such that } X \in Hom(A)\}$; i.e., the size of the smallest square containing A.

Theorem 2. *Let T and Q be sets of three and four points in the grid in general position, respectively. Then*

$$R_k(T) \leq 2s_T R_k(L), \quad R_2(Q) \leq 20 s_Q R_{40}(L).$$

Note that (1) and Theorem 2 imply that $R_2(Q) \leq 20s_Q 2^{2^{40}}$. We can also reduce the bound slightly in the case of the square S to $R_2(S) \leq 5 \cdot 2^{2^{40}}$. We prove these two Theorems in the next sections, leaving the routine case analysis for Appendix B. In the last section we compare our results with the best known density results.

2. Proof of theorem 1

When we consider 2-colorings of the grid, we assume that the codomain is the set $\{\circ, \bullet\}$. Under an arbitrary 2-coloring χ, if $\chi((x,y)) = \circ$ we say that (x,y) is colored white, and if $\chi((x,y)) = \bullet$, we say that (x,y) is colored black.

Upper bound Let $n \geq VW(8)$. Let $\chi : [n]^2 \to \{\circ, \bullet\}$ be a coloring of $[n]^2$ in two colors. By van der Waerden's Theorem, every row of $[n]^2$ contains a monochromatic 8-AP; in particular, the middle row contains an 8-AP $P = \{X, X+d, \ldots, X+7d\}$. Without loss of generality, we may assume $d = 1$ and $\chi(P) = \circ$. Let $\overline{P} = P + (0,1)$, $\underline{P} = P + (0,-1)$, and $* \in \{\circ, \bullet\}$. We consider cases according to whether either $\overline{P}$ or $\underline{P}$ have four consecutive black vertices, three consecutive black vertices in the center, two consecutive black vertices in the center, or none of the above. We show that there is a monochromatic square in each of these cases.

In the case analysis (details in appendix B), we use facts about four configurations in the grid, see Figure 5.

Case 1: $\overline{P}$ or $\underline{P}$ contains 4 consecutive black vertices.
Figure 6 deals with the case when there are three vertices to one side of these 4 consecutive vertices.
Figure 7 deals with the case when these 4 consecutive vertices are in the center.

Case 2: Case 1 does not hold and there are three consecutive black vertices in $\overline{P}$ or in $\underline{P}$ with at least two vertices on both sides.
Figure 8 deals with this case.

Case 3: Cases 1 and 2 do not hold and there are two consecutive black vertices in the center of $\overline{P}$ or in the center of $\underline{P}$.
Figure 9 deals with this case.

Case 4: Cases 1, 2, 3 do not hold.
This case implies that the two central positions above and below P are occupied by white and black vertices. Since it is impossible to have a white vertex x right above P and a white vertex exactly below x and P (see Figure 5 (2)), this case (up to reflection) gives the folowing colorings of $\overline{P}$ and $\underline{P}$:

$* * * \bullet \circ \bullet **$ and $* * \bullet \circ \bullet * **$. Figure 10 displays two grey diamonds marked 1. Figures 10, 11, and 12 deal with the case that these both have color $\circ$. Figures 13, 14, and 15 deal with the case that these both have color $\bullet$. Lastly, Figures 16 and 17 deal with the case when these vertices have different colors, completing the proof of the upper bound.

Lower bound Let $n = \lceil (VW(k,4)-1)/3 \rceil$. We construct a k-coloring χ' of $[n]^2$ which contains no monochromatic square. Let $\chi : \{0,1,\ldots,VW(k,4)-2\} \to \{1,2,\ldots,k\}$ be a coloring which admits no 4-AP. Define a k-coloring χ' on $[n]^2$ by $\chi'(x,y) = \chi(x+2y)$. If χ' admits a monochromatic square, then there exist (x,y) and $d \in \mathbb{N}$ such that $\chi'(x,y) = \chi'(x+d,y) = \chi'(x,y+d) = \chi'(x+d,y+d)$. But the definition of χ' gives that $\chi(x+2y) = \chi(x+2y+d) = \chi(x+2y+2d) = \chi(x+2y+3d)$, a 4-AP. This is a contradiction, so $R_k(S) \geq \lceil (VW(k,4)-1)/3 \rceil$, as desired. Using a 2-coloring of [34] with no 4-AP due to Chvátal [2], we can construct a specific 2-coloring of $[12]^2$ which contains no monochromatic square, and hence $R_2(S) \geq 13$; see Figure 1.

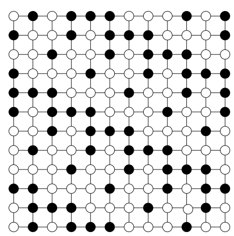

FIGURE 1. A 2-coloring of $[12]^2$ with no monochromatic square.

Using the best known lower bounds for $W(k,4)$ due to Rabung [11] and Herwig, et al. [9], we have that $R_3(S) > 97$, $R_4(S) > 349$, $R_5(S) > 751$, and $R_6(S) > 3259$.

3. Proof of Theorem 2

Again, we assume that the codomain for any 2-coloring χ is $\{\circ, \bullet\}$, and say (x,y) is colored white for $\chi((x,y)) = \circ$, and (x,y) is colored black for $\chi((x,y)) = \bullet$. Define the *diagonal* D_n of $[n]^2$ to be $D_n := \{(x,y) : x+y = n-1\}$, and the *lower triangle* $T_n = \{(x,y) : (x,y) \in [n]^2,\ x+y \leq n-1\}$. Throughout this section we shall be using a map which allows us to deal with arbitrary three point configurations as L-sets. We say that a subset $\{\mathbf{u_1}, \mathbf{u_2}, \mathbf{u_3}\}$ of three distinct elements in the grid forms a 3-AP, if, up to reordering, there is a vector $\mathbf{u}$ such that $\mathbf{u_3} = \mathbf{u_2} + \mathbf{u}$, $\mathbf{u_2} = \mathbf{u_1} + \mathbf{u}$. Given $X \subseteq \mathbb{Z}^2$ and $m, k \in \mathbb{N}$, we say that a collection of subsets $\mathcal{X} \subseteq [m]^2 \cap Hom(X)$ is a *forcing set* (with respect to parameters X, m, and k) if in any k-coloring of $[m]^2$ there is a monochromatic set from $\mathcal{X}$. Let

$forc(X, m, k)$ denote the cardinality of the smallest such collection $\mathcal{X}$. In the next two Lemmas we find bounds for $R_k(T)$, where T is a three point configuration and we prove that that for any such T and $k = 2$, there is a forcing set with 20 sets in it.

Lemma 1. $R_2(L) = 5$. *Furthermore,* $forc(L, 5, 2) \leq 20$.

Proof. To see that $R_2(L) \geq 5$, consider the coloring of $[4]^2$ with no monochromatic L-set shown in Figure 2. Consider a 2-coloring of $[5]^2$. At least 3 elements on a diagonal, D_5, are of the same color, say black. If D_5 has a 3-AP, then we immediately have a monochromatic L-set contained in the lower triangle. If D_5 has at least 4 black vertices, then either there is a 3-AP in it, or, there are exactly four black vertices on this diagonal and the central vertex is white. Then one of $\{(0, 4), (0, 3), (1, 3)\}, \{(3, 1), (3, 0), (4, 0)\}$, or $\{(0, 3), (0, 0), (3, 0)\}$ will be a monochromatic L-set. Therefore there are exactly three black vertices on the diagonal, and they do not form a 3-AP. The possible colorings (up to symmetries) of the diagonal in this case are shown in Figure 3. In each of these cases, it is easy to conclude that there is a monochromatic L-set in the lower triangle. Hence, $R_2(L) \leq 5$ and thus $R_2(L) = 5$. Since the number of L-sets in T_5 is 20, $forc(L, 5, 2) \leq 20$. $\square$

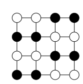

FIGURE 2. A 2-coloring of $[4]^2$ with no monochromatic L-set.

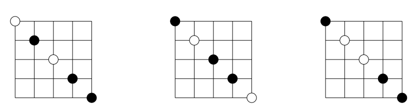

FIGURE 3. Colorings of D_5 with three black points not forming 3-AP.

For a given three point subset T of $\mathbb{Z}^2$ in general position, define the *parallelogram size* p_T to be the square size of the parallelogram defined by T. Recall that the *square size* of a set X corresponds to the size of the smallest square containing X. For example, when $T = L$, $p_T = 1$; when $T = \{(0, 0), (1, 2), (-1, 3)\}$, $p_T = 5$. Note that $p_T \leq 2s_T$. By choosing an appropriate linear transform, we find a bound on $R_k(T)$ in terms of $R_k(L)$.

Lemma 2. *If* $T \subseteq \mathbb{Z}^2$ *is in general position with* $|T| = 3$ *then* $R_k(T) \leq p_T R_k(L)$. *Furthermore,* $R_2(T) \leq 5p_T$ *and* $forc(T, 5p_T, 2) \leq 20$.

Proof. Let $T = \{\mathbf{t}_1, \mathbf{t}_2, \mathbf{t}_3\} \subset \mathbb{Z}^2$ be a set in general position with two sides corresponding to vectors $\mathbf{u} = \mathbf{t}_2 - \mathbf{t}_1$ and $\mathbf{v} = \mathbf{t}_3 - \mathbf{t}_1$, let $k \geq 2$ be an integer and let $q = R_k(L)$. Let $n = p_T q$ and Q be the parallelogram defined by T. Then qQ is contained in an $n \times n$ square grid. Formally, let $\mathbf{x} \in \mathbb{Z}^2$ such that $qQ + \mathbf{x} \subseteq [n]^2$.

Let $X = [n]^2 \cap \{k\mathbf{u} + l\mathbf{v} + \mathbf{x} : k, l \in \mathbb{N} \cup \{0\}\}$. Define $\phi : X \to [n/p_T]^2$ by $\phi(k\mathbf{u} + l\mathbf{v} + \mathbf{x}) = (k, l)$. Let χ be a k-coloring of $[n]^2$. This induces a k-coloring χ' of $[n/p_T]^2$ by $\chi'(k, l) = \chi(k\mathbf{u} + l\mathbf{v} + \mathbf{x})$. As $q = R_k(L)$, there is a monochromatic L-set under χ', say $\{(l, l'), (l + d, l'), (l, l' + d)\}$. By definition of ϕ, this corresponds to a monochromatic set $\{l\mathbf{u} + l'\mathbf{v} + \mathbf{x}, (l + d)\mathbf{u} + l'\mathbf{v} + \mathbf{x}, l\mathbf{u} + (l' + d)\mathbf{v} + \mathbf{x}\}$ which is a triangle with sides $d\mathbf{u}, d\mathbf{v}$, a homothetic image of T. Since there exists a forcing set X with parameters L, 5, 2 and $|X| \leq 20$, we may take $\phi^{-1}(X)$ to be a forcing set for T in $[p_T R_2(L)]^2 = [5p_T]^2$ to see that there exists a forcing set with respect to parameters T, $5p_T$, and 2 of cardinality at most 20. $\square$

Note that for any four point subset Q of $\mathbb{Z}^2$, there is a three point subset $T \subseteq Q$ such that $s_T = s_Q$. This is easily seen by taking T to be two points of Q with maximal Euclidean distance together with any third point of Q. This leads us to our next Lemma. First, for n an even positive integer and d any positive integer less than n, we define the *middle square of width* d of $[n]^2$ to be the $d \times d$ subgrid $\{\frac{n}{2} - \lfloor\frac{d}{2}\rfloor, \frac{n}{2} - \lfloor\frac{d}{2}\rfloor + 1, \dots, \frac{n}{2} + \lfloor\frac{d}{2}\rfloor - 1\}^2$.

Lemma 3. *Let Q be a set of four points in the grid in general position and let $T \subseteq Q$, $|T| = 3$ such that $s_T = s_Q$. Then $R_2(Q) \leq 20 s_Q R_{40}(T)$, and $R_2(S) \leq 5R_{40}(L)$.*

Proof. Let $q = 5s_T = 5s_Q$, $n = 4qR_{40}(T)$, and $\chi : [n]^2 \to \{\bullet, \circ\}$. We shall construct another coloring $\chi' : [n/q] \to \{1, 2, \dots, 40\}$ generated by χ. We shall first show that χ' has a monochromatic homothetic image T' of T in $[n/q]^2$. Using this T', we shall find a monochromatic homothetic image of Q in the original coloring.

By Lemma 2, we have that $R_2(T) \leq q$ and $forc(T, q, 2) \leq 20$. Let $\{X_1, \dots, X_{20}\}$ be a forcing set with respect to parameters T, q, and 2, and let
$(Y_1, \dots, Y_{40}) = ((X_1, \circ), (X_2, \circ), \dots, (X_{20}, \circ), (X_1, \bullet), (X_2, \bullet), \dots, (X_{20}, \bullet))$. Any 2-coloring of the $q \times q$ grid has some set X_i colored in $\circ$ or $\bullet$ which corresponds to either Y_i or Y_{20+i}, respectively, $1 \leq i \leq 20$.

Split $[n]^2$ into $q \times q$ grids $A_{(x,y)} = \{(a, b) : qx \leq a < q(x+1), qy \leq b < q(y+1), 0 \leq x, y \leq n/q - 1\}$. Let

$$\chi'((x, y)) = \min\{i : A_{(x,y)} \text{ has a colored set } Y_i \text{ under } \chi\}.$$

Note that χ' is a coloring of $[n/q]^2$ in at most 40 colors.

To allow for us to later choose additional points which belong to the grid, we consider the middle square M, of $[n/q]^2$ of width $\frac{1}{4}\frac{n}{q} = R_{40}(T)$. Then M contains, under χ', a monochromatic set $T' = \{\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3\}$, $T' \in Hom(T)$. Let $\mathbf{x}_4$ be the point such that $\{\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \mathbf{x}_4\} \in Hom(Q)$.

Since $\chi'(\mathbf{x}_1) = \chi'(\mathbf{x}_2) = \chi'(\mathbf{x}_3)$, the corresponding subgrids $A_{\mathbf{x}_1}, A_{\mathbf{x}_2}$, and $A_{\mathbf{x}_3}$ have a three element set from $Hom(T)$ in the same position and of the same color. I.e., $T'' = \{\mathbf{t}_1, \mathbf{t}_2, \mathbf{t}_3\} \in Hom(T)$, $T'' \subseteq [q]^2$, so that $T_1 = T'' + q\mathbf{x}_1 \in A_{\mathbf{x}_1}$, $T_2 = T'' + q\mathbf{x}_2 \in A_{\mathbf{x}_2}$ and $T_3 = T'' + q\mathbf{x}_3 \in A_{\mathbf{x}_3}$ are all monochromatic, (say black). Let $\mathbf{t}_4$ be the grid vertex such that $\{\mathbf{t}_1, \mathbf{t}_2, \mathbf{t}_3, \mathbf{t}_4\} \in Hom(Q)$ and $T_4 = T'' + q\mathbf{x}_4$. Since T_1, T_2, T_3 are monochromatic, T_4 is monochromatic (white), otherwise if one of its points, say $\mathbf{t}_1+q\mathbf{x}_4$ is black under χ, then $\{\mathbf{t}_1+q\mathbf{x}_1, \mathbf{t}_1+q\mathbf{x}_2, \mathbf{t}_1+q\mathbf{x}_3, \mathbf{t}_1+q\mathbf{x}_4\} \in Hom(Q)$, a monochromatic set. Similarly, we have $\chi(\mathbf{t}_4 + q\mathbf{x}_1) = \chi(\mathbf{t}_4 + q\mathbf{x}_2) = \chi(\mathbf{t}_4 + q\mathbf{x}_3) = \bullet$, and $\chi(\mathbf{t}_4 + q\mathbf{x}_4) = \circ$. Let $Q' = \{q\mathbf{x}_1 + \mathbf{t}_1, q\mathbf{x}_2 + \mathbf{t}_2, q\mathbf{x}_3 + \mathbf{t}_3, q\mathbf{x}_4 + \mathbf{t}_4\}$. (See Figure 4 for a pictorial representation.)

Claim: $Q' \in Hom(Q)$ and Q' is monochromatic under χ. Let $Q = \{\mathbf{q}_1, \mathbf{q}_2, \mathbf{q}_3, \mathbf{q}_4\}$. Then $\mathbf{x}_i = a\mathbf{q}_i + \mathbf{b}$ and $\mathbf{t}_i = a'\mathbf{q}_i + \mathbf{b}'$, $i = 1, 2, 3, 4$ for some $a, a' \in \mathbb{N}$, $\mathbf{b}, \mathbf{b}' \in \mathbb{Z}^2$. So, we have that $q\mathbf{x}_i + \mathbf{t}_i = q(a\mathbf{q}_i + \mathbf{b}) + (a'\mathbf{q}_i + \mathbf{b}') = (qa + a')\mathbf{q}_i + (q\mathbf{b} + \mathbf{b}')$. This concludes the proof of the claim.

What remains for us to check is that indeed all the selected points $\mathbf{t}_j + q\mathbf{x}_i$, $i, j = 1, 2, 3, 4$ belong to the grid $[n]^2$. Note that $q\mathbf{x}_1, q\mathbf{x}_2, q\mathbf{x}_3$ are in the middle grid M'' of $[n]^2$ of width $n/4$. Since $s_T = s_Q$, all four points $q\mathbf{x}_i$, $i = 1, 2, 3, 4$ are contained in the square of size at most $n/4$, so $q\mathbf{x}_4$ is in the middle square of $[n]^2$ of width $3n/4$. Since $\mathbf{t}_j \in [q]^2$ for $j = 1, 2, 3$, and $s_T = s_Q$, we have that $\mathbf{t}_j$ is in a $3q \times 3q$ grid for $j = 1, 2, 3, 4$. Hence, $\mathbf{t}_j + q\mathbf{x}_i$ are in the middle square of $[n]^2$ of width $3n/4 + 6q$ for $i, j = 1, 2, 3, 4$. Since $n = 4qR_{40}(T) \geq 4q \cdot 40 \geq 4q \cdot 6$, we have $6q \leq n/4$ and hence $\mathbf{t}_j + q\mathbf{x}_i$ belong to $[n]^2$ for $i, j = 1, 2, 3, 4$.

Remark: In case when $Q = S$, we can take $n = qR_{40}(L)$, instead of $4qR_{40}(T)$ because in the proof, the point $\mathbf{x}_4$ will be in the square determined by $\mathbf{x}_i$, $i = 1, 2, 3$; similarly $q\mathbf{x}_i + \mathbf{t}_4$, $i = 1, 2, 3, 4$ will be in the squares determined by corresponding $q\mathbf{x}_i + \mathbf{t}_j$, $i = 1, 2, 3, 4$, $j = 1, 2, 3$.

$\square$

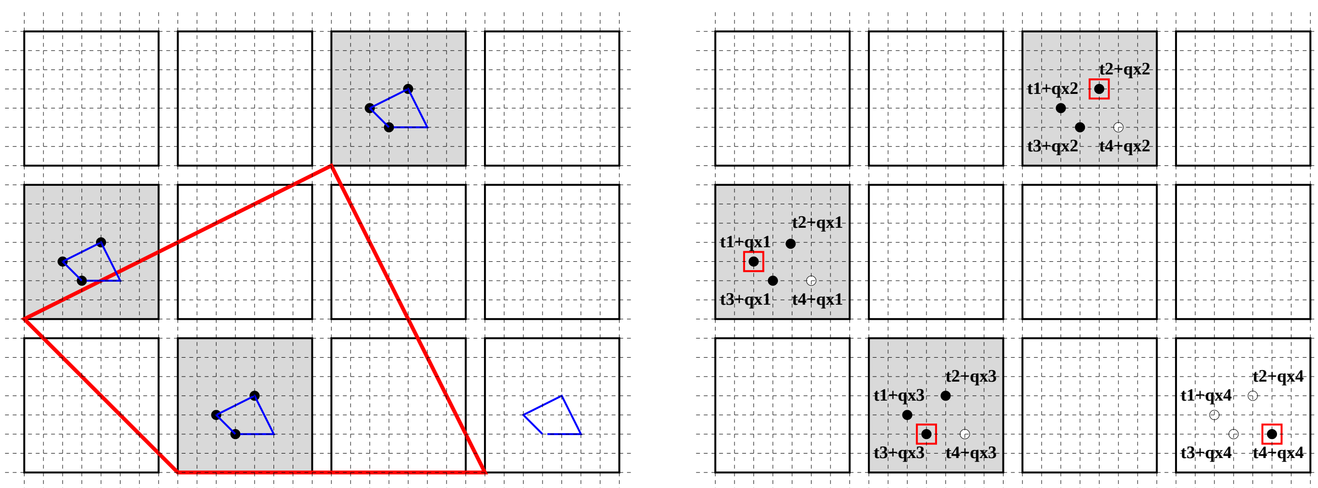

FIGURE 4. An example of the configuration from Lemma 3 is describing. In this example, the points $\mathbf{t}_j + q\mathbf{x}_i$ are elements of shaded subgrids $A_{\mathbf{x}_i}$.

Proof of Theorem 2. Let $Q \subseteq \mathbb{Z}^2$ be a set in general position with $|Q| = 4$. By Lemmas 2 and 3 together with inequality (1), we have immediately that $R_2(Q) \leq 20 s_Q \cdot 2^{2^{40}}$. $\square$

4. Density results

We now compare the bound $R_k(L) \leq 2^{2^k}$ by Graham and Solymosi [7] with the following density result of Shkredov [13]. Let $L(n) = \frac{1}{n^2}\max\{|A| : A \subseteq [n]^2$ such that A contains no L-set$\}$. Shkredov showed that $L(n) < \frac{1}{(\log\log n)^{1/73}}$. Thus, if we have a k-coloring of $[n]^2$, the pigeonhole principle implies that there is a monochromatic subset of $[n]^2$ of cardinality at least n^2/k. Assume that $n^2/k > n^2/(\log\log n)^{1/73}$, then by the above Theorem, we can conclude that this k-coloring of $[n]^2$ contains an L-set. For this to hold, we have that $k^{73} < \log\log n$, i.e., $n > 2^{2^{k^{73}}}$.

5. Appendix A

Define $C_t^n = \{(x_1, \dots, x_n) : x_i \in \{0, \dots, t-1\}\}$. Define a (combinatorial) line in C_t^n to be a set of points $\mathbf{x}_0, \dots, \mathbf{x}_{t-1} \in C_t^n$, where $\mathbf{x}_i = (x_{i1}, \dots, x_{in})$ such that, for all $1 \leq j \leq n$, either $x_{0j} = x_{1j} = \dots = x_{t-1,j}$, or $x_{sj} = s$ for $0 \leq s < t$. The Hales-Jewett Theorem [8] states that for all positive integers r and t, there exists $N_0 = HJ(r,t)$ such that for all $N \geq N_0$, any r-coloring of the vertices of C_t^N admits a monochromatic combinatorial line.

Let V be a finite subset of $\mathbb{Z}^2$. Let $C_{|V|}^N = \{(x_1, \dots, x_N) : x_i \in V\}$. Let $\phi : C_{|V|}^N \to \mathbb{Z}^2$ be an injective function with $\phi((x_1, \dots, x_N)) = \sum_{i=1}^N k_i x_i$, for some integers k_i; such an injective function exists for some choice of k_i. For any k-coloring of $\mathbb{Z}^2$, ϕ produces a k-coloing of $C_{|V|}^N$. So, if $N = HJ(k, |V|)$ then there is a monochromatic combinatorial line in $C_{|V|}^N$ under this coloring. Let, without loss of generality, in this monochromatic line $Q = (\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_{|V|})$, the first i coordinates be constant in each $\mathbf{x}_j$ and in all other coordinates $t > i$, $\mathbf{x}_j$ is equal to x_j. Let $\mathbf{b} = \sum_{j=1}^i k_j x_j$, let $c = \sum_{j=i+1}^N k_j$. Then $Q = cV + \mathbf{b}$, a homothetic image of V, which is monochromatic. Now, to find $R_2(S)$ we need to find N large enough to guarantee this function ϕ being injective.

For the square $V = S = \{(0,0), (0,1), (1,0), (1,1)\}$, we argue that choosing $k_i = 2^i$ will guarantee that ϕ is injective. Let $\mathbf{x}, \mathbf{y} \in C_{|V|}^N$, and say $\mathbf{x} = (x_1, x_2, \dots, x_N)$ and $\mathbf{y} = (y_1, y_2, \dots y_N)$ with $\mathbf{x} \neq \mathbf{y}$. Suppose $\phi(\mathbf{x}) = \phi(\mathbf{y})$, and hence $\sum_{i=1}^N 2^i (x_i - y_i) = \mathbf{0}$.

As $\mathbf{x} \neq \mathbf{y}$, we may assume (without loss of generality) that there exist x_i and y_i that differ in the first coordinate. Let, without loss of generality, indices $i_1 < i_2 < \dots < i_j$ be such that $x_{i_q} \neq y_{i_q}$ in the first coordinate, $q = 1, \dots j$. However, we have that $\sum_{\ell=1}^j 2^{i_\ell}(x_{i_\ell} - y_{i_\ell})$ is 0 in the first coordinate. Note that in the first coordinate x_{i_ℓ} and y_{i_ℓ} differ by at most 1 in absolute value. Dividing the above sum by 2^{i_1} gives an odd number, which is supposed to be equal to zero, a contradiction. Thus ϕ is injective. Thus, we have $R_2(S) \leq HJ(2,4)2^{HJ(3,4)} \leq 2^{2^{2^{\cdot^{\cdot^{\cdot^{2}}}}}} 2^{HJ(3,4)}$, where the height of the tower is 24. (The last inequality is due to Shelah; see [12].)

6. Appendix B

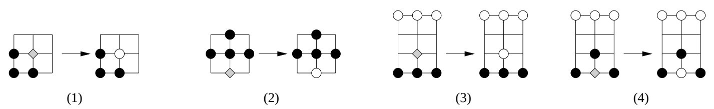

FIGURE 5. The configurations used in the case analysis. Trivially, the diamond in (1) must have color $\circ$. We refer to the Figure above labeled (2) as the *cross*; note that if the diamond in (2) has color $\bullet$, we can no longer avoid a monochromatic square. We refer to (3) as *stacked rows* and (4) as *staggered rows*. In each, the diamond must have color $\circ$.

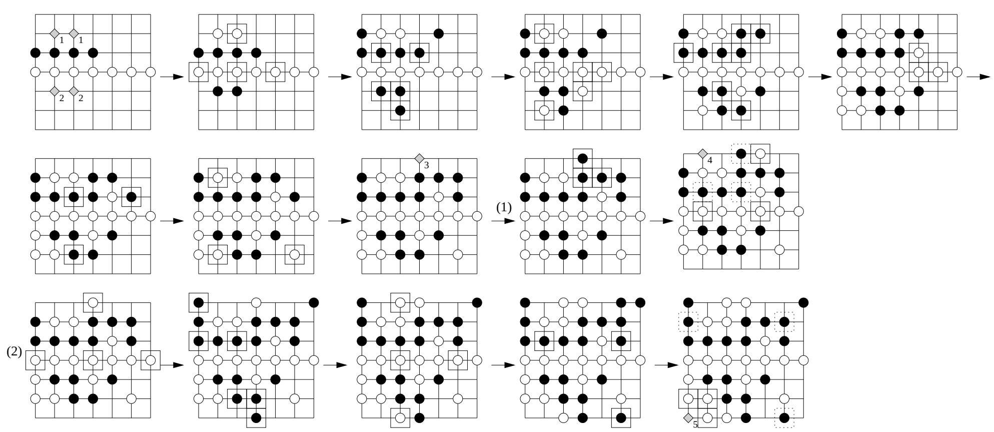

FIGURE 6. Both diamonds marked 1 must have color $\circ$, while both diamonds marked 2 must have color $\bullet$, else we have a monochromatic square. (1) examines the case where the diamond marked 3 has color $\bullet$; here, the diamond marked 4 cannot be colored. (2) examines the case where the diamond marked 3 has color $\circ$; here, the diamond marked 5 cannot be colored.

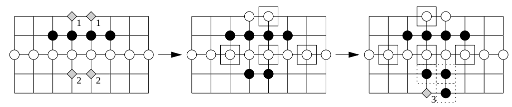

FIGURE 7. Both diamonds marked 1 must have color ○, and both diamonds marked 2 must have color •. This immediately shows that the diamond marked 3 cannot be colored, concluding the proof of case 1.

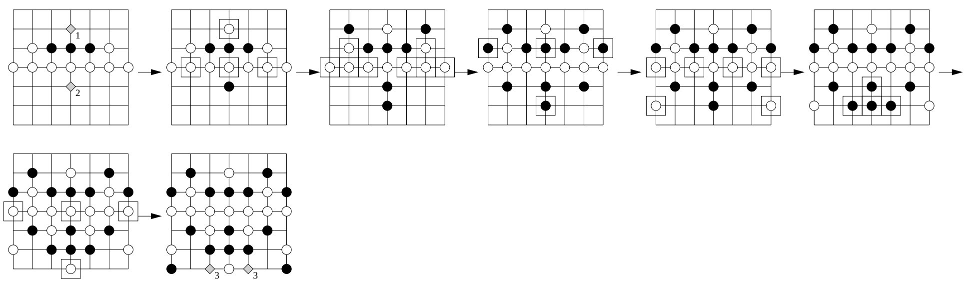

FIGURE 8. The diamond marked 1 must have color ○, and the diamond marked 2 must have color •. However, the diamonds marked 3 cannot be colored. This concludes the proof of case 2.

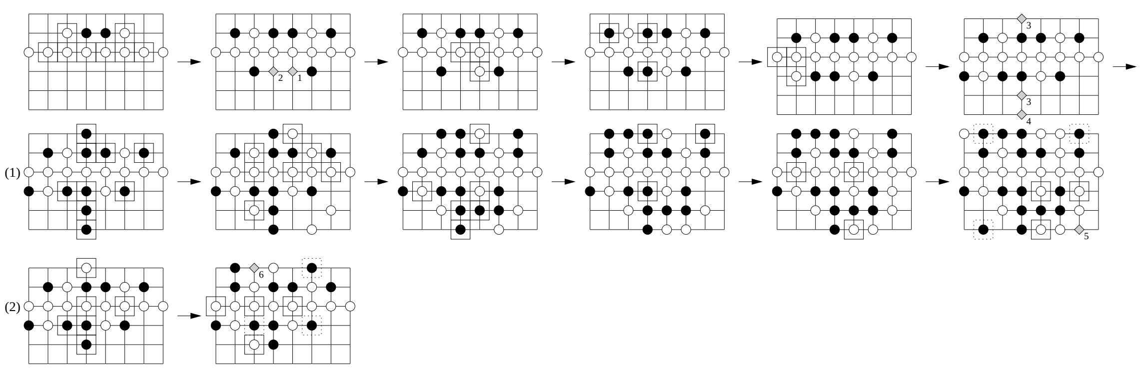

FIGURE 9. The diamonds marked 1 and 2 cannot both have color ○. Without loss of generality (due to symmetry), we color the diamond marked 1 ○. Since the diamonds marked 3 cannot both have color ○, we examine the cases where both have color • and where one has color • and the other has color ○. Similarly, either the diamond marked 4 or the vertex above the upper diamond marked 3 must have color •, so by symmetry we say that the diamond marked 4 has color •. (1) examines the case where both diamonds marked 3 have color •; here, the diamond marked 5 cannot be colored. (2) examines the case where one diamond marked 3 has color ○ and the other has color •; here, the diamond marked 6 cannot be colored. This concludes the proof of case 3.

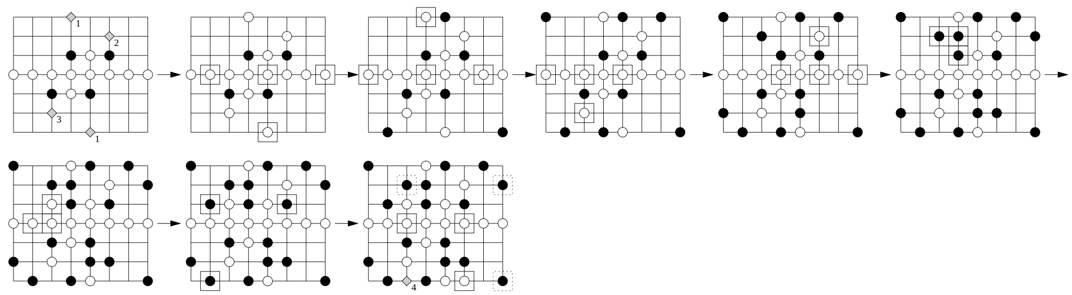

FIGURE 10. Under the hypothesis that the diamonds marked 1, 2, and 3 all have color ◦, the diamond marked 4 cannot be colored.

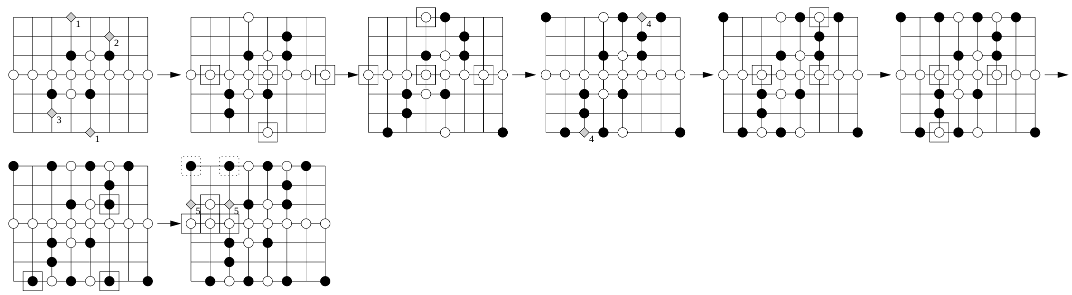

FIGURE 11. Under the hypothesis that the diamonds marked 1 have color ◦, and the diamonds marked 2 and 3 have color •, the diamond marked 4 must have color ◦ (staggered rows). The diamonds marked 5 cannot be colored.

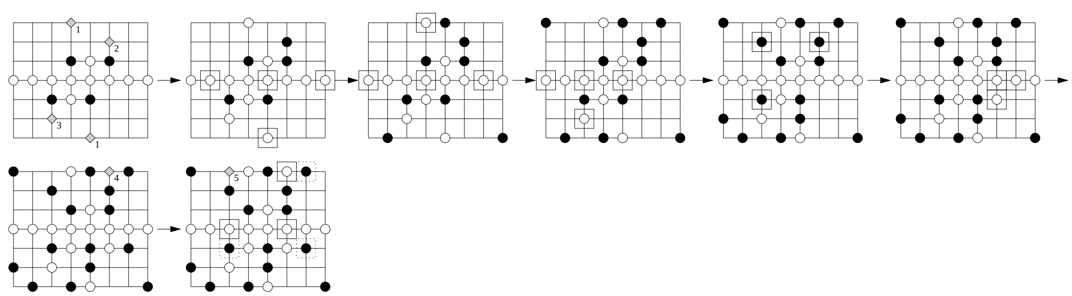

FIGURE 12. Under the hypothesis that the diamonds marked 1 have color ◦, the diamond marked 2 has color •, and the diamond marked 3 has color ◦, the diamond marked 4 must have color ◦ (staggered rows). The diamond marked 5 cannot be colored.

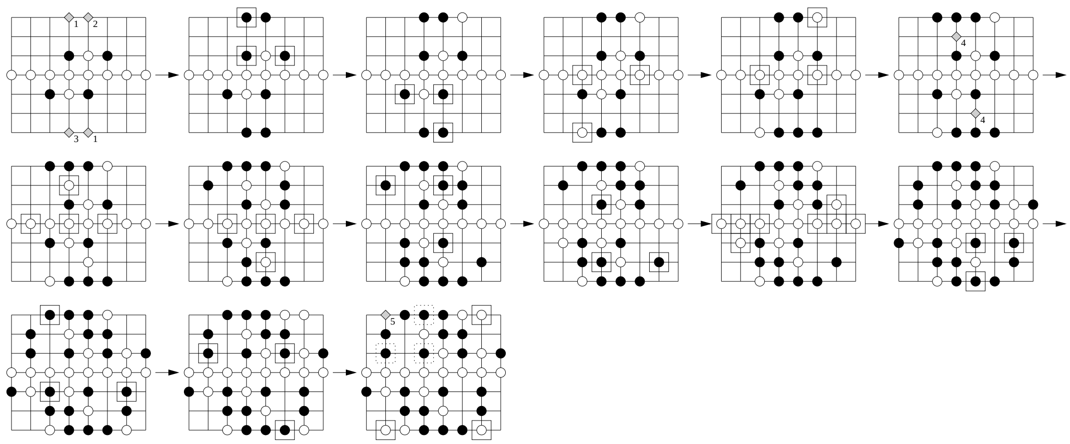

FIGURE 13. Under the hypothesis that the diamonds marked 1, 2, and 3 all have color •, the diamonds marked 4 must have color ◦ (stacked rows). The diamond marked 5 cannot be colored.

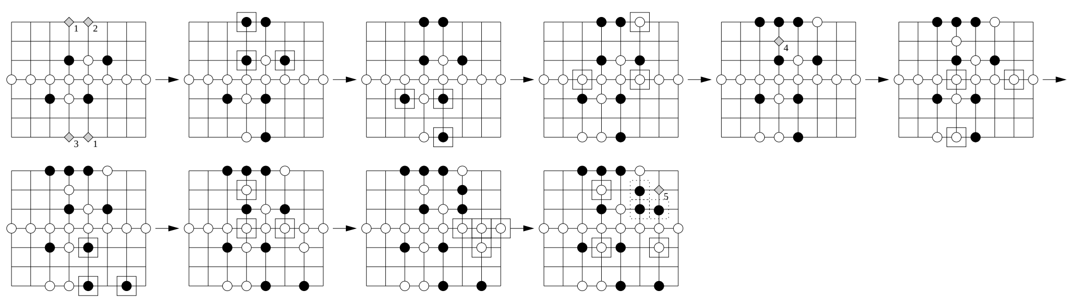

FIGURE 14. Under the hypothesis that both diamonds marked 1 have color •, the diamond marked 2 has color •, and the diamond marked 3 has color ◦, the diamond marked 4 must have color ◦ (stacked rows). This shows that the diamond marked 5 cannot be colored. (We need not consider the case where the diamond marked 2 has color ◦ and the diamond marked 3 has color •; we use symmetry to take care of this.)

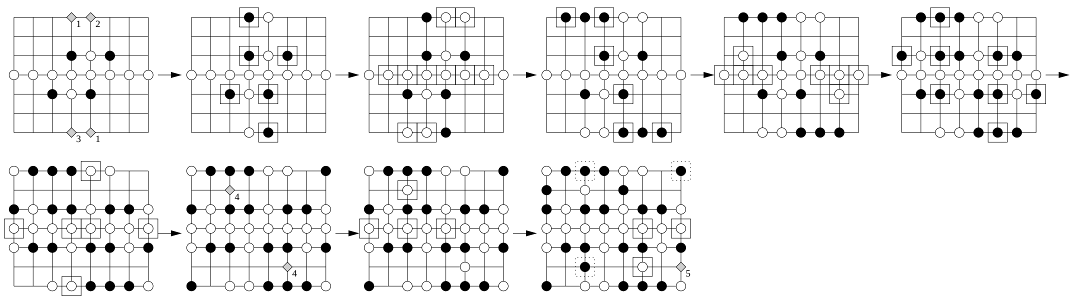

FIGURE 15. Under the hypothesis that both diamonds marked 1 have color • and both diamonds marked 2 and 3 have color ∘, the diamonds marked 4 must have color ∘ (stacked rows). This shows that the diamond marked 5 cannot be colored.

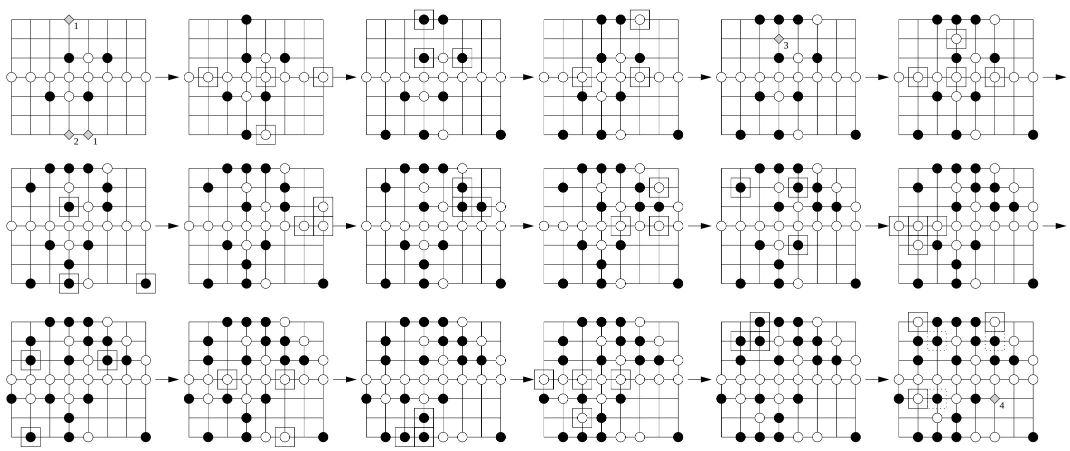

FIGURE 16. Under the hypothesis that one of the diamonds marked 1 has color ∘ and the other has color • and that the diamond marked 2 has color •, the diamond marked 3 must have color ∘ (stacked rows). The diamond marked 4 cannot be colored.

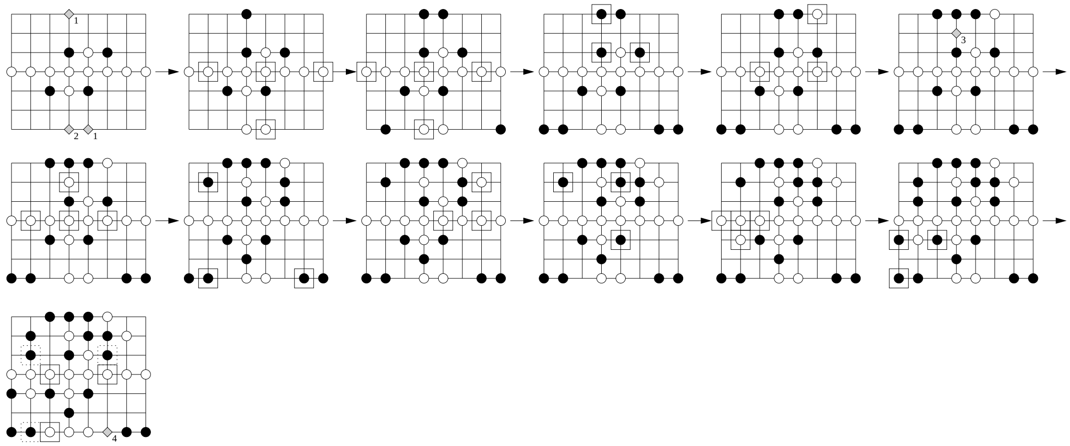

FIGURE 17. Under the hypothesis that one of the diamonds marked 1 has color ∘ and the other has color • and that the diamond marked 2 has color ∘, the diamond marked 3 must have color ∘ (stacked rows). The diamond marked 4 cannot be colored. This concludes the proof of case 4.

7. Acknowledgments

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