NOTE ON COMBINATORIAL ANALYSIS

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Consider a system of equations

\[ a_{\mu_1}x_1 + a_{\mu_2}x_2 + \ldots + a_{\mu_n}x_n = b_\mu \quad (1 \leq \mu \leq m), \]

where the \( a_{\mu}, b_\mu \) are complex numbers. Let \( A \) be a set of numbers, for instance all numbers of a number field, all complex numbers different from zero, etc. We call (1) regular in \( A \) if the following condition holds: however we split \( A \) into a finite number of subsets \( A_1, A_2, \ldots, A_k \), always at least one of these subsets \( A_\lambda \) contains a solution of (1). Roughly speaking, regularity of (1) in \( A \) means that, in a certain sense, \( A \) contains very many solutions of (1), and these solutions interlock very intimately.

In the special case where \( A \) is the set of all positive integers, I. Schur† proved the regularity of

\[ x_1 + x_2 - x_3 = 0, \]

and van der Waerden‡ proved the regularity, for every \( \ell > 0 \), of

\[ x_\ell - x_1 = x_1 - x_2 = \ldots = x_{\ell-1} - x_\ell \neq 0. \]

In a previous note|| I determined all systems (1) which are regular in the set of all positive integers. In this case necessary and sufficient conditions for regularity turned out to be certain linear relations between the \( a_{\mu}, b_\mu \). In the present note, which is an elaboration of a lecture delivered at the International Congress of Mathematicians at Oslo, 1936¶,

† "Jahresbericht der Deutschen Mathematiker-Vereinigung", 25 (1916), 114
§ Regularity of a system of equations and inequalities, in fact, of any set of conditions imposed upon certain variables, is defined in exactly the same way as above.
|| "Math. Zeitschrift", 36 (1933), 424–480, quoted as S.
I propose to consider the same problem in the case of more general sets $A$. I establish necessary and sufficient conditions for regularity expressed in terms of linear relations between the $a_{\mu v}$, $b_{\mu}$ in the following cases:

(i) The $a_{\mu v}$ are arbitrary, $b_{\mu} = 0$, $A$ is the set of all numbers different from zero contained in a given ring of complex numbers.

(ii) The $a_{\mu v}$ and $b_{\mu}$ are arbitrary numbers, $A$ is the field of all algebraic numbers.

(iii) The $a_{\mu v}$ are algebraic numbers, the $b_{\mu}$ are arbitrary, $A$ is the field of all complex numbers.

Other cases can be dealt with which are not included in this note.

The criteria are analogous to those obtained in the special case of $S$. In the cases (ii), (iii) the condition for regularity postulates that, for some number $\xi$ of $A$,

$$\sum_{\mu=1}^{n} a_{\mu v} \xi = b_{\mu} \quad (1 \leq \mu \leq m).$$

This result may be expressed as follows. *If, in every distribution of the numbers of $A$ over a finite number of classes, at least one class contains a solution of (1), then, in cases (ii) and (iii), the same is true for the extreme case of a distribution in which every number of $A$ forms a class by itself.*

Some of the proofs are extensions of proofs in $S$, others require the use of different methods. In proving the result concerning (i), we employ an extension (Theorem II) of van der Waerden's theorem quoted above. This extension was first proved by Dr. G. Grünwald, who kindly communicated it to me. It may be stated as follows. *Given any "configuration" $S$ consisting of a finite number of lattice points of a Euclidean space, and given a distribution of all lattice points of this space into a finite number of classes, there is at least one class which contains a configuration $S'$ of lattice points which is similar and parallel (homothetic) to $S$. Dr. Grünwald's proof runs parallel to van der Waerden's proof of his theorem.*

The proof given in this note is a simplification analogous to the simplification of van der Waerden's proof given in $S$ (p. 432, Satz I). In an earlier note I proved a weaker form of Grünwald's Theorem in which similarity of $S$ and $S'$ was replaced by affinity.

The last paragraph deals with regularity of systems (1) with respect to distributions which have denumerably many classes.

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† *I.e.* points with rational integral coordinates.

‡ *Berliner Sitzungsberichte* (1933), 589–596, Satz I.

1. Let $A$ be a finite or infinite aggregate. In this section letters $a, b, c, x$ denote general elements of $A$. Throughout this paper the letter $\Delta$ (and $\Delta', \Delta'', ..., \Delta_1, \Delta_2$, etc.) denotes distributions of all elements of $A$ into a finite number of classes. Occasionally we consider distributions into an infinite number of classes, in which case this is mentioned explicitly. $\Delta$ is defined by means of a relation "$\sim$" which is defined for some pairs of elements of $A$ and which has the properties:

(i) $a \sim a$ for every $a$;

(ii) $a \sim b$ implies $b \sim a$;

(iii) $a \sim b$ and $b \succ c$ imply $a \sim c$;

(iv) every infinite subset of $A$ contains two distinct elements $a, b$ such that $a \sim b$.

By $|\Delta|$ we denote the number of non-empty classes of $\Delta$. A congruence

$$a \equiv b \pmod{\Delta}$$

expresses, by definition, the fact that $a$ and $b$ belong to the objects distributed by means of $\Delta$, and, moreover, belong to the same class. In analogy with the notation for functions, we speak of a distribution $\Delta(x)$ defined for every $x$ of $A$, or, briefly, defined in $A$.

Throughout this paper $\Delta^{(k)}(x)$, for every positive integer $k$, denotes the distribution of all rational integers into classes of equal residues mod $k$. Thus

$$x \equiv y \pmod{\Delta^{(k)}}$$

is equivalent to saying that $x$ and $y$ are rational integers and

$$x \equiv y \pmod{k}.$$

$\Delta^{(0)}$ denotes that distribution of $A$ in which every element of $A$ forms a class for itself. $\Delta^{(0)}$ may have infinitely many classes. We use the same symbol $\Delta^{(0)}$ for different sets $A$.

Two methods are employed for generating new distributions from given ones†. The first is a process of multiplication. Given a finite

† Both were used in S. The notation adopted in this paper seems to be more convenient than the one used in S.
number of distributions \( \Delta_1, \Delta_2, \ldots, \Delta_n \), each defined in \( A \), we understand by their product

\[
\Delta = \Delta_1 \Delta_2 \ldots \Delta_n = \prod_{\nu=1}^{n} \Delta_{\nu}
\]

that distribution \( \Delta \) of \( A \) which is defined by the rule:

\[
x \equiv y \pmod{\Delta}
\]

if, and only if,

\[
x \equiv y \pmod{\Delta_{\nu}} \quad (1 \leq \nu \leq n).
\]

For instance, if \( k \) and \( l \) are natural numbers, then

\[
\Delta^{(k)} \Delta^{(l)} = \Delta^{(m)},
\]

where \( m \) is the least common multiple of \( k \) and \( l \).

We have

\[
|\Delta_1 \Delta_2 \ldots \Delta_n| \leq |\Delta_1| |\Delta_2| \ldots |\Delta_n|.
\]

The second process is one of \textit{inducing a distribution} in a set \( B \) by means of a distribution in \( A \) and a correspondence between every element of \( B \) and some elements of \( A \). Suppose that \( \Delta(x) \) is defined in \( A \), and that \( f(y) \) is a function defined for every element \( y \) of a set \( B \). The functional values of \( f \) are elements of \( A \). Then we define a distribution \( \Delta_1(y) \) in \( B \) by postulating that

\[
y_1 \equiv y_2 \pmod{\Delta_1}
\]

is to be equivalent to

\[
f(y_1) \equiv f(y_2) \pmod{\Delta}.
\]

We use the notation

\[
\Delta_1(y) = \Delta\left(f(y)\right).
\]

Clearly

\[
|\Delta_1| \leq |\Delta|.
\]

We have, for instance,

\[
\Delta^{(k)}(x+1) = \Delta^{(2k)}(2x) = \Delta^{(k)}(x).
\]

For

\[
x_1 \equiv x_2 \pmod{k}
\]

is equivalent to

\[
x_1 + 1 \equiv x_2 + 1 \pmod{k}
\]

and also to

\[
2x_1 \equiv 2x_2 \pmod{2k}.
\]
For real numbers \( x \neq 0 \),

\[
\Delta^{(a)} \left( \frac{x}{|x|} \right) = \Delta^{(a)} \left( \frac{x}{|x|} \right)
\]

is a distribution in which all positive numbers are in one class and all negative numbers are in a second class.

Suppose that \( \Delta(x) \) is defined in \( A \) and that the classes of \( \Delta \) are the sets

\[ A_0, A_1, \ldots, A_{k-1}. \]

Define, for every \( x \) of \( A \), a function \( f(x) \) by means of \( x < A_{f(x)} \).

Then \( \Delta(x) = \Delta^{(k)}(f(x)) \).

Now choose a natural number \( l \). Then corresponding to every \( x \) of \( A \) there are integers \( g(x), h(x) \) such that

\[
f(x) = g(x) + lh(x),
\]

\[ 0 \leq g(x) < l; \quad 0 \leq h(x) \leq (k-1)/l. \]

Therefore

\[
\Delta(x) = \Delta^{(k)}(f(x)) = \Delta^{(0)}(g(x)) \Delta^{(l)}(h(x)) = \Delta'(x) \Delta''(x),
\]
say. We have

\[
|\Delta'| \leq l, \quad |\Delta''| \leq \left[ \frac{k-1}{l} \right] + 1.
\]

Hence every \( \Delta \) can be represented in the form

\[
\Delta = \Delta' \Delta'','
\]

where

\[
|\Delta'| \leq l, \quad |\Delta''| \leq \left[ \frac{|\Delta|-1}{l} \right] + 1.
\]

In particular, putting \( l = 2 \), we see that every \( \Delta \) is a product of a finite number of distributions with not more than two classes.

If \( x, y \) are general elements of two sets \( A, B \) respectively, then \( \Delta(x, y) \) denotes a distribution of all pairs \( (x, y) \). Thus, for rational integers \( x, y \),

\[
\Delta(x, y) = \Delta^{(k)}(2x + 3y)
\]

\[ \dagger \text{ We use the symbol} \ " < \ " \text{ to denote the relation of an element to the class to which it belongs.} \]

\[ \ddagger \text{ [t]} \text{ denotes the largest integer not exceeding} \ t. \]
denotes that distribution of all pairs \((x, y)\) for which \((x_1, y_1)\) and \((x_2, y_2)\) are in the same class if, and only if,

\[2x_1 + 3y_1 \equiv 2x_2 + 3y_2 \pmod{k}.
\]

2. Let \(S(x_1, x_2, ..., x_n)\) be a system of conditions imposed upon the values of variables \(x_1, ..., x_n\). These variables are allowed to vary throughout a set \(A\). A relation

\[S(x_1^{(0)}, x_2^{(0)}, ..., x_n^{(0)}) = 0 \tag{2}
\]

expresses the fact that the elements \(x_1^{(0)}, ..., x_n^{(0)}\) of \(A\) satisfy the conditions (2); and

\[S(x_1^{(0)}, x_2^{(0)}, ..., x_n^{(0)}) \neq 0
\]

denotes the logical opposite to (2). \(S\) is called \(k\)-regular in \(A\) if, however \(A\) is split into \(k\) subsets

\[A = A_1 + A_2 + ... + A_k,
\]

there is always at least one subset \(A_1\) which contains a solution of (2). Regularity of \(S\), as mentioned in the introduction, means that \(S\) is \(k\)-regular for every \(k = 1, 2, ..., \omega\)-regularity of \(S\) means solubility in at least one class whenever \(A\) is split into denumerably many classes. Finally, we call \(S\) absolutely regular in \(A\) if, for some \(x^{(0)}\) of \(A\),

\[S(x^{(0)}, x^{(0)}, ..., x^{(0)}) = 0.
\]

If \(A\) is the set of all real numbers except zero the condition

\[(x_1 - x_2 - 1)(x_1 - x_2 - 2) ... (x_1 - x_2 - k) = 0
\]

is \(k\)-regular, but not \((k+1)\)-regular, the condition

\[x_1 + x_2 - x_3 = 0
\]

is regular but not \(\omega\)-regular†, and the condition

\[x_1 \neq x_2
\]

is \(\omega\)-regular but not absolutely regular in \(A\). The degree of regularity of \(S\) in \(A\) is the largest natural number \(k\) (if there is one) such that \(S\) is \(k\)-regular in \(A\).

Let \(S'(x_1, x_2, ..., x_n)\) be a system of conditions with \(n' \leq n\) which has the property that (2) implies

\[S'(x_1^{(0)}, x_2^{(0)}, ..., x_n^{(0)}) = 0.
\]

† See below, Theorems VII and XI.
Let \( k, k' \) be natural numbers, \( k' \leq k \), and let \( A \) be a subset of \( A' \). Then \( k \)-regularity of \( S \) in \( A \) implies \( k' \)-regularity of \( S' \) in \( A' \). Theorem I states that in certain circumstances a kind of inverse of this implication holds. In fact, many of our results are of this type.

**Theorem I.** If \( S(x_1, x_2, \ldots, x_n) \) is \( k \)-regular in a denumerable set \( A \) then \( S \) is also \( k \)-regular in a suitable finite subset \( A' \) of \( A \).

Theorem I has been proved elsewhere\(^\dagger\), but, for convenience, I reproduce the proof.

**Proof.** We may suppose that

\[ A = \{1, 2, 3, \ldots\}. \]

Put

\[ A_N = \{1, 2, \ldots, N\} \quad (N = 1, 2, 3, \ldots). \]

Let us assume that, for no value of \( N \), \( S(x_1, \ldots, x_n) \) is \( k \)-regular in \( A_N \).

We have to show that \( S \) is not \( k \)-regular in \( A \). There is a distribution \( \Delta_N \) of \( A_N \),

\[ A_N = A_{N1} + A_{N2} + \ldots + A_{Nk}, \]

such that no set \( A_{N\kappa} \) contains a solution of \( S = 0 \). Here \( N = 1, 2, \ldots; 1 \leq \kappa \leq k \). For any positive integer \( x \) we denote by \( f_N(x) \) that number \( \kappa \) \((1 \leq \kappa \leq k)\) for which

\[ x < A_{N\kappa}. \]

Then by a well-known argument (Cantor's "Diagonalverfahren") we can define a function \( f^*(x) \) such that, given any \( x_0 > 0 \), there are infinitely many \( N \)'s for which

\[
(3) \quad f_N(x) = f^*(x) \quad (1 \leq x \leq x_0).
\]

Let, for \( 1 \leq \kappa \leq k \), \( A_{\kappa}^* \) be the set of all \( x \) for which

\[ f^*(x) = \kappa, \]

and let \( \Delta^* \) be the distribution

\[ A = A_1^* + A_2^* + \ldots + A_k^*. \]

Now consider \( n \) numbers \( x_1, \ldots, x_n \) satisfying

\[
(4) \quad x_1^* \equiv x_2^* \equiv \ldots \equiv x_n^* \pmod{\Delta^*}.
\]

\(^\dagger\) S., Satz III. See also Dénes König, *Theorie der endlichen und unendlichen Graphen* (Leipzig, 1936), 84 (a) and (g).
Put 
\[ x_0 = \max (x_1', x_2', \ldots, x_n'). \]

Then, by the definition of \( f^*(x) \), there is a number \( N > x_0 \) for which (3) holds. In particular, using (4), we have
\[
\begin{align*}
  f_N(x_v') &= f^*(x_v') = \kappa_0^* (1 \leq v \leq n), \\
  x_v' &< A_{N \kappa_0} \quad (1 \leq v \leq n).
\end{align*}
\]

Hence, by the definition of \( \Delta_N \),
\[ S(x_1', x_2', \ldots, x_n') \neq 0. \]

This completes the proof of Theorem I.

As a corollary of Theorem I we prove†

\textbf{Lemma 1.} Let \( B \) be a denumerable set of numbers. Let \( R \) be the ring and \( K \) be the field generated by \( B \). Let \( R - \{0\} \) and \( K - \{0\} \) denote the sets of all non-zero numbers of \( R \) and \( K \) respectively. Then \( k \)-regularity of the system
\[
\sum_{\mu=1}^{n} a_{\mu} x_{\mu} = 0 \quad (1 \leq \mu \leq m)
\]
in \( K - \{0\} \) implies \( k \)-regularity of the same system in \( R - \{0\} \).

This lemma is another kind of inverse of the simple proposition on p. 128, immediately preceding Theorem I.

\textbf{Proof of Lemma 1.} Suppose that (5) is \( k \)-regular in \( K - \{0\} \). Then, by Theorem I, since \( K \) is denumerable, (5) is \( k \)-regular in a finite subset \( T \) of \( K - \{0\} \). Let \( N \) be the product of the "denominators" of all numbers of \( T \), or, more accurately, let \( N \) be a number of \( R - \{0\} \) for which
\[
Nt < R - \{0\}
\]
whenever \( t < T \). Take any \( \Delta(x) \) defined in \( R - \{0\} \) and satisfying \( |\Delta| \leq k \).

Put
\[ \Delta'(x) = \Delta(Nx) \quad (x < T). \]

In view of (6) this definition is significant. From \( |\Delta'| \leq |\Delta| \leq k \), and from the definition of \( T \), we deduce the existence of numbers \( t_v \) of \( T \) for which
\[
\begin{align*}
  \sum_{\nu=1}^{n} a_{\nu} t_{\nu} &= 0 \quad (1 \leq \nu \leq m), \\
  t_1 &\equiv t_2 \equiv \ldots \equiv t_n \quad (\text{mod } \Delta').
\end{align*}
\]

† S. 441.
Then
\[ \sum_{\nu} a_{\nu \nu}(N_t) = 0 \quad (1 \leq \mu \leq m), \]
\[ N_{t_1} \equiv N_{t_2} \equiv \cdots \equiv N_{t_n} \quad (\text{mod } \Delta). \]

Hence the result. Instead of (5) we might have taken any system of homogeneous conditions.

3. In this section small Latin letters denote non-negative integers. Capitals \( A, B, \ldots, X \) denote "vectors" \((x_1, x_2, \ldots, x_m)\) of a fixed dimension \( m \). If
\[ A = (x_1, x_2, \ldots, x_m), \quad A' = (x_1', x_2', \ldots, x_m'), \]
we put
\[ |A| = \max (x_1, x_2, \ldots, x_m), \quad aA = (ax_1, \ldots, ax_m), \]
\[ A + A' = (x_1 + x_1', x_2 + x_2', \ldots, x_m + x_m'). \]

0 denotes the number zero as well as the vector \((0, \ldots, 0)\).

**Theorem II.** There is a function \( f(k, R_1, R_2, \ldots, R_l) \) which has the following property. Suppose that \( R_1, R_2, \ldots, R_l \) are vectors and that \( \Delta \) is a distribution of all vectors into \( k \) classes. Then we can find \( A^*, d^*, (d^* > 0) \) such that
\[ A^* + d^* R_\lambda \equiv A^* \quad (\text{mod } \Delta) \quad (1 \leq \lambda \leq l), \]
\[ |A^* + d^* R_\lambda| \leq f \quad (1 \leq \lambda \leq l). \]

**Proof.** The theorem is true for \( l = 1 \). For there are two of the \( k + 1 \) vectors
\[ 0, R_1, 2R_1, 3R_1, \ldots, kR_1 \]
which belong to the same class of \( \Delta \), say \( aR_1 \) and \( \beta R_1 \), where
\[ 0 \leq a < \beta \leq k. \]

Put
\[ A^* = aR_1, \quad d^* = \beta - a. \]
Then
\[ A^* + d^* R_1 = \beta R_1 \equiv aR_1 = A^* \quad (\text{mod } \Delta). \]

Since
\[ |A^* + d^* R_1| = \beta |R_1| \leq k |R_1|, \]
we may put
\[ f(k, R_1) = k |R_1|. \]
Therefore we may suppose that, for some given vectors
\[ R_1, R_2, \ldots, R_l \]
(l > 1), the existence of $f(k', R_1, R_2, \ldots, R_{l-1})$ has been established for every $k' > 0$, and we deduce the existence of

$$f(k, R_1, \ldots, R_{l-1}, R_l).$$

In the proof which follows, $k, R_1, \ldots, R_l$ are constant, and these numbers are not shown in the arguments of functions. $\Delta$ is a distribution of all vectors, $|\Delta| \leq k$. We may assume, without loss of generality, that $R_1 \neq 0$.

**Lemma 2.** Under the assumptions stated there is a function $g(n)$ such that, given any $A, n$, we can find $A', d'$ ($d' > 0$) such that

(9) $A + (A' + d' R_\lambda) + B \equiv A + A' + B \pmod{\Delta}$ \hspace{1em} (1 \leq \lambda < l, \ |B| \leq n),

(10) $|A' + d' R_\lambda| \leq g(n)$ \hspace{1em} (1 \leq \lambda < l).

**Proof.** Put

$$\Delta'(X) = \prod_{|B| \leq n} \Delta(A + X + B).$$

Then

$$|\Delta'| \leq g_1(n).$$

Using the definition of

$$f\left(g_1(n), R_1, R_2, \ldots, R_{l-1}\right) = g_2(n),$$

we find that there are an $A'$ and $d'$ ($d' > 0$) such that

(11) $A' + d' R_\lambda \equiv A' \pmod{\Delta'}$ \hspace{1em} (1 \leq \lambda < l),

(12) $|A' + d' R_\lambda| \leq g_2(n)$ \hspace{1em} (1 \leq \lambda < l).

(11) and (12) are equivalent to (9) and (10) respectively if we put

$$g(n) = g_2(n).$$

Thus the lemma is proved. We note that

$$|A' + d' R_\lambda| \leq |A' + d' R_1| + |A' + d' R_1| |R_\lambda| \leq g(n) + g(n) |R_\lambda| = h(n).$$

Now, to prove Theorem II, put

$$n_\kappa = 1, \quad n_\kappa = \sum_{r=\kappa+1}^{k} h(n_r) \quad (0 \leq \kappa \leq k).$$

Apply Lemma 2, with $A = 0$, $n = n_0$. We find

$$A' = A_0, \quad d' = d_0 > 0$$
such that

\[ 0 + (A_0 + d_0 R_\lambda) + B \equiv 0 + A_0 + B \pmod{\Delta} \quad (1 \leq \lambda < l; \; |B| \leq n_0), \]

\[ |A_0 + d_0 R_\lambda| \leq h(n_0) \quad (1 \leq \lambda \leq l). \]

A second application, with \( A = A_0, \; n = n_1, \) yields \( A' = A_1, \; d' = d_1 > 0, \) such that

\[ A_0 + (A_1 + d_1 R_\lambda) + B \equiv A_0 + A_1 + B \pmod{\Delta} \quad (1 \leq \lambda \leq l; \; |B| \leq n_1), \]

\[ |A_1 + d_1 R_\lambda| \leq h(n_1) \quad (1 \leq \lambda \leq l). \]

Proceeding in this way (the next case is \( A = A_0 + A_1, \; n = n_2 \)) we find \( A' = A_x, \; d' = d_x > 0 \) such that

\[
\sum_{\kappa=0}^{\kappa=k-1} A_\kappa + (A_x + d_x R_\lambda) + B \equiv \sum_{\kappa=0}^{\kappa=k-1} A_\kappa + A_x + B \pmod{\Delta} \quad (0 \leq \kappa \leq k; \; 1 \leq \lambda \leq l; \; |B| \leq n_2), \]

\[ |A_x + d_x R_\lambda| \leq h(n_2) \quad (0 \leq \kappa \leq k; \; 1 \leq \lambda \leq l). \]

There are two among the \( k+1 \) vectors

\[ V_\kappa = \sum_{\kappa=0}^{\kappa=k-1} A_\kappa + \sum_{\kappa=0}^{\kappa=k-1} (A_x + d_x R_\lambda) \quad (0 \leq \kappa \leq k) \]

which belong to the same class of \( \Delta. \) Therefore, say,

\[ V_\kappa \equiv V_\beta \pmod{\Delta}, \]

where \( 0 \leq \alpha < \beta \leq k. \) Put

\[ A^* = V_\beta, \; d^* = \sum_{\kappa=0}^{\kappa=k} d_v. \]

Then

\[
A^* + d^* R_\lambda = \sum_{\kappa=0}^{\kappa=k} A_\kappa + \sum_{\kappa=0}^{\kappa=k} (A_x + d_x R_\lambda) + \sum_{\kappa=0}^{\kappa=k} d_x R_\lambda
\]

\[ = \sum_{\kappa=0}^{\kappa=k} A_\kappa + \sum_{\kappa=0}^{\kappa=k} (A_x + d_x R_\lambda) = V_\kappa \equiv V_\beta = A^* \pmod{\Delta}. \]

On the other hand, if \( 1 \leq \lambda < \kappa, \) then

\[ A^* + d^* R_\lambda = \sum_{\kappa=0}^{\kappa=k} A_\kappa + \sum_{\kappa=0}^{\kappa=k} (A_x + d_x R_\lambda) \]

\[ + \left( \sum_{\kappa=0}^{\kappa=k} (A_\kappa + d_x R_\lambda) + \sum_{\kappa=0}^{\kappa=k} (A_x + d_x R_\lambda) \right). \]

\[ \dagger \text{Empty sums have the value zero.} \]
Now we have, by (14),
\[
\left| \sum_{\alpha+2}^{\beta} (A_\alpha + d_\alpha R_\alpha) + \sum_{\beta+1}^{k} (A_\beta + d_\beta R_\beta) \right| \leq \sum_{\alpha+2}^{k} h(n_\alpha) = n_{\alpha+1}.
\]
Therefore, from (13),
\[
(17) \quad A^*+d^* R_\lambda = \left( \sum_{0}^{\alpha+2} (A_\alpha + d_\alpha R_\alpha) + \sum_{\alpha+3}^{\beta} (A_\alpha + d_\alpha R_\alpha) + \sum_{\beta+1}^{k} (A_\beta + d_\beta R_\beta) \right) (\text{mod } \Delta).
\]
The vector on the right-hand side of (17) is the same as the vector on the right-hand side of (16), except that \( a \) is to be replaced by \( a+1 \). When \( a+1 < \beta \) a second application of (13) leads to
\[
A^*+d^* R_\lambda = \sum_{0}^{\alpha+2} A_\alpha + \sum_{\alpha+3}^{\beta} (A_\alpha + d_\alpha R_\alpha) + \sum_{\beta+1}^{k} (A_\beta + d_\beta R_\beta) (\text{mod } \Delta),
\]
and so on. After \( \beta - a \) steps we find that
\[
A^*+d^* R_\lambda = \sum_{0}^{\alpha+2} A_\alpha + \sum_{\alpha+3}^{\beta} (A_\alpha + d_\alpha R_\alpha) = A^* \quad (\text{mod } \Delta) \quad (1 \leq \lambda < l).
\]
This result, together with (15), shows that (7) is true.
Furthermore, by (14), we have, for \( 1 \leq \lambda \leq l \),
\[
|A^*+d^* R_\lambda| = \left| \sum_{0}^{\alpha+2} A_\alpha + \sum_{\alpha+3}^{\beta} (A_\alpha + d_\alpha R_\alpha) + \sum_{\beta+1}^{k} (A_\beta + d_\beta R_\beta) \right| \leq \sum_{0}^{k} h(n_\alpha).
\]
Therefore (8) holds, with
\[
\sum_{0}^{k} h(n_\alpha),
\]
and the theorem is proved.

4. THEOREM III. There is a function \( f(k, l) \), defined for all pairs \( (k, l) \) of positive integers, which has the following property. Suppose that \( M \) is a set of objects among which a commutative and associative addition is defined. Let \( R_1, R_2, \ldots, R_l \) be elements of \( M \), and let \( \Delta \) be a distribution of \( M \) into \( k \) classes. Then there is a positive integer \( d^* \) and an element \( A^* \) of \( M \) such that
\[
(18) \quad A^*+d^* R_\lambda = A^* \quad (\text{mod } \Delta) \quad (1 \leq \lambda \leq l),
\]
\[
(19) \quad A^* = \sum_{\lambda=1}^{l} a_\lambda R_\lambda,
\]
\[
(20) \quad a_\lambda + d^* \leq f(k, l) \quad (1 \leq \lambda \leq l).
\]
\[\dagger\quad d^* R_\lambda = R_1 + R_2 + \ldots + R_\lambda \quad (d^* \text{ terms}).\]
Proof. Put, for all integers $x_i \geq 0$,

$$\Delta'(x_1, x_2, \ldots, x_l) = \Delta\left((x_1 + 1) R_1 + (x_2 + 1) R_2 + \ldots + (x_l + 1) R_l\right).$$  

Apply Theorem II to $\Delta'$ and the "unit vectors"

$$\vec{R}_1 = (1, 0, \ldots, 0),$$

$$\vec{R}_2 = (0, 1, \ldots, 0),$$

$$\ldots \ldots \ldots \ldots$$

$$\vec{R}_l = (0, 0, \ldots, 1).$$

We obtain a vector $(a_1, a_2, \ldots, a_l)$, with integral components $a_i \geq 0$, and a positive integer $d$ such that

$$a_1 + \ldots + a_l + d \vec{R}_\lambda \equiv (a_1, \ldots, a_l) \pmod{\Delta'} \quad (1 \leq \lambda \leq l),$$

$$\sum_{\nu=1}^l \left(a_\nu + 1\right) R_\nu + d \vec{R}_\lambda \equiv \sum_{\nu=1}^l \left(a_\nu + 1\right) R_\nu \pmod{\Delta} \quad (1 \leq \lambda \leq l).$$

(23) is the same as

$$a_\lambda + d \leq g(k, l) \quad (1 \leq \lambda \leq l).$$

Hence we may put

$$A* = \sum_{\nu=1}^l \left(a_\nu + 1\right) R_\nu, \quad d* = d; \quad a_\nu* = a_\nu + 1 \quad (1 \leq \nu \leq l),$$

$$f(k, l) = \bar{g}(k, l) + 1,$$

and the theorem is proved. In the special case where $M$ is the system of vectors

$$A = (x_1, x_2, \ldots, x_m)$$

whose components $x_n$ are non-negative integers, we deduce from (19) and (20) that, in the notation of Theorem II,

$$|A* + d* R_\lambda| = \left| \sum_{\nu=1}^l a_\nu* R_\nu + d* R_\lambda \right| \leq \sum_{\nu=1}^l |a_\nu* + d*| R_\nu |$$

$$\leq f(k, l) \sum_{\nu=1}^l |R_\nu| \quad (1 \leq \lambda \leq l).$$

Therefore the function $f(k, R_1, \ldots, R_l)$ of Theorem II may be assumed to be of the form

$$f(k, R_1, \ldots, R_l) = f(k, l) \sum_{\nu=1}^l |R_\nu|.$$
For a later application we note that in Theorem III we may stipulate that the elements

\[ A^\#, A^\# + d^\# R_\lambda \quad (1 \leq \lambda \leq l) \]

are either different from any one out of a finite number of given elements \( C_1, C_2, \ldots, C_s \), or are all equal to one of the elements \( C_a \). In this case (20) must be replaced by

(24) \[ a_\lambda^\# + d^\# \leq \bar{f}(k+s, l). \]

For suppose that \( \Delta \) is a distribution of \( M, |\Delta| \leq k \). Define \( \Delta' \) by means of

\[ A \equiv B \pmod{\Delta'} \]

if, and only if, either

\[ A \neq C_\sigma; \quad B \neq C_\sigma \quad (1 \leq \sigma \leq s); \quad A \equiv B \pmod{\Delta}, \]

or \( A = B = C_\sigma \) for some \( \sigma_0, 1 \leq \sigma_0 \leq s \). In other words, we remove the \( C_\sigma \)'s from their classes in \( \Delta \) and put them in separate classes. Now apply Theorem III to \( \Delta' \). We find \( A^\#, d^\# \), with \( d^\# > 0 \), such that

(25) \[ A^\# + d^\# R_\lambda \equiv A^\# \pmod{\Delta'} \quad (1 \leq \lambda \leq l). \]

Furthermore, (19) and (24) hold; (25) implies (18), and, in view of the definition of \( \Delta' \), the additional condition is satisfied.

2. Regularity of homogeneous linear equations.

1. THEOREM IV. Let \( a_1, a_2, \ldots, a_n \) be real numbers. Suppose that no non-empty subset of these numbers has the sum zero. Then the equation

(26) \[ a_1 x_1 + a_2 x_2 + \ldots + a_n x_n = 0 \]

is not regular in the set of all positive numbers. In particular, if

(27) \[ |a_{v_1} + a_{v_2} + \ldots + a_{v_k}| \geq (|a_1| + |a_2| + \ldots + |a_n|)/a > 0 \]

whenever \( 1 \leq k \leq n, \quad 1 \leq v_1 < v_2 < \ldots < v_k \leq n \),

then the degree of regularity \( D \) of (26) in the set of all positive numbers satisfies

(28) \[ D < 1 + \frac{\log b}{\log (1 + a^{-1} - b^{-1})}, \]

where \( b \) is any arbitrary number exceeding \( a \). In particular,

(29) \[ D < 1 + \frac{a}{3a} \log (2a) \quad (\text{for every } a), \]

(30) \[ D < a \log a + o(a \log a) \quad (\text{as } a \to \infty). \]
If $D'$ is the degree of regularity of (26) in the set of all real numbers $x \neq 0$, then

\begin{equation}
D' \leq 2D+1.
\end{equation}

**Proof.** (27) implies $a \geq 1$. If no non-empty subset of $a_1, a_2, \ldots, a_n$ has the sum zero, then (27) holds for some $a \geq 1$. Therefore the first assertion of the theorem follows from (28). (29) follows easily from (28) by putting $b = 2a$ and making use of the inequality

\begin{equation}
\log (1+t) = t(1-\frac{1}{2}t) + \frac{1}{3}t^3(1-\frac{2}{3}t) + \ldots > t(1-\frac{1}{2}),
\end{equation}

valid for $0 < t \leq \frac{1}{2}$.

In order to deduce (30) from (28) we put

$$b = a \log a,$$

for $a > e$. Then (28) becomes

$$D < 1 + \frac{\log (a \log a)}{\log (1+a^{-1}-(a \log a)^{-1})} = a \log a + o(a \log a)$$

as $a \to \infty$. If $D < \infty$, then there is a distribution $\Delta$ of all positive numbers such that $|\Delta| = D+1$ and

$$a_1 x_1 + \ldots + a_n x_n \neq 0$$

whenever

$$x_1 \equiv x_2 \equiv \ldots \equiv x_n \quad (\text{mod } \Delta).$$

Put

$$\Delta'(x) = \Delta(\lfloor x \rfloor) \Delta (\frac{x}{\lfloor x \rfloor}) \quad (x \text{ real, } \neq 0).$$

In other words, corresponding to every class of $\Delta$ we form a new class containing the same numbers but multiplied by $-1$. Then no class of $\Delta'$ contains a solution of (26), and (31) follows.

All that remains to be proved is (28), under the assumption (27). Suppose that (26) is $m$-regular in the set of all positive numbers, for some positive integer $m$. Choose a number $q > 1$ and put

$$\Delta(x) = \Delta(m) \left( \left\lfloor \frac{\log x}{\log q} \right\rfloor \right) \quad (x > 0).$$

Then $|\Delta| = m$. Hence, in consequence of our assumption about $m$, there are numbers $x_*$ satisfying

$$x_1 \equiv x_2 \equiv \ldots \equiv x_n \quad (\text{mod } \Delta),$$

$$a_1 x_1 + \ldots + a_n x_n = 0.$$
Put
\[
\left\lceil \frac{\log x_r}{\log q} \right\rceil = m_r \quad (1 \leq r \leq n).
\]

Then
\[
\begin{align*}
(33) & \quad m_1 \equiv m_2 \equiv \ldots \equiv m_n \pmod{m}, \\
(34) & \quad q^{m_r} \leq x_r < q^{m_{r+1}} \quad (1 \leq r \leq n).
\end{align*}
\]

Arrange the \(m_r\) in non-increasing order:
\[
m_{r_1} \geq m_{r_2} \geq \ldots \geq m_{r_k}.
\]

We have, for some suitable \(k\) \((1 \leq k \leq n)\)†,
\[
m_{r_1} = m_{r_2} = \ldots = m_{r_k} > m_{r_{k+1}}.
\]

Then, by (33),
\[
m_{r_k} \geq m_{r_{k+1}} + m.
\]

Therefore, from (34),
\[
\begin{align*}
q^{m_{r_k}} & \leq x_{r_k} < q^{m_{r_{k+1}}} \quad (1 \leq k \leq k), \\
x_{r_k} & < q^{m_{r_{k+1}}} \leq q^{m_{r_{k+1}} - m_{r_{k+1}}} \quad (k < \lambda \leq n).
\end{align*}
\]

Now, making use of (27), we deduce that
\[
0 = \left| \sum_{\mu=1}^{n} a_{x_r} x_{x_r} \right| = \left| \sum_{\kappa=1}^{k} a_{x_r} x_{x_r} + \sum_{\kappa=1}^{k} a_{x_r} (x_{r_\kappa} - x_{r_r}) + \sum_{\lambda=k+1}^{n} a_{x_r} x_{x_r} \right|
\geq x_{r_1} \left| \sum_{\kappa=1}^{k} a_{x_r} \right| - \sum_{\kappa=1}^{k} \left| a_{x_r} \right| \left| x_{r_\kappa} - x_{r_r} \right| - \sum_{\lambda=k+1}^{n} \left| a_{x_r} \right| x_{x_r}
\geq q^{m_{r_1}} a^{-1} \sum_{\mu=1}^{n} \left| a_{x_r} \right| - \left( q^{m_{r_{k+1}}} - q^{m_{r_1}} \right) \sum_{\mu=1}^{n} \left| a_{x_r} \right| - q^{m_{r_{k+1}} - m_{r_{k+1}}} \sum_{\mu=1}^{n} \left| a_{x_r} \right|
= q^{m_{r_1}} \sum_{\mu=1}^{n} \left| a_{x_r} \right| (a^{-1} - (q - 1)q^{-(m_{r_1})}).
\]

Therefore
\[
a^{-1} - q + 1 - q^{-(m_{r_1})} < 0,
\]

i.e.,
\[
q^{-(m_{r_1})} > a^{-1} - q + 1,
\]

\[
m - 1 < -\frac{\log (a^{-1} - q + 1)}{\log q},
\]

provided that
\[
a^{-1} - q + 1 > 0.
\]

† In the case \(k = n\) everything relating to \(k+1, k+2, \ldots\) has to be omitted, and similarly later.
Put 

\[ q = 1 + a^{-1} - b^{-1}, \]

where \( b \) is any number exceeding \( a \). Then it follows that

\[ m - 1 < \frac{\log b}{\log (1 + a^{-1} - b^{-1})}. \]

Hence (26) is not regular in the set of all positive numbers, and (28) holds. This completes the proof of Theorem IV.

**Theorem V.** Let \( a_1, a_2, \ldots, a_n \) be complex numbers. Suppose that no non-empty subset of these numbers has the sum zero. Then the equation

\[ a_1 x_1 + \ldots + a_n x_n = 0 \]

is not regular in the set of all complex numbers different from zero. In particular, if

\[ |a_{v_1} + a_{v_2} + \ldots + a_{v_k}| \geq (|a_1| + |a_2| + \ldots + |a_n|)/a > 0 \]

whenever \( 1 \leq k \leq n; \ 1 \leq v_1 < v_2 < \ldots < v_k \leq n, \)

then the degree of regularity \( D \) of (35) in the set of all complex numbers different from zero satisfies

\[ D < 1 + \frac{1}{\delta} \frac{\log (1 + a^{-1} - q^{1+\delta} - 2\pi \delta)}{\delta \log q}, \]

where \( \delta, q \) are any real numbers such that

\[ 0 < 2\pi \delta < a^{-1}, \]

\[ 1 < q^{1+\delta} < 1 + a^{-1} - 2\pi \delta. \]

In particular

\[ D < 1 + 6\pi a + 28\pi a^2 \log (3a) \quad (\text{for every } a), \]

\[ D < 8\pi a^2 \log a + o(a^2 \log a) \quad (\text{as } a \to \infty). \]

**Proof.** (36) implies \( a \geq 1 \). The first part of the theorem follows as in the case of Theorem IV. (40) follows from (37) by putting

\[ 2\pi \delta = \frac{1}{3a}, \quad q^{1+\delta} = 1 + \frac{1}{3a}, \]

and using (32). (41) follows from (37) if we put, for every sufficiently large \( a \),

\[ 2\pi \delta = \frac{1}{2a}, \quad q = 1 + \frac{1}{2a} - \frac{1}{a \log a}. \]

† In other words, \( \delta, q \) satisfy \( 0 < \delta, \ 1 < q, \) and have such values as to give a real value to the right-hand side of (37).
Then, as \( a \to \infty \),

\[
1 + \frac{1}{a} - q^{1+q} - 2\pi \delta = 1 + \frac{1}{2a} \left( 1 + \frac{1}{2a} - \frac{1}{\alpha \log a} \right)^{1+\log(\pi a^{-1})} - \exp \left[ \left( 1 + \frac{1}{4\pi a} \right) \left( \frac{1}{2a} - \frac{1}{\alpha \log a} + O \left( \frac{1}{a^2} \right) \right) \right] - \exp \left\{ \frac{1}{2a} - \frac{1}{\alpha \log a} + O \left( \frac{1}{a^2} \right) \right\} = 1 + \frac{1}{2a} \left( 1 + \frac{1}{2a} - \frac{1}{\alpha \log a} + O \left( \frac{1}{a^2} \right) \right) = \frac{1}{\alpha \log a} + O \left( \frac{1}{a^2} \right).
\]

Hence, by (37),

\[
D < 1 + 4\pi a - 4\pi a \frac{\log \left( (a \log a)^{-1} + O(a^{-2}) \right)}{\log \left( 1 + \frac{1}{4a^{-1}} + O(a \log a)^{-1} \right)} = 1 + 4\pi a + 4\pi a \frac{\log a + o(\log a)}{\frac{1}{a^{-1}} + o(a^{-1})} = 8\pi a^2 \log a + o(\log a).
\]

In order to prove the theorem we have to show that (37) holds, provided that (36) is true for all \( k, v_1, \ldots, v_k \). We assume that (35) is \( m \)-regular in the set of all complex numbers different from zero. Choose real numbers \( q, \delta \) such that \( q > 1, \delta > 0 \). Then every complex \( x \neq 0 \) has a unique representation

\[
x = q^r e^{2\pi i t}
\]

where \( r = r(x), t = t(x) \) are real numbers satisfying

\[
0 \leq r - t < 1.
\]

For (42) determines \( r \) uniquely and \( t \) uniquely mod \( 1 \), and (43) fixes \( t \). Put

\[
\Delta(x) = \Delta^m \left( \left\lfloor \frac{t}{\delta} \right\rfloor \right) \quad (x \neq 0).
\]

Then, according to the choice of \( m \), there are numbers \( x_r \neq 0 \) for which

\[
x_1 \equiv x_2 \equiv \cdots \equiv x_n \pmod{\Delta}, \quad a_1 x_1 + \cdots + a_n x_n = 0.
\]

Let

\[
r(x_r) = r; \quad t(x_r) = t_r; \quad \left\lfloor \frac{t_r}{\delta} \right\rfloor = m_r.
\]

Then

\[
x_r = q^{r_r} e^{2\pi i t_r},
\]

\[
l_r < r_r + 1, \quad m_1 \equiv m_2 \equiv \cdots \equiv m_n \pmod{m},
\]

\[
m_r \delta < l_r < m_r, m_r \delta + \delta \quad (1 \leq r \leq n).
\]
Also
\[ m_{r_1} = m_{r_2} = \ldots = m_{r_k} > m_{r_{k+1}} \geq \ldots \geq m_{r_n}, \]
where
\[ (46) \quad r_1, r_2, \ldots, r_n \]
is a permutation of 1, 2, ..., \( n \) and \( k \) an integer, \( 1 \leq k \leq n \). Moreover, using (45) and (44), we can choose the permutation (46) in such a way that
\[ m_{r_1} \delta \leq t_{r_1} \leq r_{r_2} \leq \ldots \leq r_{r_k} \leq t_{r_k+1} < m_{r_k} \delta + \delta + 1 = m_{r_1} \delta + \delta + 1. \]
Then we have, for every \( \kappa \) (1 \( \leq \kappa \leq k \)),
\[ (47) \quad |x_{r_\kappa} - x_{r_{\kappa+1}}| = |(q^{r_{\kappa}}e^{2\pi it_{\kappa}} - q^{r_{\kappa+1}}e^{2\pi it_{\kappa+1}}) + (q^{r_{\kappa+1}}e^{2\pi it_{\kappa+1}} - q^{r_{\kappa}}e^{2\pi it_{\kappa}})| \]
\[ \leq |q^{r_{\kappa}} - q^{r_{\kappa+1}}| + q^{r_{\kappa}} \times 2\pi |t_{r_{\kappa}} - t_{r_{\kappa+1}}| \]
\[ < q^{r_{\kappa}}(q^{1+\delta} - 1) + 2\pi \delta q^{r_{\kappa}} \quad (1 \leq \kappa \leq k). \]
If \( k < \lambda \leq n \), then
\[ m_{r_1} \leq m_{r_{k+1}} \leq m_{r_k} - m = m_{r_1} - m, \]
and therefore
\[ (48) \quad |x_{r_\lambda}| = q^{r_{\lambda}} < q^{r_{\lambda+1}} < q^{m_{r_k} \delta + \delta + 1} \leq q^{m_{r_1} - m + \delta + 1} \]
\[ \leq q^{r_{\lambda} - m + \delta + 1} \leq q^{r_{\lambda} - (m-1)\delta + 1} \quad (k < \lambda \leq n). \]

Just as in the proof of Theorem IV, we deduce from (36), (47), (48), (38) and (39) that
\[ 0 = \sum_{\mu=1}^{n} a_{\mu} x_{\mu} = \left| \sum_{\kappa=1}^{k} a_{r_{\kappa}} x_{r_{\kappa}} + \sum_{\kappa=1}^{k} a_{r_{\kappa}} (x_{r_{\kappa}} - x_{r_{\kappa+1}}) + \sum_{\lambda=k+1}^{n} a_{r_{\lambda}} x_{r_{\lambda}} \right| \]
\[ \geq |x_{r_1}| \left| \sum_{\kappa=1}^{k} a_{r_{\kappa}} \right| - \left| \sum_{\kappa=1}^{k} a_{r_{\kappa}} x_{r_{\kappa}} - \sum_{\lambda=k+1}^{n} a_{r_{\lambda}} x_{r_{\lambda}} \right| \]
\[ > q^{r_1} a_1 \sum_{\mu=1}^{n} a_{\mu} - q^{r_1} (q^{1+\delta} - 1 + 2\pi \delta) \sum_{\mu=1}^{n} a_{\mu} - q^{r_1 - (m-1)\delta + 1} \sum_{\mu=1}^{n} a_{\mu} \]
\[ = q^{r_1} \sum_{\mu=1}^{n} a_{\mu} \left( a_1 - q^{1+\delta} + 1 - 2\pi \delta - q^{-(m-1)\delta + 1} \right), \]
\[ q^{-(m-1)\delta + 1} > a_1 - q^{1+\delta} + 1 - 2\pi \delta, \]
\[ (m-1)\delta - 1 < -\log (a_1 - q^{1+\delta} + 1 - 2\pi \delta), \]
\[ m < 1 + \frac{1}{\delta} - \log \left( 1 + \frac{a_1 - q^{1+\delta} - 2\pi \delta}{\delta \log q} \right). \]
If this holds for every $m$, then the admissible values of $m$ are bounded, and the largest of them, i.e., $D$, satisfies (37). Thus Theorem V is proved. It is obvious from the foregoing proofs that the numerical constants in (29) and (40) can be improved.

2. Consider a system of homogeneous linear equations

$$(49) \quad \sum_{\nu=1}^{n} a_{\nu \nu} x_{\nu} = 0 \quad (1 \leq \mu \leq m)$$

with arbitrary complex coefficients $a_{\mu \nu}$. Let $B$ be a set of numbers. We say that (49) satisfies the condition $\Gamma(B)$ if it is possible to divide the set $\{1, 2, ..., n\}$ into non-empty, non-overlapping subsets $S_1, S_2, ..., S_l$ such that, corresponding to every $\lambda (1 \leq \lambda \leq l)$, there exists a solution $x_{\nu} = \xi^{(\lambda)}_{\nu}$ of (49) for which

$$\xi^{(\lambda)}_{\nu} < B \quad (\nu \in S_1, S_2, ..., S_\lambda),$$

$$\xi^{(\lambda)}_{\nu} = 0 \quad (\nu \not\in S_{\lambda+1}, S_{\lambda+2}, ..., S_l),$$

and, moreover, all numbers $\xi^{(\lambda)}_{\nu}$ for $\nu$ in $S_\lambda$ have the same value $\xi^{(\lambda)}_{\nu} \neq 0$. In other words, if we number the variables suitably we can find a matrix of the type (in the case $l = 4$)

$$\begin{pmatrix}
\xi^{(1)}_1, & \xi^{(1)}_2, & \cdots, & \xi^{(1)}_\gamma, & 0, & \cdots & \cdots & \cdots & \cdots & \xi^{(1)}_n \\
\xi^{(2)}_1, & \xi^{(2)}_2, & \cdots, & \xi^{(2)}_\gamma, & \xi^{(2)}_\delta, & \cdots & \cdots & \cdots & \cdots & \xi^{(2)}_n \\
\xi^{(3)}_1, & \xi^{(3)}_2, & \cdots, & \xi^{(3)}_\gamma, & \xi^{(3)}_\delta, & \cdots & \cdots & \cdots & \cdots & \xi^{(3)}_n \\
\xi^{(4)}_1, & \xi^{(4)}_2, & \cdots, & \xi^{(4)}_\gamma, & \xi^{(4)}_\delta, & \cdots & \cdots & \cdots & \cdots & \xi^{(4)}_n
\end{pmatrix}$$

in which every row constitutes a solution of (49), every $\xi^{(\lambda)}_{\nu}$ and $\xi^{(\lambda)}_{\nu}$ belongs to $B$, and $\xi^{(\lambda)}_{\nu} \neq 0$. We have

$$0 < a < \beta < \gamma < n.$$

For instance, the system of equations

$$x_2 - x_1 = x_3 - x_2 = \cdots = x_{n-1} - x_{n-2} = x_n$$

satisfies $\Gamma(B)$, where $B$ is the set of all integers. For we may put

$$l = 2; \quad S_1 = \{1, 2, ..., n-1\}; \quad S_2 = \{n\}.$$
The $\xi^{(t)}$ are

$$
\begin{pmatrix}
1, 1, 1, ..., 1, 0 \\
1, 2, 3, ..., n-1, 1
\end{pmatrix}.
$$

If (49) satisfies the condition $\Gamma(B)$ for some set $B$, then (49) satisfies $\Gamma(R_0)$, where $R_0$ is the ring generated by all coefficients $w_{\nu}$. This follows at once from well-known properties of systems of linear equations. Subsequently (pp. 149 et seq.), the structure of systems satisfying $\Gamma(B)$ will be further elucidated.

Let $K$ be a number field, and denote by $K - \{0\}$ the set of all numbers of $K$ except zero.

**Theorem VI.** If (49) is regular in $K - \{0\}$ then (49) satisfies $\Gamma(K)$.

Before we prove this theorem we establish a simple lemma which is geometrically obvious.

**Lemma 3.** Let

$$L_1(t), L_2(t), ..., L_r(t), M_1(t), ..., M_s(t)$$

be $r+s$ linear forms in $(t) = (t_1, t_2, ..., t_N)$. Let $r \geq 0$; $s > 0$. Suppose that

$$(50) \quad L_\rho(t') = 0 \quad (1 \leq \rho \leq r)$$

implies that, for at least one $\sigma_0 = \sigma_0(t')$,

$$M_{\sigma_0}(t') = 0.$$

Then at least one of the forms $M_\sigma(t)$ is a linear combination of the forms $L_\rho(t)$.

**Proof of Lemma 3.** We may assume that no proper subset of the system $M_1(t), ..., M_s(t)$ has the same property as the whole system of forms $M_\sigma(t)$. Then $s = 1$. For if $s > 1$, then, corresponding to every $\sigma (1 \leq \sigma \leq s)$, there is a vector $(t')$ for which

$$L_\rho(t^{(\sigma)}) = 0 \quad (1 \leq \rho \leq n), \quad M_\sigma(t^{(\sigma)}) = 1, \quad M_{\sigma'}(t^{(\sigma)}) = 0 \quad (\sigma' \neq \sigma).$$

Then the vector

$$(t') = \sum_{\sigma=1}^s (t^{(\sigma)})$$

satisfies (50). But, on the other hand,

$$M_\sigma(t') = M_\sigma(t^{(\sigma)}) = 1 \quad (1 \leq \sigma \leq s),$$

† We exclude the trivial case where all $a_{\nu}$ vanish.
which contradicts our hypothesis. Therefore $s = 1$. In this case the lemma is an immediate consequence of well-known facts concerning linear forms.

For a later application we add the remark that (50) is required only for those $(t') = (t'_1, t'_2, \ldots, t'_n)$ whose components $t'_v$ belong to the field generated by the coefficients of all forms $L_p, M_q$.

**Proof of Theorem VI.** We suppose that (49) is regular in $K$. Then every linear combination of the equations (49) is regular in the set of all complex numbers different from zero. Therefore in every such combination certain coefficients have the sum zero (Theorem V, p. 138). If we exploit this fact for suitable linear combinations of (49), we obtain the condition $\Gamma(K)$. But before starting along these lines we replace (49) by a system of equations whose coefficients belong to $K$.

Let $x_v = x_v^{(e)} \quad (1 \leq v \leq n; \ 1 \leq a \leq g)$

be a system of a maximal number of linearly independent solutions of (49) which belong to $K$. Such solutions exist because (49) is regular in $K - \{0\}$. Now let

\[ \sum_{v=0}^{n} b_{\mu v} x_v = 0 \quad (1 \leq \mu \leq m') \]

be a system of linear equations whose general solution is an arbitrary linear combination of the vectors $(x_v^{(e)}) \quad (1 \leq a \leq g)$. Then, as far as solutions in $K$ are concerned, (49) and (51) have exactly the same solutions. In particular, (51) is regular in $K - \{0\}$ and therefore, a fortiori, regular in the set of all complex numbers different from zero. Since the $x_v^{(e)}$ belong to $K$, it is possible to choose (51) so that the $b_{\mu v}$ belong to $K$.

Choose $m'$ parameters $t_\mu$ and consider the equation

\[ \sum_{\mu=1}^{m'} t_\mu \sum_{v=1}^{n} b_{\mu v} x_v = 0, \]

i.e.,

\[ \sum_{r=1}^{n} R_r(t) x_r = 0, \]

where

\[ R_r(t) = \sum_{\mu=1}^{m'} b_{\mu v} t_\mu \quad (1 \leq v \leq n). \]

Let $M_1(t), M_2(t), \ldots, M_s(t) \quad (s = 2^n - 1)$

be all the linear forms which are sums of forms $R_r(t)$ corresponding to any choice of distinct $v$. For any values of the $t_\mu$, (52) is regular in the set of all complex numbers different from zero. Hence, by Theorem V, at
least one of the numbers $M_{a}(t)$ vanishes. Therefore, by Lemma 3 above (p. 142), with $r = 0$; $s = 2^{n} - 1$, at least one of the forms $M_{a}(t)$ vanishes identically in $(t)$. If the $x_\nu$ are suitably numbered, we may assume that, identically in $(t)$,

\begin{equation}
\sum_{\nu = 1}^{a} R_{\nu}(t) = 0.
\end{equation}

Here $a$ is an integer and $1 \leq a \leq n$.

If $a = n$, we stop at this stage. If $a < n$, we let the parameters $t_\nu$ be subject to the conditions

\begin{equation}
R_{\nu}(t) = 0 \quad (1 \leq \nu \leq a).
\end{equation}

Then (52) becomes

\begin{equation}
\sum_{\nu = a+1}^{n} R_{\nu}(t)x_\nu = 0.
\end{equation}

Suppose that

\begin{align*}
M_1'(t), \ M_2'(t), \ldots, \ M_{a'}'(t) \quad (s' = 2^{n-a} - 1)
\end{align*}

are all the sums of any number of forms $R_{\nu}(t)$ corresponding to distinct values of $\nu$ with $a < \nu \leq n$. Again Theorem V shows that, for every choice of $(t)$ satisfying (55), at least one of the numbers $M_{a'}(t)$ vanishes, and this implies, by Lemma 3 ($r = a$; $s = s'$), that at least one of the forms $M_{a'}'(t)$ $(1 \leq a' \leq s')$ is a linear combination of the $R_{\nu}(t)$ $(1 \leq \nu \leq a)$. We can number the variables $x_{a+1}, x_{a+2}, \ldots, x_{n}$ so that this linear relation becomes

\begin{equation}
\sum_{\nu = 1}^{a} f_{\nu}R_{\nu}(t) + \sum_{\nu = a+1}^{s'} R_{\nu}(t) = 0,
\end{equation}

identically in $(t)$. Here $\beta$ is some number satisfying $a < \beta \leq n$, and the $f_{\nu}$ are constants.

If we proceed in this way, treating $\beta$ as we treated $a$, we find (in the case $\beta < n$) a relation

\begin{equation}
\sum_{\nu = 1}^{a} g_{\nu}R_{\nu}(t) + \sum_{\nu = \beta+1}^{s} R_{\nu}(t) = 0,
\end{equation}

where $a < \beta < \gamma \leq n$, etc. The process stops, say, with

\begin{equation}
\sum_{\nu = 1}^{s} j_{\nu}R_{\nu}(t) + \sum_{\nu = \delta+1}^{n} R_{\nu}(t) = 0.
\end{equation}

We have

\begin{equation}
1 \leq a < \beta < \gamma < \ldots < \delta < n.
\end{equation}

Since the coefficients of the $R_{\nu}$ belong to $K$, it is possible to choose the relations (56), (57), ..., (58) so that their coefficients $f_{\nu}, g_{\nu}, \ldots, j_{\nu}$ belong to $K$. 

\[\text{R. Rado} \quad [\text{Dec. 15,} \]
In view of (53), the relations (54), (56), (57), ..., (58) are equivalent to

\[ \sum_{\nu=1}^{n} b_{\mu\nu} \times 1 = 0, \]

\[ \sum_{\nu=1}^{a} b_{\mu\nu} f_{\nu} + \sum_{\nu=a+1}^{b} b_{\mu\nu} \times 1 = 0, \]

\[ \sum_{\nu=1}^{b} b_{\mu\nu} g_{\nu} + \sum_{\nu=b+1}^{c} b_{\mu\nu} \times 1 = 0, \]

\[ \ldots \quad \ldots \quad \ldots \quad \ldots \]

\[ \sum_{\nu=1}^{t} b_{\mu\nu} j_{\nu} + \sum_{\nu=t+1}^{n} b_{\mu\nu} \times 1 = 0, \]

where \( \mu \) takes all values 1, 2, ..., \( m' \). This means that the rows of the matrix

\[
\begin{pmatrix}
1, 1, \ldots, 1, 0, & \ldots & \ldots & \ldots & \ldots & 0 \\
 f_{1}, f_{2}, \ldots, f_{a}, 1, & \ldots & 1, & 0, & \ldots & \ldots & \ldots & \ldots & 0 \\
g_{1}, g_{2}, \ldots, \ldots, g_{b}, 1, & \ldots & 1, & 0, & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & j_{1}, j_{2}, \ldots, \ldots, j_{t}, 1, \ldots, 1
\end{pmatrix}
\]

are solutions of (51). The elements of this matrix belong to \( K \). Therefore, in view of the connection between (51) and (49), its rows are, at the same time, solutions of (49), and this amounts to saying that (49) satisfies \( \Gamma(K) \). The subsets \( S_{1}, S_{2}, \ldots, S_{t} \) of p. 141 are

\[ \{1, 2, \ldots, a\}, \{a+1, a+2, \ldots, \beta\}, \{\beta+1, \ldots, \gamma\}, \ldots, \{\delta+1, \ldots, n\}. \]

Thus Theorem VI is proved.

3. Let \( R \) be a ring of numbers, and let \( R - \{0\} \) denote the set of all numbers of \( R \) except zero. The main result of this paragraph is the following theorem.

**Theorem VII.** **A system of equations**

(59) \[ \sum_{\nu=1}^{n} a_{\nu} x_{\nu} = 0 \quad (1 \leq \mu \leq m) \]

is regular in \( R - \{0\} \) if, and only if, (59) satisfies \( \Gamma(R) \). In particular,
the equation
\[ a_1 x_1 + a_2 x_2 + \ldots + a_n x_n = 0, \]
where \( a_v < R - \{0\} \), is regular in \( R - \{0\} \) if, and only if, some of the numbers \( a_1, a_2, \ldots, a_n \) have the sum zero.

**Proof.** Let \( K \) be the field generated by the numbers of \( R \). To begin with the easier half of the theorem, let us assume that (59) is regular in \( R - \{0\} \). Then, a fortiori, (59) is regular in \( K - \{0\} \). Therefore, by Theorem VI, (59) satisfies \( \Gamma(K) \). Let \( \xi_v^{(b)} \) be the \( l \) solutions of (59) mentioned in the definition of \( \Gamma(K) \). Then
\[ \xi_v^{(b)} = \eta_v^{(b)}/\zeta_v^{(b)}, \]
where
\[ \eta_v^{(b)} < R; \quad \zeta_v^{(b)} < R - \{0\} \quad (1 \leq v \leq n; \ 1 \leq \lambda \leq l). \]

Let
\[ N = \prod_{1 < v < n, 1 < \lambda < l} \xi_v^{(b)}; \quad \bar{\xi}_v^{(b)} = N \xi_v^{(b)}. \]

Then the numbers \( \bar{\xi}_v^{(b)} \) belong to \( R \), and it is obvious that (59) satisfies \( \Gamma(R) \).

Now let us assume that (59) satisfies \( \Gamma(R) \). We have to show that (59) is regular in \( R - \{0\} \). We exclude the trivial case where all \( a_v \), vanish. In accordance with the definition of \( \Gamma(R) \) there are numbers \( \xi_v^{(b)} \) (1 \( \leq v \leq n; \ 1 \leq \lambda \leq l \) of \( R \) which have the properties stated on p. 141. Let \( \mathcal{B} \) be the system of these numbers \( \xi_v^{(b)} \), and denote by \( R', K' \) the ring and the field respectively generated by the numbers of \( \mathcal{B} \). We shall show that (59) is regular in \( K' - \{0\} \). If this fact is established, then, by Lemma 1 on p. 129, regularity in \( R' - \{0\} \) will follow, and since \( R' \) is contained in \( R \), regularity in \( R - \{0\} \).

Suppose, first of all, that the number \( l \) occurring in the definition of \( \Gamma(R) \) has the value 1. Then
\[ \sum_{v=1}^{n} a_{v \mu} \times 1 = 0 \quad (1 \leq \mu \leq m), \]
and therefore (59) is regular, in fact, absolutely regular, in \( K' - \{0\} \). Now suppose that \( l = l' > 1 \). We may assume that it has already been proved that all systems (59) satisfying \( \Gamma(R) \) with a value \( l = l' - 1 \) are regular in the corresponding \( K' - \{0\} \). We may suppose without loss of generality that
\[ S_l = \{n' + 1, n' + 2, \ldots, n\}, \]
where \( 1 \leq n' < n \). Then, clearly, the system
\[ \sum_{v=1}^{n'} a_{v \mu} x_v = 0 \quad (1 \leq \mu \leq m) \]
satisfies $\Gamma(R)$ with $l=l'-1$. For we can use the same numbers $\xi^{(x)}_v$ as for (59) but restrict $v, \lambda$ to the ranges $1 \leq v \leq n', 1 \leq \lambda \leq l'-1$. Therefore (60) is regular in $K'\setminus\{0\}$. Now let $k$ be the least positive integer, if there is one, such that (59) is not $k$-regular in $K'\setminus\{0\}$. We have to deduce a contradiction.

According to $\Gamma(R)$ we have

$$\sum_{v=1}^{n'} a_{\nu^v} \xi^{(x)}_v + \sum_{v=n'+1}^{n} a_{\nu^v} \xi^{(x)}_v = 0 \quad (1 \leq \mu \leq m),$$

where $\xi^{(x)}_v < R'\setminus\{0\}$. By Theorem I (p. 128) there are finite subsets $M, M'$ of $K'\setminus\{0\}$ such that (59) is $(k-1)$-regular in $M$ and (60) is $k$-regular in $M'$\dagger. Since (59) is not $k$-regular in $K'\setminus\{0\}$, there is a distribution $\Delta$ of $K'\setminus\{0\}$ into $k$ classes which has the property that (59) has no solution $x_v$ with

$$x_1 = x_2 = \ldots = x_n \pmod{\Delta}.$$

Let $M''$ be the set consisting of all numbers

$$\xi^{(x)}_v t y^{-1} \quad (1 \leq v \leq n'; \; t \in M; \; y \in M')$$

and of the number 1. Define $\Delta'(x)$ in $K'$ as follows:

$$\Delta'(x) = \prod_{y < M'} \Delta(xy) \quad (x < K'\setminus\{0\}),$$

$$0 \not\equiv x \pmod{\Delta'} \quad (x < K'\setminus\{0\}).$$

By Theorem III there exists a number $a$ of $K'$ and a positive integer $d$ for which

$$a + dz \equiv a \pmod{\Delta'} \quad (z < M'').$$

In view of a remark made above (p. 135) we may postulate that the numbers $a$ and $a + dz (z < M'')$ differ from zero or else are all equal to zero. The second possibility is ruled out, since we included the number 1 in $M''$. Therefore, by (62),

$$(a + dz)y \equiv ay \pmod{\Delta} \quad (z < M''; \; y < M').$$

Put

$$\Delta''(y) = \Delta(ay) \quad (y < M').$$

\dagger Strictly speaking, $K'$ should be replaced by the field $K''$ belonging to (60). But $K''$ is contained in $K'$, and therefore regularity in $K'\setminus\{0\}$ holds a fortiori.

\ddagger In the case $k = 1$ statements about $k-1$ have to be omitted.
By definition of $M'$ there are numbers $y_v$ of $M'$ such that

\[(66)\quad y_1 \equiv y_2 \equiv \ldots \equiv y_{n'} \pmod{A''},\]

\[(67)\quad \sum_{v=1}^{n'} a_{\mu v} y_v = 0 \quad (1 \leq \mu \leq m);\]

(64), (66) and (65) imply that

\[(68)\quad (a+dz)y_v \equiv ay_v \equiv ay_1 \pmod{\Delta} \quad (1 \leq v \leq n'; \ z < M''').\]

Also, from (67) and (61),

\[(69)\quad \sum_{v=1}^{n'} a_{\mu v} (ay_v + d\xi_v^{(r)} t) + \sum_{v=n'+1}^{n} a_{\mu v} \xi_v^{(r)} dt = 0 \quad (1 \leq \mu \leq m; \ t < M').\]

Now

$$ay_v + d\xi_v^{(r)} t = (a+dz)ty_v^{-1}y_v,$$

where

$$\xi_v^{(r)} ty_v^{-1} < M'' \quad (1 \leq v \leq n'; \ t < M').$$

Therefore, by (68),

\[(70)\quad ay_v + d\xi_v^{(r)} t \equiv ay_1 \pmod{\Delta} \quad (1 \leq v \leq n'; \ t < M).\]

We now show that this implies that

\[(71)\quad d\xi_v^{(r)} t \equiv ay_1 \pmod{\Delta} \quad (t < M).\]

If (71) is not true, then, for some number $t_0$ of $M$,

\[(72)\quad d\xi_v^{(r)} t_0 \equiv ay_1 \pmod{\Delta}.\]

But then (69), (70) (with $t = t_0$) and (72) show that, contrary to our initial assumption about $\Delta$, the system (59) has a solution whose numbers belong to the same class of $\Delta$. Therefore (71) is established.

If $k = 1$, then (71) is plainly impossible; for then all numbers belong to the same class of $\Delta$. When $k > 1$ put

$$\Delta'''(x) = \Delta(d\xi_x^{(r)} x) \quad (x < M).$$

Then, by (71),

$$|\Delta'''| \leq k-1.$$

Hence, from the definition of $M$, there are numbers $x_v$ of $M$ satisfying (59) and

$$x_1 \equiv x_2 \equiv \ldots \equiv x_n \pmod{\Delta'''}.\]

Then

$$\sum_{v=1}^{n} a_{\mu v} d\xi_v^{(r)} x_v = 0 \quad (1 \leq \mu \leq m),$$

$$d\xi_v^{(r)} x_v \equiv d\xi_v^{(r)} x_1 \pmod{\Delta} \quad (1 \leq v \leq n)$$

and this, again, contradicts the definition of $\Delta$. This completes the proof of Theorem VII.
4. We note a corollary of Theorem VII.

A system

\[ \sum_{\nu=1}^{n} a_{\nu} x_{\nu} = 0 \quad (1 \leq \mu \leq m) \]

is regular in the set of all complex numbers other than 0 if, and only if, it is regular in \( R_0 - \{0\} \), where \( R_0 \) is the ring generated by all coefficients \( a_{\nu} \).

(We have to exclude the trivial case where all \( a_{\nu} = 0 \).)

It is, however, not true that, corresponding to every system of equations which is regular in some set, there exists a smallest ring \( R^* \) such that it is regular in \( R^* - \{0\} \). For consider the equation (\( m = 1; \ n = 4 \))

\[ (73) \quad ax_1 + \beta x_2 - (a + \beta) x_3 - a\beta x_4 = 0, \]

where \( a = \sqrt{2}; \beta = \sqrt{3} \). The only possible sets \( S_1, S_2, \ldots \), consistent with the definition of \( \Gamma \) (p. 141) are

\[ S_1 = \{1, 2, 3\}; \quad S_2 = \{4\}. \]

The existence of the solutions

\[ (1, 1, 1, 0); \quad (0, 0, 0, 1); \quad (\beta, 0, 0, 1) \]

of (73) shows that (73) is regular in \( R_a - \{0\} \) as well as in \( R_\beta - \{0\} \), where \( R_a, R_\beta \) are the rings generated by \( a, 1 \) and \( \beta, 1 \) respectively. The common numbers of \( R_a \) and \( R_\beta \) form the ring \( R_0 \) of all rational integers. But (73) is not regular in \( R_0 - \{0\} \). For this would imply that \( \Gamma(R_0) \) holds, i.e.

that there are rational integers \( x_1, x_2, x_3, x_4 \), with \( x_4 \neq 0 \), such that

\[ ax_1 + \beta x_2 - (a + \beta) x_3 - a\beta x_4 = 0, \]

\[ (x_1 - x_2) \sqrt{2} + (x_2 - x_3) \sqrt{3} = x_4 \sqrt{6}, \]

\[ 2(x_1 - x_3)(x_2 - x_3) \sqrt{6} = 6x_4^2 - 2(x_1 - x_3)^2 - 3(x_2 - x_4)^2. \]

which is easily seen to be impossible.

5. Let \( R \) be a ring of numbers (not consisting of zero only), and \( K \) be the field generated by \( R \). We want to bring out more clearly the significance of the condition \( \Gamma(R) \). From considerations at the beginning of the proof of Theorem VII, it is obvious that \( \Gamma(R) \) and \( \Gamma(K) \) are equivalent. We can easily write down a general class of systems of linear equations which satisfy \( \Gamma(K) \), viz.

\[ \sum_{\nu=1}^{n-m} c_{\mu} x_{\nu} - x_{n-m+\mu} = 0 \quad (1 \leq \mu \leq m). \]
Here \( 1 \leq m < n \), and the \( c_{\mu, \nu} \) are arbitrary numbers of \( K \) subject to the two conditions:

(i) no row

\begin{equation}
\tag{75}
c_{\mu_1}, c_{\mu_2}, \ldots, c_{\mu, n-m} \quad (1 \leq \mu \leq m)
\end{equation}

contains zeros only;

(ii) the first non-zero number of every row (75) has the value 1.

Let us call systems (74) of this kind \( T \)-systems.

In order to prove that (74) satisfies \( \Gamma(K) \), we may assume (74) to be of the form

\begin{equation}
\begin{aligned}
x_{s_1} + \sum_{\nu=s_1+1}^{n-m} c_{\mu, \nu} x_{\nu} - x_{n-m+\mu} &= 0 \quad (1 \leq \mu \leq a_1), \\
x_{s_2} + \sum_{\nu=s_2+1}^{n-m} c_{\mu, \nu} x_{\nu} - x_{n-m+\mu} &= 0 \quad (a_1 < \mu \leq a_2), \\
& \vdots \\
x_{s_{l-1}} + \sum_{\nu=s_{l-1}+1}^{n-m} c_{\mu, \nu} x_{\nu} - x_{n-m+\mu} &= 0 \quad (a_{l-2} < \mu \leq a_{l-1}),
\end{aligned}
\end{equation}

(76)

where \( l \geq 2 \); \( 1 \leq s_1 < s_2 < \ldots < s_{l-1} \leq n-m \);

\[ 1 \leq a_1 < a_2 < \ldots < a_{l-1} = n. \]

Then we put

\[ S_1 = \{ s_1, n-m+1, n-m+2, \ldots, n-m+a_1 \} \]

\[ S_2 = \{ s_2, n-m+a_1+1, n-m+a_1+2, \ldots, n-m+a_2 \} \]

\[ \ldots \]

\[ S_{l-1} = \{ s_{l-1}, n-m+a_{l-2}+1, n-m+a_{l-2}+2, \ldots, n-m+a_{l-1} \} \]

while the last set \( S_l \) consists of all indices out of 1, 2, \ldots, \( n \) which do not occur in any of the sets \( S_1, S_2, \ldots, S_{l-1} \). A moment’s consideration shows that we can choose numbers \( \xi^{(\nu)} \) which are consistent with the definition of \( \Gamma(K) \).

In a certain sense (76) is the most general system which satisfies \( \Gamma(K) \).

For we have the following result.

Corresponding to every system

\begin{equation}
\sum_{\nu=1}^{n} a_{\mu, \nu} x_{\nu} = 0 \quad (1 \leq \mu \leq m')
\end{equation}

(77)
which satisfies $\Gamma(K)$, there is a $T$-system (74) such that every solution of (74) is at the same time a solution of (77).

In other words, by adding suitable equations to those of (77), which is supposed to satisfy $\Gamma(K)$, we can obtain a $T$-system (or a system equivalent to a $T$-system).

In proving this statement we may assume, without loss of generality, that the sets $S_\alpha$ corresponding to (77) are

$$
S_1 = \{1, 2, \ldots, \beta_1\},
$$

$$
S_2 = \{\beta_1+1, \beta_1+2, \ldots, \beta_2\},
$$

$$
\ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots
$$

$$
S_l = \{\beta_{l-1}+1, \beta_{l-1}+2, \ldots, \beta_l\},
$$

where $l \geq 1$; $1 \leq \beta_1 < \beta_2 < \ldots < \beta_l = n$.

Consequently there is a matrix

$$
(78) \left( \begin{array}{ccccccccccc}
1, & 1, & \ldots, & 1, & 0, & \ldots & \ldots & \ldots & \ldots & \ldots & 0 \\
\xi^{(3)}_1, & \xi^{(3)}_2, & \ldots, & \xi^{(3)}_d, & 1, & 0, & \ldots & \ldots & \ldots & \ldots & 0 \\
\ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\
\xi^{(l)}_1, & \xi^{(l)}_2, & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \xi^{(l)}_{\beta_{l-1}}, & 1, & \ldots, & 1
\end{array} \right)
$$

every row of which is a solution of (77). The $\xi^{(\alpha)}_i$ belong to $K$. In (78) the columns with indices $\beta_1, \beta_2, \ldots, \beta_l$ are linearly independent, while every one of the remaining columns is a linear combination of these $l$ columns. Moreover, these linear relations are expressible in the form

$$
(79) \begin{cases}
  x_\mu = x_{\beta_1} + \sum_{\lambda=2}^{l} c_{\mu\lambda} x_{\beta_\lambda} & (1 \leq \mu < \beta_1), \\
  x_\mu = x_{\beta_2} + \sum_{\lambda=3}^{l} c_{\mu\lambda} x_{\beta_\lambda} & (\beta_1 < \mu < \beta_2), \\
  \quad \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \\
  x_\mu = x_{\beta_l} & (\beta_{l-1} < \mu < \beta_l),
\end{cases}
$$

where $c_{\mu\lambda} < K$. (79) is a complete set of linearly independent relations between the columns of (78). Hence every vector $(x_1, x_2, \ldots, x_n)$ which satisfies (79) is a linear combination of the rows of (78) and is therefore a solution of (77). But (79) is a $T$-system, as is seen by ordering the variables as follows:

$$
x_{\beta_1}, x_{\beta_2}, \ldots, x_{\beta_l}, x_1, x_2, \ldots, x_{\beta_{l-1}}, x_{\beta_{l+1}}, x_{\beta_{l+2}}, \ldots, x_{\beta_1-1}, x_{\beta_1+1}, \ldots, x_{\beta_{l-1}}.
$$
This proves our statement about the connection between (77) and 
$T$-systems.

Questions about the regularity of a system of homogeneous linear 
equations in the set of all positive rational integers or in the set of all 
complex numbers $re^{i\phi}$, where $r > 0$ and $\phi$ belongs to a given interval 
$\phi_0 < \phi < \phi_1$, and many more, are settled by the following theorem.

**Theorem VIII.** Let $R$ be a ring of complex numbers, and let $A$ be a 
subset of $R - \{0\}$ which has the following property. There is a finite number 
of elements $d_1, d_2, \ldots, d_k$ of $R - \{0\}$ such that every $x$ of $R - \{0\}$ is expressible as 

$\sum_{r=1}^{n} a_r x_r = 0 \quad (1 \leq \mu \leq m)$

where $a < A$; $1 \leq \kappa \leq k$. Then regularity of 

$x = ad_\kappa$

in $A$ is equivalent to regularity of (81) in $R - \{0\}$.

For instance, in the case of all numbers $re^{i\phi}$, where $r > 0$; $\phi_0 < \phi < \phi_1$, 
we choose an integer $k > 2\pi/(\phi_1 - \phi_0)$ and put 

$d_\kappa = e^{2\pi i \kappa/k} \quad (1 \leq \kappa \leq k)$.

Of course, the representation (80) need not be unique.

**Proof.** Suppose that (81) is regular in $R - \{0\}$. We have to show 
that (81) is regular even in $A$. Let $\Delta(x)$ be defined in $A$. Given any $x$ 
of $R - \{0\}$, we define $\lambda = \lambda(x)$ as being the least index $\kappa$ for which $xd_\kappa^{-1}$ 
belongs to $A$: $\lambda$ exists. Put 

$\Delta'(x) = \Delta^{(0)}(\lambda(x)) \Delta(xd_{\lambda(x)}^{-1}) \quad (x < R - \{0\})$.

Then there are numbers $x$, satisfying (81) and 

$x_1 \equiv x_2 \equiv \ldots \equiv x_n \pmod{\Delta'}$.

(82) implies that 

$\lambda(x_1) = \lambda(x_2) = \ldots = \lambda(x_n) = \lambda_0,$

say, and 

$x_1 d_{\lambda_0}^{-1} \equiv x_2 d_{\lambda_0}^{-1} \equiv \ldots \equiv x_n d_{\lambda_0}^{-1} \pmod{\Delta}$;

while (81) yields 

$\sum_{r} a_r x_r d_{\lambda_0}^{-1} = 0 \quad (1 \leq \mu \leq m)$.

This proves the theorem,
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3. \textit{Non-homogeneous linear equations.}

1. Let $A_0$ be the field of all algebraic numbers and $C_0$ that of all complex numbers.

\textbf{Theorem IX.} \hspace{1em} A system

\begin{equation}
\sum_{\nu=1}^{n} a_{\mu,\nu} x_{\nu} = b_{\mu} \quad (1 \leq \mu \leq m)
\end{equation}

is regular in $A_0$ if, and only if, it is absolutely regular in $A_0$.

\textbf{Theorem X.} \hspace{1em} A system (83) with algebraic coefficients $a_{\mu,\nu}$ (and with the $b_{\mu}$ arbitrary complex numbers) is regular in $C_0$ if, and only if, it is absolutely regular in $C_0$.

In proving the last two theorems it is sufficient to show that regularity implies absolute regularity in the sets in question.

Theorem IX follows from Theorem X. For let us assume that (83) is regular in $A_0$. Then there are algebraic numbers $x^{(0)}$ satisfying

\[ \sum_{\nu=1}^{n} a_{\mu,\nu} x^{(0)}_{\nu} = b_{\mu} \quad (1 \leq \mu \leq m). \]

This simply means that (83) is 1-regular in $A_0$. Therefore (83) is equivalent to

\[ \sum_{\nu=1}^{n} a_{\mu,\nu} (x_{\nu} - x^{(0)}_{\nu}) = 0 \quad (1 \leq \mu \leq m). \]

If we use the argument which leads from (49) to (51) we find a system

\begin{equation}
\sum_{\nu=1}^{n} a'_{\mu,\nu} y_{\nu} = 0 \quad (1 \leq \mu \leq m')
\end{equation}

with algebraic coefficients $a'_{\mu,\nu}$ such that (84) and

\begin{equation}
\sum_{\nu=1}^{n} a_{\mu,\nu} y_{\nu} = 0 \quad (1 \leq \mu \leq m)
\end{equation}

have exactly the same solutions in $A_0$. Therefore

\begin{equation}
\sum_{\nu=1}^{n} a'_{\mu,\nu} (x_{\nu} - x^{(0)}_{\nu}) = 0 \quad (1 \leq \mu \leq m')
\end{equation}

and (83) have the same solutions in $A_0$. In particular, (85) is regular in $A_0$ and \textit{a fortiori} regular in $C_0$. Now apply Theorem X to (85). It follows that (85) is absolutely regular in $C_0$. Since $a'_{\mu,\nu}, x^{(0)}_{\nu}$ are numbers of $A_0$ we conclude further that (85) is absolutely regular even in $A_0$, and finally that (83) is absolutely regular in $A_0$.

\[ \dagger \] Absolute regularity was defined on p. 127.
It remains to prove Theorem X. We note that absolute regularity in a set $M$ of a single equation

$$a_1 x_1 + a_2 x_2 + \ldots + a_n x_n = b$$

means that either

$$a_1 + a_2 + \ldots + a_n = 0$$

or

$$s = a_1 + \ldots + a_n \neq 0; \quad b/s < M.$$

**Lemma 4.** Suppose that (83) has the following property. For every choice of numbers $t_\mu$ belonging to the field $K$ generated by the $a_\nu$, the equation

$$\sum_{\mu=1}^{m} t_\mu \sum_{\nu=1}^{n} a_{\mu \nu} x_\nu = \sum_{\mu=1}^{m} t_\mu b_\mu$$

is absolutely regular in $M$. Then (83) is absolutely regular in $M$.

To prove Lemma 4, put $s_\mu = \sum_{\nu} a_{\mu \nu}$. We may assume that

$$s_\mu \neq 0 \quad (1 \leq \mu \leq m'), \quad s_\mu = 0 \quad (m' < \mu \leq m),$$

where $0 \leq m' \leq m$. By hypothesis, any numbers $t_\mu$ of $K$ for which

$$\sum_{\mu} s_\mu t_\mu = 0$$

satisfy

$$\sum_{\mu} b_\mu t_\mu = 0.$$

Therefore, by Lemma 3 on p. 142 ($r = s = 1$: see the remark at the end of the proof of Lemma 3),

$$x' \sum_{\mu} s_\mu t_\mu = \sum_{\mu} b_\mu t_\mu,$$

identically in the $t_\mu$. Here $x'$ is some constant. Hence

$$\sum_{\nu} a_{\mu \nu} x' = b_\mu \quad (1 \leq \mu \leq m).$$

If $m' > 0$ then

$$x' = b_1/s_1 < M.$$

For the equation

$$\sum_{\nu} a_{1 \nu} x_\nu = b_1$$

is absolutely regular in $M$. If $m' = 0$, i.e. if

$$s_\mu = 0 \quad (1 \leq \mu \leq m),$$

then

$$b_\mu = 0 \quad (1 \leq \mu \leq m);$$

and therefore

$$\sum_{\nu} a_{\mu \nu} x'' = b_\mu \quad (1 \leq \mu \leq m),$$

where $x''$ is any arbitrary number of $M$. Thus the lemma is proved.
In view of Lemma 4, it is sufficient to consider the case \( m = 1 \) of Theorem X. Let us therefore assume that \( a_1, a_2, \ldots, a_n \) belong to \( A_0 \), that \( b \) is an arbitrary number and that
\[
(86) \quad a_1 + a_2 + \ldots + a_n = 0; \quad b \neq 0.
\]
We have to show that
\[
(87) \quad a_1x_1 + \ldots + a_nx_n = b
\]
is not regular in \( C_0 \). We may assume that no \( a_x \) vanishes.

By means of transfinite induction we can find a "basis" \( \xi_1, \xi_2, \ldots, \xi_n, \ldots \) of all complex numbers with respect to the field \( K \) generated by \( a_1, a_2, \ldots, a_n \). This means that every complex number \( x \) is uniquely representable in the form
\[
(88) \quad x = \sum_a x_a' \xi_a,
\]
where \( a \) runs through certain ordinal numbers and \( x_a' \) belongs to \( K \). For every \( x \) only a finite number of "coordinates" \( x_a' \) differ from zero. Let
\[
(89) \quad b = \sum_a x_a' \xi_a
\]
be the representation (88) in the case \( x = b \). Since \( b \neq 0 \), at least one of the numbers \( b_a' \) differs from zero. There is no loss of generality in assuming that \( b_1' \neq 0 \). The numbers \( a_1, a_2, \ldots, a_n, b_1' \) are algebraic. If we multiply (87) by a suitable number of \( K \), we can obtain a case where these \( n+1 \) numbers are algebraic integers of \( K \). We may therefore assume that (87) is such that these numbers are algebraic integers. Let \( \omega_1, \omega_2, \ldots, \omega_p \) be a minimal basis of \( K \), so that every algebraic integer of \( K \) has a unique representation in the form
\[
(90) \quad \sum r_\lambda \omega_\lambda = \sum r_\lambda \omega_\lambda + \omega_p\omega_p,
\]
where the \( r_\lambda \) are rational integers. Then every number of \( K \) has a unique representation (89) with rational coefficients \( r_\lambda \).

Let \( p \) be a prime ideal in \( K \) which is not a divisor of \( b_1' \). Define \( \Delta(x) \), for all algebraic integers of \( K \), by means of the rule that
\[
x \equiv y \pmod{\Delta} \quad \text{if, and only if,} \quad x \equiv y \pmod{p}.
\]
Then no class of \( \Delta \) contains a solution of
\[
(90) \quad a_1x_1 + \ldots + a_nx_n = b_1'.
\]
For (90) and
\[ x_1 \equiv x_2 \equiv \ldots \equiv x_n \pmod{\Delta} \tag{91} \]
would lead to the contradiction
\[ b'_1 = \sum a_r x_r = \sum a_r (x_r - x_i) = 0 \pmod{p}. \]

Define, for every \( x \) of \( K \), a function \( f(x) \) by means of
\[ f(x) = \sum_{\lambda=1}^{p} |r_{\lambda}|, \]
where
\[ x = \sum_{\lambda=1}^{p} r_{\lambda} \omega_{\lambda} \ (r_{\lambda} \text{ rational}). \tag{92} \]

Then, for any \( x \), satisfying (91),
\[ f(\sum a_r x_r - b'_1) \geq 1. \tag{93} \]

Now choose an integer \( N \) satisfying
\[ N > \sum_{1 \leq r \leq n, 1 \leq \lambda \leq p} f(a_r \omega_{\lambda}); \tag{94} \]
and put, for every \( x \) of \( K \) satisfying (92),
\[ \Delta'(x) = \Delta \left( \sum_{\lambda} [r_{\lambda}] \omega_{\lambda} \right) \prod_{\lambda=1}^{p} \Delta^{(N)} ([N r_{\lambda}]) \dagger. \tag{95} \]

We now show that no class of \( \Delta' \) contains a solution of (90). Suppose that
\[ x_1 \equiv x_2 \equiv \ldots \equiv x_n \pmod{\Delta'}, \tag{96} \]
\[ a_1 x_1 + \ldots + a_n x_n = b'_1, \tag{97} \]
\[ x_r = \sum_{\lambda} r_{\lambda} \omega_{\lambda} \ (r_{\lambda} \text{ rational}). \]

(96) and (95) imply that
\[ \sum_{\lambda} [r_{\lambda}] \omega_{\lambda} \equiv \sum_{\lambda} [r_{\lambda}] \omega_{\lambda} \pmod{\Delta} \ (1 \leq r \leq n), \tag{98} \]
\[ [N r_{\lambda}] \equiv [N r_{\lambda}] \pmod{N} \ (1 \leq r \leq n; 1 \leq \lambda \leq p), \]
\[ \frac{1}{N} [N r_{\lambda}] - [r_{\lambda}] = \frac{1}{N} [N r_{\lambda}] - \left[ \frac{1}{N} [N r_{\lambda}] \right] \]
\[ = \frac{1}{N} [N r_{\lambda}] - \left[ \frac{1}{N} [N r_{\lambda}] \right] = \frac{1}{N} [N r_{\lambda}] - r_{\lambda} = d_{\lambda}, \]

\dagger \Delta^{(N)} \text{ was defined on p. 124.}
say, for \( 1 \leq v \leq n, \ 1 \leq \lambda \leq p, \)

\[
\sum_{\lambda} \alpha_{\lambda} \sum_{r_{\lambda \alpha}} \omega_{\lambda} = \sum_{\lambda} \alpha_{\lambda} \sum_{r_{\lambda \alpha}} \frac{1}{N} [N r_{\lambda \alpha}] \omega_{\lambda} - \sum_{\lambda} \alpha_{\lambda} \sum_{r_{\lambda \alpha}} \omega_{\lambda} = \sum_{\lambda} \alpha_{\lambda} \sum_{r_{\lambda \alpha}} \omega_{\lambda} - \sum_{\lambda} \alpha_{\lambda} \sum_{r_{\lambda \alpha}} \omega_{\lambda},
\]

where \( r'_{\lambda \alpha} \) is rational, \( 0 < r'_{\lambda \alpha} < 1/N \). Hence, from (98) and (93),

\[
1 \leq f \left( \sum_{\lambda} \alpha_{\lambda} \sum_{r_{\lambda \alpha}} \omega_{\lambda} - b_{1} \right) = f \left( \sum_{\lambda} \alpha_{\lambda} \sum_{r_{\lambda \alpha}} \omega_{\lambda} - \sum_{\lambda} \alpha_{\lambda} \sum_{r'_{\lambda \alpha}} \omega_{\lambda} - b_{1} \right) = f \left( - \sum_{\lambda} \alpha_{\lambda} \sum_{r'_{\lambda \alpha}} \omega_{\lambda} \right) \leq \sum_{r'_{\lambda \alpha}} | f(\alpha_{\omega_{\lambda}}) \leq \frac{1}{N} \sum_{r'_{\lambda \alpha}} f(\alpha_{\omega_{\lambda}}).
\]

The last inequality contradicts (94). Therefore, as stated above, no class of \( \Delta' \) contains a solution of (90).

We now proceed to define a distribution of \( C_{0} \). Put, for every complex \( x \),

\[
g(x) = x_{1}',
\]

where \( x_{1}' \) is the first "coordinate" in the representation (88). \( g(x) \) has the following properties:

\[
g(x) < K, \ g(b) = b_{1}' \neq 0, \ g \left( \sum_{x=1}^{n} \alpha_{x_{v}} \right) = \sum_{x=1}^{n} \alpha_{x_{v}} g(x_{v})
\]

for all complex \( x_{v} \).

Put

\[
\Delta''(x) = \Delta' \left( g(x) \right) \quad (x < C_{0}).
\]

Suppose that, for some \( x_{v} \) of \( C_{0} \),

\[
x_{1} \equiv x_{2} \equiv \ldots \equiv x_{n} \quad (\text{mod } \Delta'').
\]

Then

\[
g(x_{v}) \equiv g(x_{1}) \quad (\text{mod } \Delta') \quad (1 \leq v \leq n).
\]

Hence, by the definition of \( \Delta' \),

\[
\sum_{v} \alpha_{x_{v}} g(x_{v}) \neq g(b),
\]

i.e.,

\[
g(\sum_{v} \alpha_{x_{v}}) \neq g(b);
\]

and therefore

\[
\sum_{v} \alpha_{x_{v}} \neq b.
\]

This completes the proofs of Theorems IX and X.
2. The foregoing proof made use of transfinite induction in order to define a certain distribution of all complex numbers. In the case of rational coefficients \( a_\mu \) (the \( b_\mu \) may be arbitrary complex numbers) it is possible to eliminate transfinite induction. In this section we give a proof of the assertion of Theorem X in the special case of rational coefficients \( a_\mu \) or even in the slightly more general case where

\[
a_\mu = a_\mu' + ia_\mu'' \quad (a_\mu', a_\mu'' \text{ rational}).
\]

In view of Lemma 4 it is sufficient to prove the following proposition.

Let \( a_\nu = a_\nu' + ia_\nu'' \quad (1 \leq \nu \leq n) \),

where \( a_\nu' \), \( a_\nu'' \) are rational integers. Suppose that

\[
a_1 + \ldots + a_n = 0,
\]

while \( b \) is a complex number different from zero. Then

\[
a_1 x_1 + \ldots + a_n x_n = b
\]

is not regular in \( C_0 \).

Define, for any

\[
x = x' + ix'' \quad (x', x'' \text{ real}),
\]

\[
[x] = [x'] + i[x''].
\]

Choose positive integers \( m', m'' \) such that

\[
m' > \sqrt{2} \sum |a_\nu| + |b|; \quad m'' > \sqrt{2} |b|^{-1} \sum |a_\nu|.
\]

Put, for every such \( x \),

\[
\Delta^*(x) = \Delta^{(m' m'')}(m'' x') \Delta^{(m' m'')}(m'' x'').
\]

Then our proof is complete when we can show that no class of \( \Delta^* \) contains a solution of (99). Suppose that

\[
x_1 \equiv x_2 \equiv \ldots \equiv x_n \pmod{\Delta^*}.
\]

Then (using an obvious notation)

\[
[m'' x_v] \equiv [m'' x_1] \pmod{m' m''},
\]

(100)

\[
[m'' x_v] - [m'' x_1] = k_\nu m'm'' \quad (1 \leq \nu \leq n),
\]

where

\[
k_\nu = k_\nu' + ik_\nu'' \quad (k_\nu', k_\nu'' \text{ rational integers}).
\]
(100) implies that
\[
\frac{1}{m''} [m'' x_r] - \frac{1}{m''} [m'' x_1] = k_v m',
\]
\[
\left[ \frac{1}{m''} [m'' x_r] \right] - \left[ \frac{1}{m''} [m'' x_1] \right] = k_v m',
\]
\[
[x_r] - [x_1] = k_v m'.
\]
Also,
\[
m'' x_r - m'' x_1 = k_v m' m'' + r_v,
\]
where
\[
|r_v| < \sqrt{2}.
\]

Case (1): Let \( \Sigma a_v[x_r] \neq 0. \)
Then
\[
|\Sigma a_v[x_r]| = |\Sigma a_v([x_r] - [x_1])| = |\Sigma a_v k_v m'| \geq m',
\]
\[
|\Sigma a_v x_r| \geq |\Sigma a_v[x_r] - |\Sigma a_v(x_r - [x_r])| \geq m' - \Sigma |a_v| \sqrt{2} > b.
\]

Case (2): Let \( \Sigma a_v[x_r] = 0. \)
Then
\[
|\Sigma a_v x_r| = |\Sigma a_v((x_r - x_1) - ([x_r] - [x_1]))| = |\Sigma a_v\left(\frac{k_v m' m'' + r_v - k_v m'}{m''}\right)| \leq \frac{1}{m''} \Sigma |a_v| \sqrt{2} < b.
\]
Hence in either case
\[
a_1 x_1 + a_2 x_2 + \ldots + a_n x_n \neq b.
\]

4. \( \omega \)-regularity.

A system of conditions
\[(101) \quad S(x_1, \ldots, x_n) = 0\]
was called \( \omega \)-regular in a set \( M \) if, given any distribution \( \Delta \) of \( M \) into denumerably many classes, there is always a solution of (101) satisfying
\[
x_1 \equiv x_2 \equiv \ldots \equiv x_n \pmod{\Delta}.
\]
Of the three propositions:
(i) (101) is \( k \)-regular in \( C_0 \) for every \( k = 1, 2, 3, \ldots; \)
(ii) (101) is \( \omega \)-regular in \( C_0; \)
(iii) (101) is absolutely regular in \( C_0. \)
Note on Combinatorial Analysis.

(i) is weaker than (ii), and (ii) is weaker than (iii). For the equation

\[ x_1 + x_2 - x_3 = 0 \]

satisfies (i) but not (ii) (as follows from Theorem XI below), and the condition

\[ x_1 \neq x_2 \]

satisfies (ii) but not (iii). We prove that in the case of linear equations or, more generally, of conditions whose solutions form a closed set, (ii) and (iii) are equivalent.

Theorem XI. A system of equations

\[ \sum_{\nu=1}^{n} a_{\mu\nu} x_{\nu} = b_{\mu} \quad (1 \leq \mu \leq m) \]

is \( \omega \)-regular in a set \( M \) of complex numbers if, and only if, (102) is absolutely regular in \( M \).

Proof. Suppose that (102) is not absolutely regular in \( M \). We have to show that (102) is not \( \omega \)-regular in \( M \). We call a circle

\[ |x - (c' + ic'')| < r \]

a rational circle if \( c', c'', r \) are rational numbers. Every point \( z \) of \( M \) is contained in a rational circle which does not contain any solution of (102). For, if this is not true for some \( z = z_0 \) of \( M \), then every rational circle (103) which contains \( z_0 \) also contains a solution \( x_v \) of (102), and making \( r \to 0 \) shows that

\[ \sum_{\nu} a_{\mu\nu} z_0 = b_{\mu} \quad (1 \leq \mu \leq m), \]

i.e. that (102) is absolutely regular in \( M \). The rational circles are enumerable. Let \( R_1, R_2, \ldots \) be a sequence containing every rational circle. Define, for every \( z \) of \( M \), \( f(z) \) as being the least \( \lambda \) such that \( z \) lies in \( R_\lambda \) but, at the same time, \( R_\lambda \) contains no solution of (102). Then, obviously no class of

\[ \Delta^{*}(z) = \Delta^{(0)} \left( (f(z) \right) \quad (z < M) \]

contains a solution of (102). Therefore (102) is not \( \omega \)-regular in \( M \).

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\[ \dagger \Delta^{(0)} \] was defined on p. 124.