NOTE ON COMBINATORIAL ANALYSIS

By R. RADO.

[Received 16 September, 1939.—Read 15 December, 1939.]

Consider a system of equations

$$(1)\qquad a_{\mu 1}x_1+a_{\mu 2}x_2+\ldots+a_{\mu n}x_n=b_\mu \quad (1\leqslant\mu\leqslant m),$$

where the $a_{\mu\nu}$, b_μ are complex numbers. Let A be a set of numbers, for instance all numbers of a number field, all complex numbers different from zero, etc. We call (1) *regular in* A if the following condition holds: however we split A into a finite number of subsets $A_1, A_2, \ldots, A_k$, always at least one of these subsets A_κ contains a solution of (1). Roughly speaking, regularity of (1) in A means that, in a certain sense, A contains very many solutions of (1), and these solutions interlock very intimately.

In the special case where A is the set of all positive integers, I. Schur† proved the regularity of

$$x_1+x_2-x_3=0,$$

and van der Waerden‡ proved the regularity, for every $l>0$, of

$$x_0-x_1=x_1-x_2=\ldots=x_{l-1}-x_l\neq 0§.$$

In a previous note|| I determined all systems (1) which are regular in the set of all positive integers. In this case necessary and sufficient conditions for regularity turned out to be certain linear relations between the $a_{\mu\nu}$, b_μ. In the present note, which is an elaboration of a lecture delivered at the International Congress of Mathematicians at Oslo, 1936¶,

† *Jahresbericht der Deutschen Mathematiker-Vereinigung*, 25 (1916), 114

‡ *Nieuw Archief voor Wiskunde*, 15 (1927), 212–216.

§ Regularity of a system of equations and inequalities, in fact, of any set of conditions imposed upon certain variables, is defined in exactly the same way as above.

|| *Math. Zeitschrift*, 36 (1933), 424–480, quoted as S.

¶ *Comptes Rendus, Oslo*, 2 (1936), 20–21.

I propose to consider the same problem in the case of more general sets A. I establish necessary and sufficient conditions for regularity expressed in terms of linear relations between the $a_{\mu\nu}$, b_{μ} in the following cases:

(i) The $a_{\mu\nu}$ are arbitrary, $b_{\mu}=0$, A is the set of all numbers different from zero contained in a given ring of complex numbers.

(ii) The $a_{\mu\nu}$ and b_{μ} are arbitrary numbers, A is the field of all algebraic numbers.

(iii) The $a_{\mu\nu}$ are algebraic numbers, the b_{μ} are arbitrary, A is the field of all complex numbers.

Other cases can be dealt with which are not included in this note.

The criteria are analogous to those obtained in the special case of S. In the cases (ii), (iii) the condition for regularity postulates that, for some number ξ of A,

$$\sum_{\nu=1}^{n} a_{\mu\nu}\,\xi = b_{\mu} \quad (1 \leqslant \mu \leqslant m).$$

This result may be expressed as follows. *If, in every distribution of the numbers of A over a finite number of classes, at least one class contains a solution of* (1), *then, in cases* (ii) *and* (iii), *the same is true for the extreme case of a distribution in which every number of A forms a class by itself.*

Some of the proofs are extensions of proofs in S, others require the use of different methods. In proving the result concerning (i), we employ an extension (Theorem II) of van der Waerden's theorem quoted above. This extension was first proved by Dr. G. Grünwald, who kindly communicated it to me. It may be stated as follows. *Given any "configuration" S consisting of a finite number of lattice points*† *of a Euclidean space, and given a distribution of all lattice points of this space into a finite number of classes, there is at least one class which contains a configuration S' of lattice points which is similar and parallel* (*homothetic*) *to S.* Dr. Grünwald's proof runs parallel to van der Waerden's proof of his theorem. The proof given in this note is a simplification analogous to the simplification of van der Waerden's proof given in S (p. 432, Satz I). In an earlier note‡ I proved a weaker form of Grünwald's Theorem in which similarity of S and S' was replaced by affinity.

The last paragraph deals with regularity of systems (1) with respect to distributions which have denumerably many classes.

† *I.e.* points with rational integral coordinates.

‡ *Berliner Sitzungsberichte* (1933), 589–596, Satz I.

1. *Preliminaries. Generalisation of van der Waerden's Theorem.*

1. Let A be a finite or infinite aggregate. In this section letters a, b, c, x denote general elements of A. Throughout this paper the letter Δ (and Δ', Δ'', ..., Δ_1, Δ_2, etc.) denotes distributions of all elements of A into a finite number of classes. Occasionally we consider distributions into an infinite number of classes, in which case this is mentioned explicitly. Δ is defined by means of a relation "$\sim$" which is defined for some pairs of elements of A and which has the properties:

(i) $a \sim a$ for every a;

(ii) $a \sim b$ implies $b \sim a$;

(iii) $a \sim b$ and $b \sim c$ imply $a \sim c$;

(iv) every infinite subset of A contains two distinct elements a, b such that $a \sim b$.

By $|\Delta|$ we denote the number of non-empty classes of Δ. A congruence

$$a \equiv b \pmod{\Delta}$$

expresses, by definition, the fact that a and b belong to the objects distributed by means of Δ, and, moreover, belong to the same class. In analogy with the notation for functions, we speak of a distribution $\Delta(x)$ defined for every x of A, or, briefly, defined in A.

Throughout this paper $\Delta^{(k)}(x)$, for every positive integer k, denotes the distribution of all rational integers into classes of equal residues mod k. Thus

$$x \equiv y \pmod{\Delta^{(k)}}$$

is equivalent to saying that x and y are rational integers and

$$x \equiv y \pmod{k}.$$

$\Delta^{(0)}$ denotes that distribution of A in which every element of A forms a class for itself. $\Delta^{(0)}$ may have infinitely many classes. We use the same symbol $\Delta^{(0)}$ for different sets A.

Two methods are employed for generating new distributions from given ones†. The first is a process of multiplication. Given a finite

† Both were used in S. The notation adopted in this paper seems to be more convenient than the one used in S.

number of distributions $\Delta_1, \Delta_2, \ldots, \Delta_n$, each defined in A, we understand by their *product*

$$\Delta = \Delta_1 \Delta_2 \ldots \Delta_n = \prod_{\nu=1}^{n} \Delta_\nu$$

that distribution Δ of A which is defined by the rule:

$$x \equiv y \pmod{\Delta}$$

if, and only if,

$$x \equiv y \pmod{\Delta_\nu} \quad (1 \leqslant \nu \leqslant n).$$

For instance, if k and l are natural numbers, then

$$\Delta^{(k)} \Delta^{(l)} = \Delta^{(m)},$$

where m is the least common multiple of k and l.

We have

$$|\Delta_1 \Delta_2 \ldots \Delta_n| \leqslant |\Delta_1| |\Delta_2| \ldots |\Delta_n|.$$

The second process is one of *inducing a distribution* in a set B by means of a distribution in A and a correspondence between every element of B and some elements of A. Suppose that $\Delta(x)$ is defined in A, and that $f(y)$ is a function defined for every element y of a set B. The functional values of f are elements of A. Then we define a distribution $\Delta_1(y)$ in B by postulating that

$$y_1 \equiv y_2 \pmod{\Delta_1}$$

is to be equivalent to

$$f(y_1) \equiv f(y_2) \pmod{\Delta}.$$

We use the notation

$$\Delta_1(y) = \Delta\big(f(y)\big).$$

Clearly

$$|\Delta_1| \leqslant |\Delta|.$$

We have, for instance,

$$\Delta^{(k)}(x+1) = \Delta^{(2k)}(2x) = \Delta^{(k)}(x).$$

For

$$x_1 \equiv x_2 \pmod{k}$$

is equivalent to

$$x_1 + 1 \equiv x_2 + 1 \pmod{k}$$

and also to

$$2x_1 \equiv 2x_2 \pmod{2k}.$$

For real numbers $x \neq 0$,

$$\Delta^{(0)}\left(\frac{x}{|x|}\right) = \Delta^{(3)}\left(\frac{x}{|x|}\right)$$

is a distribution in which all positive numbers are in one class and all negative numbers are in a second class.

Suppose that $\Delta(x)$ is defined in A and that the classes of Δ are the sets

$$A_0, A_1, \ldots, A_{k-1}.$$

Define, for every x of A, a function $f(x)$ by means of†

$$x \prec A_{f(x)}.$$

Then

$$\Delta(x) = \Delta^{(k)}\big(f(x)\big).$$

Now choose a natural number l. Then corresponding to every x of A there are integers $g(x)$, $h(x)$ such that

$$f(x) = g(x) + lh(x),$$

$$0 \leqslant g(x) < l; \quad 0 \leqslant h(x) \leqslant (k-1)/l.$$

Therefore

$$\Delta(x) = \Delta^{(k)}\big(f(x)\big) = \Delta^{(l)}\big(g(x)\big)\,\Delta^{(k)}\big(h(x)\big) = \Delta'(x)\,\Delta''(x),$$

say. We have

$$|\Delta'| \leqslant l, \quad |\Delta''| \leqslant \left[\frac{k-1}{l}\right] + 1‡.$$

Hence every Δ can be represented in the form

$$\Delta = \Delta'\Delta'',$$

where

$$|\Delta'| \leqslant l, \quad |\Delta''| \leqslant \left[\frac{|\Delta|-1}{l}\right] + 1.$$

In particular, putting $l = 2$, we see that every Δ is a product of a finite number of distributions with not more than two classes.

If x, y are general elements of two sets A, B respectively, then $\Delta(x, y)$ denotes a distribution of all pairs (x, y). Thus, for rational integers x, y,

$$\Delta(x, y) = \Delta^{(k)}(2x + 3y)$$

† We use the symbol "$\prec$" to denote the relation of an element to the class to which it belongs.

‡ $[t]$ denotes the largest integer not exceeding t.

denotes that distribution of all pairs (x, y) for which (x_1, y_1) and (x_2, y_2) are in the same class if, and only if,

$$2x_1+3y_1 \equiv 2x_2+3y_2 \pmod{k}.$$

2. Let $S(x_1, x_2, \ldots, x_n)$ be a system of conditions imposed upon the values of variables $x_1, \ldots, x_n$. These variables are allowed to vary throughout a set A. A relation

$$S(x_1^{(0)}, x_2^{(0)}, \ldots, x_n^{(0)}) = 0 \tag{2}$$

expresses the fact that the elements $x_1^{(0)}, \ldots, x_n^{(0)}$ of A satisfy the conditions (2); and

$$S(x_1^{(0)}, x_2^{(0)}, \ldots, x_n^{(0)}) \neq 0$$

denotes the logical opposite to (2). S is called *k-regular in A* if, however A is split into k subsets

$$A = A_1+A_2+\ldots+A_k,$$

there is always at least one subset A_κ which contains a solution of (2). *Regularity* of S, as mentioned in the introduction, means that S is k-regular for every $k = 1, 2, \ldots$. *ω-regularity* of S means solubility in at least one class whenever A is split into denumerably many classes. Finally, we call S *absolutely regular in* A if, for some $x^{(0)}$ of A,

$$S(x^{(0)}, x^{(0)}, \ldots, x^{(0)}) = 0.$$

If A is the set of all real numbers except zero the condition

$$(x_1-x_2-1)(x_1-x_2-2)\ldots(x_1-x_2-k) = 0$$

is k-regular, but not $(k+1)$-regular, the condition

$$x_1+x_2-x_3 = 0$$

is regular but not ω-regular†, and the condition

$$x_1 \neq x_2$$

is ω-regular but not absolutely regular in A. The *degree of regularity* of S in A is the largest natural number k (if there is one) such that S is k-regular in A.

Let $S'(x_1, x_2, \ldots, x_{n'})$ be a system of conditions with $n' \leqslant n$ which has the property that (2) implies

$$S'(x_1^{(0)}, x_2^{(0)}, \ldots, x_{n'}^{(0)}) = 0.$$

† See below, Theorems VII and XI.

Let k, k' be natural numbers, $k' \leqslant k$, and let A be a subset of A'. Then k-regularity of S in A implies k'-regularity of S' in A'. Theorem I states that in certain circumstances a kind of inverse of this implication holds. In fact, many of our results are of this type.

THEOREM I. *If $S(x_1, x_2, \ldots, x_n)$ is k-regular in a denumerable set A then S is also k-regular in a suitable finite subset A' of A.*

Theorem I has been proved elsewhere†, but, for convenience, I reproduce the proof.

Proof. We may suppose that

$$A = \{1, 2, 3, \ldots\}.$$

Put
$$A_N = \{1, 2, \ldots, N\} \qquad (N = 1, 2, 3, \ldots).$$

Let us assume that, for no value of N, $S(x_1, \ldots, x_n)$ is k-regular in A_N. We have to show that S is not k-regular in A. There is a distribution Δ_N of A_N,

$$A_N = A_{N1} + A_{N2} + \ldots + A_{Nk},$$

such that no set $A_{N\kappa}$ contains a solution of $S = 0$. Here $N = 1, 2, \ldots$; $1 \leqslant \kappa \leqslant k$. For any positive integer x we denote by $f_N(x)$ that number κ $(1 \leqslant \kappa \leqslant k)$ for which

$$x \prec A_{N\kappa}.$$

Then by a well-known argument (Cantor's "Diagonalverfahren") we can define a function $f^*(x)$ such that, given any $x_0 > 0$, there are infinitely many N's for which

$$f_N(x) = f^*(x) \quad (1 \leqslant x \leqslant x_0). \tag{3}$$

Let, for $1 \leqslant \kappa \leqslant k$, A_κ^* be the set of all x for which

$$f^*(x) = \kappa,$$

and let Δ^* be the distribution

$$A = A_1^* + A_2^* + \ldots + A_k^*.$$

Now consider n numbers x_ν' satisfying

$$x_1' \equiv x_2' \equiv \ldots \equiv x_n' \quad (\text{mod } \Delta^*) \tag{4}$$

† S., Satz III. See also Dénes König, *Theorie der endlichen und unendlichen Graphen* (Leipzig, 1936), 84 (α) and (β).

Put $$x_0 = \max(x_1', x_2', \ldots, x_n').$$

Then, by the definition of $f^*(x)$, there is a number $N > x_0$ for which (3) holds. In particular, using (4), we have

$$f_N(x_\nu') = f^*(x_\nu') = \kappa_0 \quad (1 \leqslant \nu \leqslant n),$$
$$x_\nu' \prec A_{N\kappa_0} \qquad (1 \leqslant \nu \leqslant n).$$

Hence, by the definition of Δ_N,

$$S(x_1', x_2', \ldots, x_n') \neq 0.$$

This completes the proof of Theorem I.

As a corollary of Theorem I we prove†

LEMMA 1. *Let B be a denumerable set of numbers. Let R be the ring and K be the field generated by B. Let $R-\{0\}$ and $K-\{0\}$ denote the sets of all non-zero numbers of R and K respectively. Then k-regularity of the system*

$$\sum_{\nu=1}^{n} a_{\mu\nu} x_\nu = 0 \quad (1 \leqslant \mu \leqslant m) \tag{5}$$

in $K-\{0\}$ implies k-regularity of the same system in $R-\{0\}$.

This lemma is another kind of inverse of the simple proposition on p. 128, immediately preceding Theorem I.

Proof of Lemma 1. Suppose that (5) is k-regular in $K-\{0\}$. Then, by Theorem I, since K is denumerable, (5) is k-regular in a finite subset T of $K-\{0\}$. Let N be the product of the "denominators" of all numbers of T, or, more accurately, let N be a number of $R-\{0\}$ for which

$$Nt \prec R-\{0\} \tag{6}$$

whenever $t \prec T$. Take any $\Delta(x)$ defined in $R-\{0\}$ and satisfying $|\Delta| \leqslant k$. Put

$$\Delta'(x) = \Delta(Nx) \quad (x \prec T).$$

In view of (6) this definition is significant. From $|\Delta'| \leqslant |\Delta| \leqslant k$, and from the definition of T, we deduce the existence of numbers t_ν of T for which

$$\sum_{\nu=1}^{n} a_{\mu\nu} t_\nu = 0 \qquad (1 \leqslant \mu \leqslant m),$$
$$t_1 \equiv t_2 \equiv \ldots \equiv t_n \qquad (\text{mod } \Delta').$$

† S, 441.

Then
$$\sum_{\nu} a_{\mu\nu}(Nt_\nu) = 0 \qquad (1 \leqslant \mu \leqslant m),$$
$$Nt_1 \equiv Nt_2 \equiv \ldots \equiv Nt_n \pmod{\Delta}.$$

Hence the result. Instead of (5) we might have taken any system of homogeneous conditions.

3. In this section small Latin letters denote non-negative integers. Capitals A, B, ..., X denote "vectors" $(x_1, x_2, \ldots, x_m)$ of a fixed dimension m. If
$$A = (x_1, x_2, \ldots, x_m), \quad A' = (x_1', x_2', \ldots, x_m'),$$
we put
$$|A| = \max(x_1, x_2, \ldots, x_m), \quad aA = (ax_1, \ldots, ax_m),$$
$$A + A' = (x_1 + x_1', x_2 + x_2', \ldots, x_m + x_m').$$
0 denotes the number zero as well as the vector $(0, \ldots, 0)$.

THEOREM II. *There is a function $f(k, R_1, R_2, \ldots, R_l)$ which has the following property. Suppose that $R_1, R_2, \ldots, R_l$ are vectors and that Δ is a distribution of all vectors into k classes. Then we can find A^*, d^*, $(d^* > 0)$ such that*

$$A^* + d^* R_\lambda \equiv A^* \pmod{\Delta} \quad (1 \leqslant \lambda \leqslant l), \tag{7}$$

$$|A^* + d^* R_\lambda| \leqslant f \quad (1 \leqslant \lambda \leqslant l). \tag{8}$$

Proof. The theorem is true for $l = 1$. For there are two of the $k+1$ vectors
$$0, R_1, 2R_1, 3R_1, \ldots, kR_1$$
which belong to the same class of Δ, say αR_1 and βR_1, where
$$0 \leqslant \alpha < \beta \leqslant k.$$
Put
$$A^* = \alpha R_1, \quad d^* = \beta - \alpha.$$
Then
$$A^* + d^* R_1 = \beta R_1 \equiv \alpha R_1 = A^* \pmod{\Delta}.$$
Since
$$|A^* + d^* R_1| = \beta |R_1| \leqslant k |R_1|,$$
we may put
$$f(k, R_1) = k |R_1|.$$
Therefore we may suppose that, for some given vectors
$$R_1, R_2, \ldots, R_l$$

($l > 1$), the existence of $f(k', R_1, R_2, \ldots, R_{l-1})$ has been established for every $k' > 0$, and we deduce the existence of

$$f(k, R_1, \ldots, R_{l-1}, R_l).$$

In the proof which follows, k, $R_1, \ldots, R_l$ are constant, and these numbers are not shown in the arguments of functions. Δ is a distribution of all vectors, $|\Delta| \leqslant k$. We may assume, without loss of generality, that $R_1 \neq 0$.

LEMMA 2. *Under the assumptions stated there is a function $g(n)$ such that, given any A, n, we can find A', d' ($d' > 0$) such that*

$$A+(A'+d' R_\lambda)+B \equiv A+A'+B \pmod{\Delta} \quad (1 \leqslant \lambda < l,\ |B| \leqslant n), \tag{9}$$

$$|A'+d' R_\lambda| \leqslant g(n) \quad (1 \leqslant \lambda < l). \tag{10}$$

Proof. Put

$$\Delta'(X) = \prod_{|B| \leqslant n} \Delta(A+X+B).$$

Then

$$|\Delta'| \leqslant g_1(n).$$

Using the definition of

$$f\big(g_1(n), R_1, R_2, \ldots, R_{l-1}\big) = g_2(n),$$

we find that there are an A' and d' ($d' > 0$) such that

$$A'+d' R_\lambda \equiv A' \pmod{\Delta'} \quad (1 \leqslant \lambda < l), \tag{11}$$

$$|A'+d' R_\lambda| \leqslant g_2(n) \quad (1 \leqslant \lambda < l). \tag{12}$$

(11) and (12) are equivalent to (9) and (10) respectively if we put

$$g(n) = g_2(n).$$

Thus the lemma is proved. We note that

$$|A'+d' R_l| \leqslant |A'+d' R_1| + |A'+d' R_1|\,|R_l| \leqslant g(n)+g(n)\,|R_l| = h(n).$$

Now, to prove Theorem II, put

$$n_k = 1, \quad n_\kappa = \sum_{\nu=\kappa+1}^{k} h(n_\nu) \quad (0 \leqslant \kappa < k).$$

Apply Lemma 2, with $A = 0$, $n = n_0$. We find

$$A' = A_0, \quad d' = d_0 > 0$$

such that

$$0+(A_0+d_0 R_\lambda)+B \equiv 0+A_0+B \pmod{\Delta} \quad (1 \leqslant \lambda < l;\ |B| \leqslant n_0),$$

$$|A_0+d_0 R_\lambda| \leqslant h(n_0) \quad (1 \leqslant \lambda \leqslant l).$$

A second application, with $A = A_0$, $n = n_1$, yields $A' = A_1$, $d' = d_1 > 0$, such that

$$A_0+(A_1+d_1 R_\lambda)+B \equiv A_0+A_1+B \pmod{\Delta} \quad (1 \leqslant \lambda < l;\ |B| \leqslant n_1),$$

$$|A_1+d_1 R_\lambda| \leqslant h(n_1) \qquad (1 \leqslant \lambda \leqslant l).$$

Proceeding in this way (the next case is $A = A_0+A_1$, $n = n_2$) we find $A' = A_\kappa$, $d' = d_\kappa > 0$ such that

$$\sum_{\nu=0}^{\kappa-1} A_\nu+(A_\kappa+d_\kappa R_\lambda)+B \equiv \sum_{\nu=0}^{\kappa-1} A_\nu+A_\kappa+B \pmod{\Delta}\dagger \tag{13}$$

$$(0 \leqslant \kappa \leqslant k;\ 1 \leqslant \lambda < l;\ |B| \leqslant n_\kappa),$$

$$|A_\kappa+d_\kappa R_\lambda| \leqslant h(n_\kappa) \quad (0 \leqslant \kappa \leqslant k;\ 1 \leqslant \lambda \leqslant l). \tag{14}$$

There are two among the $k+1$ vectors

$$V_\kappa = \sum_{\nu=0}^{\kappa} A_\nu + \sum_{\nu=\kappa+1}^{k} (A_\nu+d_\nu R_l) \quad (0 \leqslant \kappa \leqslant k)$$

which belong to the same class of Δ. Therefore, say,

$$V_\alpha \equiv V_\beta \pmod{\Delta},$$

where $0 \leqslant \alpha < \beta \leqslant k$. Put

$$A^* = V_\beta, \quad d^* = \sum_{\nu=\alpha+1}^{\beta} d_\nu.$$

Then

$$\begin{aligned} A^*+d^* R_l &= \sum_0^\beta A_\nu + \sum_{\beta+1}^{k} (A_\nu+d_\nu R_l) + \sum_{\alpha+1}^{\beta} d_\nu R_l \\ &= \sum_0^\alpha A_\nu + \sum_{\alpha+1}^{k} (A_\nu+d_\nu R_l) = V_\alpha \equiv V_\beta = A^* \pmod{\Delta}. \end{aligned} \tag{15}$$

On the other hand, if $1 \leqslant \lambda < l$, then

$$\begin{aligned} A^*+d^* R_\lambda = \sum_0^\alpha A_\nu &+ (A_{\alpha+1}+d_{\alpha+1} R_\lambda) \\ &+ \left(\sum_{\alpha+2}^{\beta} (A_\nu+d_\nu R_\lambda) + \sum_{\beta+1}^{k} (A_\nu+d_\nu R_l)\right). \end{aligned} \tag{16}$$

† Empty sums have the value zero.

Now we have, by (14),

$$\left| \sum_{\alpha+2}^{\beta} (A_\nu + d_\nu R_\lambda) + \sum_{\beta+1}^{k} (A_\nu + d_\nu R_l) \right| \leqslant \sum_{\alpha+2}^{k} h(n_\nu) = n_{\alpha+1}.$$

Therefore, from (13),

$$(17) \qquad A^* + d^* R_\lambda \equiv \sum_0^\alpha A_\nu + A_{\alpha+1} + \left(\sum_{\alpha+2}^{\beta} (A_\nu + d_\nu R_\lambda) + \sum_{\beta+1}^{k} (A_\nu + d_\nu R_l) \right) \pmod{\Delta}.$$

The vector on the right-hand side of (17) is the same as the vector on the right-hand side of (16), except that α is to be replaced by $\alpha+1$. When $\alpha+1 < \beta$ a second application of (13) leads to

$$A^* + d^* R_\lambda \equiv \sum_0^{\alpha+2} A_\nu + \sum_{\alpha+3}^{\beta} (A_\nu + d_\nu R_\lambda) + \sum_{\beta+1}^{k} (A_\nu + d_\nu R_l) \pmod{\Delta},$$

and so on. After $\beta - \alpha$ steps we find that

$$A^* + d^* R_\lambda \equiv \sum_0^\beta A_\nu + \sum_{\beta+1}^{h} (A + d_\nu R_l) = V_\beta = A^* \pmod{\Delta} \quad (1 \leqslant \lambda < l).$$

This result, together with (15), shows that (7) is true.

Furthermore, by (14), we have, for $1 \leqslant \lambda \leqslant l$,

$$|A^* + d^* R_\lambda| = \left| \sum_0^\alpha A_\nu + \sum_{\alpha+1}^{\beta} (A_\nu + d_\nu R_\lambda) + \sum_{\beta+1}^{k} (A_\nu + d_\nu R_l) \right| \leqslant \sum_0^k h(n_\nu).$$

Therefore (8) holds, with

$$f(k, R_1, R_2, \ldots, R_l) = \sum_0^k h(n_\nu),$$

and the theorem is proved.

4. THEOREM III. *There is a function $\bar{f}(k, l)$, defined for all pairs (k, l) of positive integers, which has the following property. Suppose that M is a set of objects among which a commutative and associative addition is defined. Let $R_1, R_2, \ldots, R_l$ be elements of M, and let Δ be a distribution of M into k classes. Then there is a positive integer d^* and an element A^* of M such that*

$$(18) \qquad A^* + d^* R_\lambda \equiv A^* \pmod{\Delta} \quad (1 \leqslant \lambda \leqslant l)\dagger,$$

$$(19) \qquad A^* = \sum_{\lambda=1}^{l} a_\lambda^* R_\lambda,$$

$$(20) \qquad a_\lambda^* + d^* \leqslant \bar{f}(k, l) \quad (1 \leqslant \lambda \leqslant l).$$

† $d^* R_\lambda = R_\lambda + R_\lambda + \ldots + R_\lambda$ (d^* terms).

Proof. Put, for all integers $x_\lambda \geqslant 0$,

(21) $\Delta'(x_1, x_2, \ldots, x_l) = \Delta\big((x_1+1) R_1 + (x_2+1) R_2 + \ldots + (x_l+1) R_l\big).$

Apply Theorem II to Δ' and the "unit vectors"

$$\bar{R}_1 = (1, 0, \ldots, 0),$$
$$\bar{R}_2 = (0, 1, \ldots, 0),$$
$$\ldots \quad \ldots \quad \ldots \quad \ldots$$
$$\bar{R}_l = (0, 0, \ldots, 1).$$

We obtain a vector $(a_1, a_2, \ldots, a_l)$, with integral components $a_\nu \geqslant 0$, and a positive integer d such that

(22) $$(a_1, \ldots, a_l) + d\bar{R}_\lambda \equiv (a_1, \ldots, a_l) \pmod{\Delta'} \quad (1 \leqslant \lambda \leqslant l),$$

(23) $$|(a_1, \ldots, a_l) + d\bar{R}_\lambda| \leqslant f(k, \bar{R}_1, \ldots, \bar{R}_l) = \bar{g}(k, l) \quad (1 \leqslant \lambda \leqslant l),$$

say. (22), in view of (21), is the same as

$$\sum_{\nu=1}^{l} (a_\nu+1) R_\nu + d R_\lambda \equiv \sum_{\nu=1}^{l} (a_\nu+1) R_\nu \pmod{\Delta} \quad (1 \leqslant \lambda \leqslant l).$$

(23) is the same as

$$a_\lambda + d \leqslant \bar{g}(k, l) \quad (1 \leqslant \lambda \leqslant l).$$

Hence we may put

$$A^* = \sum_{\nu=1}^{l} (a_\nu+1) R_\nu, \quad d^* = d; \quad a_\nu^* = a_\nu + 1 \ (1 \leqslant \nu \leqslant l),$$
$$\bar{f}(k, l) = \bar{g}(k, l) + 1,$$

and the theorem is proved. In the special case where M is the system of vectors

$$A = (x_1, x_2, \ldots, x_m)$$

whose components x_n are non-negative integers, we deduce from (19) and (20) that, in the notation of Theorem II,

$$|A^* + d^* R_\lambda| = \left| \sum_{\nu=1}^{l} a_\nu^* R_\nu + d^* R_\lambda \right| \leqslant \sum_\nu (a_\nu^* + d^*) |R_\nu|$$
$$\leqslant \bar{f}(k, l) \sum_\nu |R_\nu| \quad (1 \leqslant \lambda \leqslant l).$$

Therefore the function $f(k, R_1, \ldots, R_l)$ of Theorem II may be assumed to be of the form

$$f(k, R_1, \ldots, R_l) = \bar{f}(k, l) \sum_{\nu=1}^{l} |R_\nu|.$$

For a later application we note that in Theorem III we may stipulate that the elements

$$A^*,\ A^*+d^*R_\lambda \quad (1 \leqslant \lambda \leqslant l)$$

are either different from any one out of a finite number of given elements $C_1, C_2, \ldots, C_s$, or are all equal to one of the elements C_σ. In this case (20) must be replaced by

$$a_\lambda^* + d^* \leqslant \bar{f}(k+s,\ l). \tag{24}$$

For suppose that Δ is a distribution of M, $|\Delta| \leqslant k$. Define Δ' by means of

$$A \equiv B \pmod{\Delta'}$$

if, and only if, either

$$A \neq C_\sigma; \quad B \neq C_\sigma\ (1 \leqslant \sigma \leqslant s); \quad A \equiv B \pmod{\Delta},$$

or $A = B = C_{\sigma_0}$ for some σ_0, $1 \leqslant \sigma_0 \leqslant s$. In other words, we remove the C_σ's from their classes in Δ and put them in separate classes. Now apply Theorem III to Δ'. We find A^*, d^*, with $d^* > 0$, such that

$$A^* + d^* R_\lambda \equiv A^* \pmod{\Delta'} \quad (1 \leqslant \lambda \leqslant l). \tag{25}$$

Furthermore, (19) and (24) hold; (25) implies (18), and, in view of the definition of Δ', the additional condition is satisfied.

2. *Regularity of homogeneous linear equations.*

1. THEOREM IV. *Let $a_1, a_2, \ldots, a_n$ be real numbers. Suppose that no non-empty subset of these numbers has the sum zero. Then the equation*

$$a_1 x_1 + a_2 x_2 + \ldots + a_n x_n = 0 \tag{26}$$

is not regular in the set of all positive numbers. In particular, if

$$|a_{\nu_1} + a_{\nu_2} + \ldots + a_{\nu_k}| \geqslant (|a_1| + |a_2| + \ldots + |a_n|)/a > 0 \tag{27}$$

whenever $\quad 1 \leqslant k \leqslant n, \quad 1 \leqslant \nu_1 < \nu_2 < \ldots < \nu_k \leqslant n,$

then the degree of regularity D of (26) *in the set of all positive numbers satisfies*

$$D < 1 + \frac{\log b}{\log(1 + a^{-1} - b^{-1})}, \tag{28}$$

where b is any arbitrary number exceeding a. In particular,

$$D < 1 + \tfrac{8}{3} a \log(2a) \qquad (\textit{for every } a), \tag{29}$$

$$D < a \log a + o(a \log a) \quad (\textit{as } a \to \infty). \tag{30}$$

If D' is the degree of regularity of (26) *in the set of all real numbers $x \neq 0$, then*

$$D' \leqslant 2D+1. \tag{31}$$

Proof. (27) implies $a \geqslant 1$. If no non-empty subset of $a_1, a_2, \ldots, a_n$ has the sum zero, then (27) holds for some $a \geqslant 1$. Therefore the first assertion of the theorem follows from (28). (29) follows easily from (28) by putting $b = 2a$ and making use of the inequality

$$\log(1+t) = t(1-\tfrac{1}{2}t)+\tfrac{1}{3}t^3(1-\tfrac{3}{4}t)+\ldots > t(1-\tfrac{1}{4}), \tag{32}$$

valid for $0 < t \leqslant \frac{1}{2}$.

In order to deduce (30) from (28) we put

$$b = a \log a,$$

for $a > e$. Then (28) becomes

$$D < 1+\frac{\log(a \log a)}{\log\{1+a^{-1}-(a \log a)^{-1}\}} = a \log a + o(a \log a)$$

as $a \to \infty$. If $D < \infty$, then there is a distribution Δ of all positive numbers such that $|\Delta| = D+1$ and

$$a_1 x_1 + \ldots + a_n x_n \neq 0$$

whenever $$x_1 \equiv x_2 \equiv \ldots \equiv x_n \pmod{\Delta}.$$

Put $$\Delta'(x) = \Delta(|x|)\,\Delta^{(3)}\left(\frac{x}{|x|}\right) \quad (x \text{ real}, \ \neq 0).$$

In other words, corresponding to every class of Δ we form a new class containing the same numbers but multiplied by -1. Then no class of Δ' contains a solution of (26), and (31) follows.

All that remains to be proved is (28), under the assumption (27). Suppose that (26) is m-regular in the set of all positive numbers, for some positive integer m. Choose a number $q > 1$ and put

$$\Delta(x) = \Delta^{(m)}\left(\left[\frac{\log x}{\log q}\right]\right) \quad (x > 0).$$

Then $|\Delta| = m$. Hence, in consequence of our assumption about m, there are numbers x_ν satisfying

$$x_1 \equiv x_2 \equiv \ldots \equiv x_n \pmod{\Delta},$$

$$a_1 x_1 + \ldots + a_n x_n = 0,$$

Put
$$\left[\frac{\log x_\nu}{\log q}\right] = m_\nu \quad (1 \leqslant \nu \leqslant n).$$

Then

(33)
$$m_1 \equiv m_2 \equiv \ldots \equiv m_n \pmod{m},$$

(34)
$$q^{m_\nu} \leqslant x_\nu < q^{m_\nu+1} \quad (1 \leqslant \nu \leqslant n).$$

Arrange the m_ν in non-increasing order:

$$m_{\nu_1} \geqslant m_{\nu_2} \geqslant \ldots \geqslant m_{\nu_n}.$$

We have, for some suitable k $(1 \leqslant k \leqslant n)$†,

$$m_{\nu_1} = m_{\nu_2} = \ldots = m_{\nu_k} > m_{\nu_{k+1}}.$$

Then, by (33),
$$m_{\nu_k} \geqslant m_{\nu_{k+1}} + m.$$

Therefore, from (34),

$$q^{m_{\nu_1}} \leqslant x_{\nu_\kappa} < q^{m_{\nu_1}+1} \qquad (1 \leqslant \kappa \leqslant k),$$

$$x_{\nu_\lambda} < q^{m_{\nu_\lambda}+1} \leqslant q^{m_{\nu_1}-m+1} \quad (k < \lambda \leqslant n).$$

Now, making use of (27), we deduce that

$$0 = \left| \sum_{\mu=1}^{n} a_\mu x_\mu \right| = \left| \sum_{\kappa=1}^{k} a_{\nu_\kappa} x_{\nu_1} + \sum_{\kappa=1}^{k} a_{\nu_\kappa}(x_{\nu_\kappa} - x_{\nu_1}) + \sum_{\lambda=k+1}^{n} a_{\nu_\lambda} x_{\nu_\lambda} \right|$$

$$\geqslant x_{\nu_1} \left| \sum_{\kappa=1}^{k} a_{\nu_\kappa} \right| - \sum_{\kappa=1}^{k} |a_{\nu_\kappa}| \, |x_{\nu_\kappa} - x_{\nu_1}| - \sum_{\lambda=k+1}^{n} |a_{\nu_\lambda}| x_{\nu_\lambda}$$

$$> q^{m_{\nu_1}} a^{-1} \sum_1^n |a_\mu| - (q^{m_{\nu_1}+1} - q^{m_{\nu_1}}) \sum_1^n |a_\mu| - q^{m_{\nu_1}-m+1} \sum_1^n |a_\mu|$$

$$= q^{m_{\nu_1}} \sum_1^n |a_\mu| \, (a^{-1} - \{q-1\} - q^{-(m-1)}).$$

Therefore
$$a^{-1} - q + 1 - q^{-(m-1)} < 0,$$

i.e.,
$$q^{-(m-1)} > a^{-1} - q + 1,$$

$$m - 1 < -\frac{\log(a^{-1} - q + 1)}{\log q},$$

provided that
$$a^{-1} - q + 1 > 0.$$

† In the case $k = n$ everything relating to $k+1$, $k+2$, ... has to be omitted, and similarly later.

Put
$$q = 1 + a^{-1} - b^{-1},$$
where b is any number exceeding a. Then it follows that
$$m - 1 < \frac{\log b}{\log(1 + a^{-1} - b^{-1})}.$$

Hence (26) is not regular in the set of all positive numbers, and (28) holds. This completes the proof of Theorem IV.

THEOREM V. *Let $a_1, a_2, \ldots, a_n$ be complex numbers. Suppose that no non-empty subset of these numbers has the sum zero. Then the equation*

(35) $$a_1 x_1 + \ldots + a_n x_n = 0$$

is not regular in the set of all complex numbers different from zero. In particular, if

(36) $$|a_{\nu_1} + a_{\nu_2} + \ldots + a_{\nu_k}| \geqslant (|a_1| + |a_2| + \ldots + |a_n|)/a > 0$$

whenever $1 \leqslant k \leqslant n;\quad 1 \leqslant \nu_1 < \nu_2 < \ldots < \nu_k \leqslant n,$

then the degree of regularity D of (35) *in the set of all complex numbers different from zero satisfies*

(37) $$D < 1 + \frac{1}{\delta} - \frac{\log(1 + a^{-1} - q^{1+\delta} - 2\pi\delta)}{\delta \log q},$$

where δ, q are any real numbers such that

(38) $$0 < 2\pi\delta < a^{-1},$$

(39) $$1 < q^{1+\delta} < 1 + a^{-1} - 2\pi\delta\dagger.$$

In particular

(40) $$D < 1 + 6\pi a + 28\pi a^2 \log(3a) \quad (\textit{for every } a),$$

(41) $$D < 8\pi a^2 \log a + o(a^2 \log a) \quad (\textit{as } a \to \infty).$$

Proof. (36) implies $a \geqslant 1$. The first part of the theorem follows as in the case of Theorem IV. (40) follows from (37) by putting
$$2\pi\delta = \frac{1}{3a}, \quad q^{1+\delta} = 1 + \frac{1}{3a},$$
and using (32). (41) follows from (37) if we put, for every sufficiently large a,
$$2\pi\delta = \frac{1}{2a}, \quad q = 1 + \frac{1}{2a} - \frac{1}{a \log a}.$$

† In other words, δ, q satisfy $0 < \delta$, $1 < q$, and have such values as to give a real value to the right-hand side of (37).

Then, as $a \to \infty$,

$$1+\frac{1}{a}-q^{1+\delta}-2\pi\delta = 1+\frac{1}{2a}-\left(1+\frac{1}{2a}-\frac{1}{a\log a}\right)^{1+(4\pi a)^{-1}}$$

$$= 1+\frac{1}{2a}-\exp\left[\left(1+\frac{1}{4\pi a}\right)\left\{\frac{1}{2a}-\frac{1}{a\log a}+O\left(\frac{1}{a^2}\right)\right\}\right]$$

$$= 1+\frac{1}{2a}-\exp\left\{\frac{1}{2a}-\frac{1}{a\log a}+O\left(\frac{1}{a^2}\right)\right\}$$

$$= 1+\frac{1}{2a}-\left(1+\frac{1}{2a}-\frac{1}{a\log a}+O\left(\frac{1}{a^2}\right)\right)=\frac{1}{a\log a}+O\left(\frac{1}{a^2}\right).$$

Hence, by (37),

$$D < 1+4\pi a-4\pi a\,\frac{\log\{(a\log a)^{-1}+O(a^{-2})\}}{\log\{1+\frac{1}{2}a^{-1}+O(a\log a)^{-1}\}}$$

$$= 1+4\pi a+4\pi a\,\frac{\log a+o(\log a)}{\frac{1}{2}a^{-1}+o(a^{-1})} = 8\pi a^2\log a+o(a^2\log a).$$

In order to prove the theorem we have to show that (37) holds, provided that (36) is true for all k, $\nu_1, \ldots, \nu_k$. We assume that (35) is m-regular in the set of all complex numbers different from zero. Choose real numbers q, δ such that $q > 1$, $\delta > 0$. Then every complex $x \neq 0$ has a unique representation

$$x = q^r e^{2\pi i t} \tag{42}$$

where $r = r(x)$, $t = t(x)$ are real numbers satisfying

$$0 \leqslant r-t < 1. \tag{43}$$

For (42) determines r uniquely and t uniquely mod 1, and (43) fixes t. Put

$$\Delta(x) = \Delta^{(m}\left(\left[\frac{t}{\delta}\right]\right) \quad (x \neq 0).$$

Then, according to the choice of m, there are numbers $x_\nu \neq 0$ for which

$$x_1 \equiv x_2 \equiv \ldots \equiv x_n \pmod{\Delta}, \quad a_1 x_1+\ldots+a_n x_n = 0.$$

Let $$r(x_\nu) = r_\nu; \quad t(x_\nu) = t_\nu; \quad \left[\frac{t_\nu}{\delta}\right] = m_\nu.$$

Then $$x_\nu = q^{r_\nu} e^{2\pi i t_\nu},$$

$$t_\nu \leqslant r_\nu < t_\nu+1, \quad m_1 \equiv m_2 \equiv \ldots \equiv m_n \pmod{m}, \tag{44}$$

$$m_\nu\delta \leqslant t_\nu < m_\nu\delta+\delta \quad (1 \leqslant \nu \leqslant n). \tag{45}$$

Also
$$m_{\nu_1} = m_{\nu_2} = \ldots = m_{\nu_k} > m_{\nu_{k+1}} \geqslant \ldots \geqslant m_{\nu_n},$$
where

(46) $$\nu_1, \nu_2, \ldots, \nu_n$$

is a permutation of 1, 2, ..., n and k an integer, $1 \leqslant k \leqslant n$. Moreover, using (45) and (44), we can choose the permutation (46) in such a way that
$$m_{\nu_1}\delta \leqslant t_{\nu_1} \leqslant r_{\nu_1} \leqslant r_{\nu_2} \leqslant \ldots \leqslant r_{\nu_k} < t_{\nu_k}+1 < m_{\nu_k}\delta+\delta+1 = m_{\nu_1}\delta+\delta+1.$$
Then we have, for every κ $(1 \leqslant \kappa \leqslant k)$,

(47) $$\begin{aligned} |x_{\nu_\kappa} - x_{\nu_1}| &= |(q^{r_{\nu_\kappa}} e^{2\pi i t_{\nu_\kappa}} - q^{r_{\nu_1}} e^{2\pi i t_{\nu_\kappa}}) + (q^{r_{\nu_1}} e^{2\pi i t_{\nu_\kappa}} - q^{r_{\nu_1}} e^{2\pi i t_{\nu_1}})| \\ &\leqslant (q^{r_{\nu_\kappa}} - q^{r_{\nu_1}}) + q^{r_{\nu_1}} \times 2\pi |t_{\nu_\kappa} - t_{\nu_1}| \\ &< q^{r_{\nu_1}}(q^{1+\delta}-1) + 2\pi\delta q^{r_{\nu_1}} \quad (1 \leqslant \kappa \leqslant k). \end{aligned}$$

If $k < \lambda \leqslant n$, then
$$m_{\nu_\lambda} \leqslant m_{\nu_{k+1}} \leqslant m_{\nu_k} - m = m_{\nu_1} - m,$$
and therefore

(48) $$\begin{aligned} |x_{\nu_\lambda}| &= q^{r_{\nu_\lambda}} < q^{t_{\nu_\lambda}+1} < q^{m_{\nu_\lambda}\delta+\delta+1} \leqslant q^{(m_{\nu_1}-m)\delta+\delta+1} \\ &\leqslant q^{t_{\nu_1}-m\delta+\delta+1} \leqslant q^{r_{\nu_1}-(m-1)\delta+1} \quad (k < \lambda \leqslant n). \end{aligned}$$

Just as in the proof of Theorem IV, we deduce from (36), (47), (48), (38) and (39) that
$$\begin{aligned} 0 &= \left| \sum_{\mu=1}^{n} a_\mu x_\mu \right| = \left| \sum_{\kappa=1}^{k} a_{\nu_\kappa} x_{\nu_1} + \sum_{\kappa=1}^{k} a_{\nu_\kappa}(x_{\nu_\kappa} - x_{\nu_1}) + \sum_{\lambda=k+1}^{n} a_{\nu_\lambda} x_{\nu_\lambda} \right| \\ &\geqslant |x_{\nu_1}| \left| \sum_{\kappa=1}^{k} a_{\nu_\kappa} \right| - \sum_{\kappa=1}^{k} |a_{\nu_\kappa}| |x_{\nu_\kappa} - x_{\nu_1}| - \sum_{\lambda=k+1}^{n} |a_{\nu_\lambda}| |x_{\nu_\lambda}| \\ &> q^{r_{\nu_1}} a^{-1} \sum_{\mu=1}^{n} |a_\mu| - q^{r_{\nu_1}}(q^{1+\delta} - 1 + 2\pi\delta) \sum_{\mu=1}^{n} |a_\mu| - q^{r_{\nu_1}-(m-1)\delta+1} \sum_{\mu=1}^{n} |a_\mu| \\ &= q^{r_{\nu_1}} \sum_{1}^{n} |a_\mu| \, (a^{-1} - q^{1+\delta} + 1 - 2\pi\delta - q^{-(m-1)\delta+1}), \end{aligned}$$
$$q^{-(m-1)\delta+1} > a^{-1} - q^{1+\delta} + 1 - 2\pi\delta,$$
$$(m-1)\delta - 1 < -\frac{\log(a^{-1} - q^{1+\delta} + 1 - 2\pi\delta)}{\log q},$$
$$m < 1 + \frac{1}{\delta} - \frac{\log(1 + a^{-1} - q^{1+\delta} - 2\pi\delta)}{\delta \log q}.$$

If this holds for every m, then the admissible values of m are bounded, and the largest of them, *i.e.*, D, satisfies (37). Thus Theorem V is proved. It is obvious from the foregoing proofs that the numerical constants in (29) and (40) can be improved.

2. Consider a system of homogeneous linear equations

$$\sum_{\nu=1}^{n} a_{\mu\nu} x_\nu = 0 \quad (1 \leqslant \mu \leqslant m) \tag{49}$$

with arbitrary complex coefficients $a_{\mu\nu}$. Let B be a set of numbers. We say that (49) satisfies the *condition* $\Gamma(B)$ if it is possible to divide the set

$$\{1, 2, \ldots, n\}$$

into non-empty, non-overlapping subsets $S_1, S_2, \ldots, S_l$ such that, corresponding to every λ $(1 \leqslant \lambda \leqslant l)$, there exists a solution $x_\nu = \xi_\nu^{(\lambda)}$ of (49) for which

$$\xi_\nu^{(\lambda)} \prec B \quad (\nu \prec S_1, S_2, \ldots, S_\lambda),$$

$$\xi_\nu^{(\lambda)} = 0 \quad (\nu \prec S_{\lambda+1}, S_{\lambda+2}, \ldots, S_l),$$

and, moreover, all numbers $\xi_\nu^{(\lambda)}$ for ν in S_λ have the same value $\xi^{(\lambda)} \neq 0$. In other words, if we number the variables suitably we can find a matrix of the type (in the case $l = 4$)

$$\begin{pmatrix}
\xi^{(1)}, & \xi^{(1)}, & \ldots, & \xi^{(1)}, & 0, & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots, & 0 \\
\xi_1^{(2)}, & \xi_2^{(2)}, & \ldots, & \xi_\alpha^{(2)}, & \xi^{(2)}, & \ldots, & \xi^{(2)}, & 0, & \ldots & \ldots & \ldots & \ldots, & 0 \\
\xi_1^{(3)}, & \xi_2^{(3)}, & \ldots & \ldots & \ldots & \ldots & \xi_\beta^{(3)}, & \xi^{(3)}, & \ldots, & \xi^{(3)}, & 0, & \ldots, & 0 \\
\xi_1^{(4)}, & \xi_2^{(4)}, & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \xi_\gamma^{(4)}, & \xi^{(4)}, & \ldots, & \xi^{(4)}
\end{pmatrix}$$

in which every row constitutes a solution of (49), every $\xi_\nu^{(\lambda)}$ and $\xi^{(\lambda)}$ belongs to B, and $\xi^{(\lambda)} \neq 0$. We have

$$0 < \alpha < \beta < \gamma < n.$$

For instance, the system of equations

$$x_2 - x_1 = x_3 - x_2 = \ldots = x_{n-1} - x_{n-2} = x_n$$

satisfies $\Gamma(B)$, where B is the set of all integers. For we may put

$$l = 2; \quad S_1 = \{1, 2, \ldots, n-1\}; \quad S_2 = \{n\}.$$

The $\xi_\nu^{(\lambda)}$ are

$$\begin{pmatrix} 1, & 1, & 1, & \ldots, & 1, & 0 \\ 1, & 2, & 3, & \ldots, & n-1, & 1 \end{pmatrix}.$$

If (49) satisfies the condition $\Gamma(B)$ for some set B, then (49) satisfies $\Gamma(R_0)$, where R_0 is the ring generated by all coefficients $a_{\mu\nu}$†. This follows at once from well-known properties of systems of linear equations. Subsequently (pp. 149 *et seq.*), the structure of systems satisfying $\Gamma(B)$ will be further elucidated.

Let K be a number field, and denote by $K-\{0\}$ the set of all numbers of K except zero.

THEOREM VI. *If* (49) *is regular in* $K-\{0\}$ *then* (49) *satisfies* $\Gamma(K)$.

Before we prove this theorem we establish a simple lemma which is geometrically obvious.

LEMMA 3. *Let*

$$L_1(t),\ L_2(t),\ \ldots,\ L_r(t),\ M_1(t),\ \ldots,\ M_s(t)$$

be $r+s$ *linear forms in* $(t)=(t_1, t_2, \ldots, t_N)$. *Let* $r \geqslant 0$; $s>0$. *Suppose that*

$$L_\rho(t')=0 \quad (1 \leqslant \rho \leqslant r) \tag{50}$$

implies that, for at least one $\sigma_0=\sigma_0(t')$,

$$M_{\sigma_0}(t')=0.$$

Then at least one of the forms $M_\sigma(t)$ *is a linear combination of the forms* $L_\rho(t)$.

Proof of Lemma 3. We may assume that no proper subset of the system $M_1(t), \ldots, M_s(t)$ has the same property as the whole system of forms $M_\sigma(t)$. Then $s=1$. For if $s>1$, then, corresponding to every σ $(1 \leqslant \sigma \leqslant s)$, there is a vector (t^σ) for which

$$L_\rho(t^{(\sigma)})=0 \ \ (1 \leqslant \rho \leqslant n), \qquad M_\sigma(t^{(\sigma)})=1, \qquad M_{\sigma'}(t^{(\sigma)})=0 \ \ (\sigma' \neq \sigma).$$

Then the vector
$$(t')=\sum_{\sigma=1}^{s}(t^{(\sigma)})$$

satisfies (50). But, on the other hand,

$$M_\sigma(t')=M_\sigma(t^{(\sigma)})=1 \quad (1 \leqslant \sigma \leqslant s),$$

† We exclude the trivial case where all $a_{\mu\nu}$ vanish.

which contradicts our hypothesis. Therefore $s = 1$. In this case the lemma is an immediate consequence of well-known facts concerning linear forms.

For a later application we add the remark that (50) is required only for those $(t') = (t_1', t_2', \ldots, t_N')$ whose components t_ν' belong to the field generated by the coefficients of all forms L_ρ, M_σ.

Proof of Theorem VI. We suppose that (49) is regular in K. Then every linear combination of the equations (49) is regular in the set of all complex numbers different from zero. Therefore in every such combination certain coefficients have the sum zero (Theorem V, p. 138). If we exploit this fact for suitable linear combinations of (49), we obtain the condition $\Gamma(K)$. But before starting along these lines we replace (49) by a system of equations whose coefficients belong to K.

Let $$x_\nu = x_\nu^{(a)} \quad (1 \leqslant \nu \leqslant n;\ 1 \leqslant a \leqslant g)$$

be a system of a maximal number of linearly independent solutions of (49) which belong to K. Such solutions exist because (49) is regular in $K-\{0\}$. Now let

(51) $$\sum_{\nu=0}^{n} b_{\mu\nu} x_\nu = 0 \quad (1 \leqslant \mu \leqslant m')$$

be a system of linear equations whose general solution is an arbitrary linear combination of the vectors $(x_\nu^{(a)})$ $(1 \leqslant a \leqslant g)$. Then, as far as solutions in K are concerned, (49) and (51) have exactly the same solutions. In particular, (51) is regular in $K-\{0\}$ and therefore, *a fortiori*, regular in the set of all complex numbers different from zero. Since the $x_\nu^{(a)}$ belong to K, it is possible to choose (51) so that the $b_{\mu\nu}$ belong to K.

Choose m' parameters t_μ and consider the equation

$$\sum_{\mu=1}^{m'} t_\mu \sum_{\nu=1}^{n} b_{\mu\nu} x_\nu = 0,$$

i.e.,

(52) $$\sum_{\nu=1}^{n} R_\nu(t)\, x_\nu = 0,$$

where

(53) $$R_\nu(t) = \sum_{\mu=1}^{m'} b_{\mu\nu} t_\mu \quad (1 \leqslant \nu \leqslant n).$$

Let $$M_1(t),\ M_2(t),\ \ldots,\ M_s(t) \quad (s = 2^n - 1)$$

be all the linear forms which are sums of forms $R_\nu(t)$ corresponding to any choice of distinct ν. For any values of the t_μ, (52) is regular in the set of all complex numbers different from zero. Hence, by Theorem V, at least one of the numbers $M_\sigma(t)$ vanishes. Therefore, by Lemma 3 above (p. 142), with $r=0$; $s=2^n-1$, at least one of the forms $M_\sigma(t)$ vanishes identically in (t). If the x_ν are suitably numbered, we may assume that, identically in (t),

$$\sum_{\nu=1}^{a} R_\nu(t) = 0. \tag{54}$$

Here a is an integer and $1 \leqslant a \leqslant n$.

If $a=n$, we stop at this stage. If $a<n$, we let the parameters t_μ be subject to the conditions

$$R_\nu(t) = 0 \quad (1 \leqslant \nu \leqslant a). \tag{55}$$

Then (52) becomes

$$\sum_{\nu=a+1}^{n} R_\nu(t)\, x_\nu = 0.$$

Suppose that

$$M_1'(t),\ M_2'(t),\ \ldots,\ M'_{s'}(t) \quad (s' = 2^{n-a}-1)$$

are all the sums of any number of forms $R_\nu(t)$ corresponding to distinct values of ν with $a<\nu\leqslant n$. Again Theorem V shows that, for every choice of (t) satisfying (55), at least one of the numbers $M_\sigma'(t)$ vanishes, and this implies, by Lemma 3 ($r=a$; $s=s'$), that at least one of the forms $M_\sigma'(t)$ $(1\leqslant\sigma\leqslant s')$ is a linear combination of the $R_\nu(t)$ $(1\leqslant\nu\leqslant a)$. We can number the variables $x_{a+1}, x_{a+2}, \ldots, x_n$ so that this linear relation becomes

$$\sum_{\nu=1}^{a} f_\nu R_\nu(t) + \sum_{\nu=a+1}^{\beta} R_\nu(t) = 0, \tag{56}$$

identically in (t). Here β is some number satisfying $a<\beta\leqslant n$, and the f_ν are constants.

If we proceed in this way, treating β as we treated a, we find (in the case $\beta<n$) a relation

$$\sum_{\nu=1}^{\beta} g_\nu R_\nu(t) + \sum_{\nu=\beta+1}^{\gamma} R_\nu(t) = 0, \tag{57}$$

where $a<\beta<\gamma\leqslant n$, etc. The process stops, say, with

$$\sum_{\nu=1}^{\delta} j_\nu R_\nu(t) + \sum_{\nu=\delta+1}^{n} R_\nu(t) = 0. \tag{58}$$

We have $$1 \leqslant a < \beta < \gamma < \ldots < \delta < n.$$

Since the coefficients of the R_ν belong to K, it is possible to choose the relations (56), (57), ..., (58) so that their coefficients $f_\nu, g_\nu, \ldots, j_\nu$ belong to K.

In view of (53), the relations (54), (56), (57), ..., (58) are equivalent to

$$\begin{aligned}
&\sum_{\nu=1}^{\alpha} b_{\mu\nu}\times 1 = 0,\\
&\sum_{\nu=1}^{\alpha} b_{\mu\nu} f_\nu + \sum_{\nu=\alpha+1}^{\beta} b_{\mu\nu}\times 1 = 0,\\
&\sum_{\nu=1}^{\beta} b_{\mu\nu} g_\nu + \sum_{\nu=\beta+1}^{\gamma} b_{\mu\nu}\times 1 = 0,\\
&\dots \quad \dots \quad \dots \quad \dots \quad \dots\\
&\sum_{\nu=1}^{\delta} b_{\mu\nu} j_\nu + \sum_{\nu=\delta+1}^{n} b_{\mu\nu}\times 1 = 0,
\end{aligned}$$

where μ takes all values 1, 2, ..., m'. This means that the rows of the matrix

$$\begin{pmatrix}
1, & 1, & \dots, & 1, & 0, & \dots & \dots & \dots & \dots & \dots & \dots & 0\\
f_1, & f_2, & \dots, & f_\alpha, & 1, & \dots, & 1, & 0, & \dots & \dots & \dots & 0\\
g_1, & g_2, & \dots & \dots & \dots & g_\beta, & 1, & \dots, & 1, & 0, & \dots & 0\\
\dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots\\
j_1, & j_2, & \dots & \dots & \dots & \dots & \dots & \dots & j_\delta, & 1, & \dots, & 1
\end{pmatrix}$$

are solutions of (51). The elements of this matrix belong to K. Therefore, in view of the connection between (51) and (49), its rows are, at the same time, solutions of (49), and this amounts to saying that (49) satisfies $\Gamma(K)$. The subsets $S_1, S_2, \dots, S_l$ of p. 141 are

$$\{1, 2, \dots, \alpha\}, \quad \{\alpha+1, \alpha+2, \dots, \beta\}, \quad \{\beta+1, \dots, \gamma\}, \quad \dots, \quad \{\delta+1, \dots, n\}.$$

Thus Theorem VI is proved.

3. Let R be a ring of numbers, and let $R-\{0\}$ denote the set of all numbers of R except zero. The main result of this paragraph is the following theorem.

THEOREM VII. *A system of equations*

$$\sum_{\nu=1}^{n} a_{\mu\nu} x_\nu = 0 \quad (1 \leqslant \mu \leqslant m) \tag{59}$$

is regular in $R-\{0\}$ *if, and only if,* (59) *satisfies* $\Gamma(R)$†. *In particular,*

† The condition $\Gamma(R)$ was defined on p. 141.

the equation

$$a_1 x_1 + a_2 x_2 + \ldots + a_n x_n = 0,$$

where $a_\nu \prec R - \{0\}$, *is regular in* $R - \{0\}$ *if, and only if, some of the numbers* $a_1, a_2, \ldots, a_n$ *have the sum zero.*

Proof. Let K be the field generated by the numbers of R. To begin with the easier half of the theorem, let us assume that (59) is regular in $R - \{0\}$. Then, *a fortiori*, (59) is regular in $K - \{0\}$. Therefore, by Theorem VI, (59) satisfies $\Gamma(K)$. Let $\xi_\nu^{(\lambda)}$ be the l solutions of (59) mentioned in the definition of $\Gamma(K)$. Then

$$\xi_\nu^{(\lambda)} = \eta_\nu^{(\lambda)} / \zeta_\nu^{(\lambda)},$$

where $$\eta_\nu^{(\lambda)} \prec R;\ \zeta_\nu^{(\lambda)} \prec R - \{0\} \quad (1 \leqslant \nu \leqslant n;\ 1 \leqslant \lambda \leqslant l).$$

Let $$N = \prod_{1 \leqslant \nu \leqslant n,\ 1 \leqslant \lambda \leqslant l} \zeta_\nu^{(\lambda)};\quad \bar{\xi}_\nu^{(\lambda)} = N \xi_\nu^{(\lambda)}.$$

Then the numbers $\bar{\xi}_\nu^{(\lambda)}$ belong to R, and it is obvious that (59) satisfies $\Gamma(R)$.

Now let us assume that (59) satisfies $\Gamma(R)$. We have to show that (59) is regular in $R - \{0\}$. We exclude the trivial case where all $a_{\mu\nu}$ vanish. In accordance with the definition of $\Gamma(R)$ there are numbers $\xi_\nu^{(\lambda)}$ $(1 \leqslant \nu \leqslant n;\ 1 \leqslant \lambda \leqslant l)$ of R which have the properties stated on p. 141. Let B be the system of these numbers $\xi_\nu^{(\lambda)}$, and denote by R', K' the ring and the field respectively generated by the numbers of B. We shall show that (59) is regular in $K' - \{0\}$. If this fact is established, then, by Lemma 1 on p. 129, regularity in $R' - \{0\}$ will follow, and since R' is contained in R, regularity in $R - \{0\}$.

Suppose, first of all, that the number l occurring in the definition of $\Gamma(R)$ has the value 1. Then

$$\sum_{\nu=1}^{n} a_{\mu\nu} \times 1 = 0 \quad (1 \leqslant \mu \leqslant m),$$

and therefore (59) is regular, in fact, absolutely regular, in $K' - \{0\}$. Now suppose that $l = l' > 1$. We may assume that it has already been proved that all systems (59) satisfying $\Gamma(R)$ with a value $l = l' - 1$ are regular in the corresponding $K' - \{0\}$. We may suppose without loss of generality that

$$S_{l'} = \{n'+1, n'+2, \ldots, n\},$$

where $1 \leqslant n' < n$. Then, clearly, the system

$$\sum_{\nu=1}^{n'} a_{\mu\nu} x_\nu = 0 \quad (1 \leqslant \mu \leqslant m) \tag{60}$$

satisfies $\Gamma(R)$ with $l = l'-1$. For we can use the same numbers $\xi_\nu^{(\lambda)}$ as for (59) but restrict ν, λ to the ranges $1 \leqslant \nu \leqslant n'$, $1 \leqslant \lambda \leqslant l'-1$. Therefore (60) is regular in $K'-\{0\}$†. Now let k be the least positive integer, if there is one, such that (59) is not k-regular in $K'-\{0\}$. We have to deduce a contradiction.

According to $\Gamma(R)$ we have

$$(61) \qquad \sum_{\nu=1}^{n'} a_{\mu\nu}\, \xi_\nu^{(l')} + \sum_{\nu=n'+1}^{n} a_{\mu\nu}\, \xi^{(l')} = 0 \quad (1 \leqslant \mu \leqslant m),$$

where $\xi^{(l')} \prec R'-\{0\}$. By Theorem I (p. 128) there are finite subsets M, M' of $K'-\{0\}$ such that (59) is $(k-1)$-regular in M and (60) is k-regular in M'‡. Since (59) is not k-regular in $K'-\{0\}$, there is a distribution Δ of $K'-\{0\}$ into k classes which has the property that (59) has no solution x_ν with

$$x_1 \equiv x_2 \equiv \ldots \equiv x_n \pmod{\Delta}.$$

Let M'' be the set consisting of all numbers

$$\xi_\nu^{(l')}\, t y^{-1} \quad (1 \leqslant \nu \leqslant n';\ t \prec M;\ y \prec M')$$

and of the number 1. Define $\Delta'(x)$ in K' as follows:

$$(62) \qquad \Delta'(x) = \prod_{y \prec M'} \Delta(xy) \quad (x \prec K'-\{0\}),$$

$$(63) \qquad 0 \not\equiv x \pmod{\Delta'} \quad (x \prec K'-\{0\}).$$

By Theorem III there exists a number a of K' and a positive integer d for which

$$a+dz \equiv a \pmod{\Delta'} \quad (z \prec M'').$$

In view of a remark made above (p. 135) we may postulate that the numbers a and $a+dz$ $(z \prec M'')$ differ from zero or else are all equal to zero. The second possibility is ruled out, since we included the number 1 in M''. Therefore, by (62),

$$(64) \qquad (a+dz)\,y \equiv ay \pmod{\Delta} \quad (z \prec M'';\ y \prec M').$$

Put

$$(65) \qquad \Delta''(y) = \Delta(ay) \quad (y \prec M').$$

---

† Strictly speaking, K' should be replaced by the field K'' belonging to (60). But K'' is contained in K', and therefore regularity in $K'-\{0\}$ holds *a fortiori*.

‡ In the case $k=1$ statements about $k-1$ have to be omitted.

By definition of M' there are numbers y_ν of M' such that

$$(66)\qquad y_1 \equiv y_2 \equiv \ldots \equiv y_{n'} \pmod{\Delta''},$$

$$(67)\qquad \sum_{\nu=1}^{n'} a_{\mu\nu} y_\nu = 0 \quad (1 \leqslant \mu \leqslant m);$$

(64), (66) and (65) imply that

$$(68)\qquad (a+dz)y_\nu \equiv ay_\nu \equiv ay_1 \pmod{\Delta} \quad (1 \leqslant \nu \leqslant n';\ z \prec M'').$$

Also, from (67) and (61),

$$(69)\qquad \sum_{\nu=1}^{n'} a_{\mu\nu}(ay_\nu + d\xi_\nu^{(l')} t) + \sum_{\nu=n'+1}^{n} a_{\mu\nu} \xi^{(l')} dt = 0 \quad (1 \leqslant \mu \leqslant m;\ t \prec M).$$

Now
$$ay_\nu + d\xi_\nu^{(l')} t = (a + d\xi_\nu^{(l')} t y_\nu^{-1}) y_\nu,$$

where
$$\xi_\nu^{(l')} t y_\nu^{-1} \prec M'' \quad (1 \leqslant \nu \leqslant n';\ t \prec M).$$

Therefore, by (68),

$$(70)\qquad ay_\nu + d\xi_\nu^{(l')} t \equiv ay_1 \pmod{\Delta} \quad (1 \leqslant \nu \leqslant n';\ t \prec M).$$

We now show that this implies that

$$(71)\qquad d\xi^{(l')} t \not\equiv ay_1 \pmod{\Delta} \quad (t \prec M).$$

If (71) is not true, then, for some number t_0 of M,

$$(72)\qquad d\xi^{(l')} t_0 \equiv ay_1 \pmod{\Delta}.$$

But then (69), (70) (with $t = t_0$) and (72) show that, contrary to our initial assumption about Δ, the system (59) has a solution whose numbers belong to the same class of Δ. Therefore (71) is established.

If $k = 1$, then (71) is plainly impossible; for then all numbers belong to the same class of Δ. When $k > 1$ put

$$\Delta'''(x) = \Delta(d\xi^{(l')} x) \quad (x \prec M).$$

Then, by (71),
$$|\Delta'''| \leqslant k-1.$$

Hence, from the definition of M, there are numbers x_ν of M satisfying (59) and

$$x_1 \equiv x_2 \equiv \ldots \equiv x_n \pmod{\Delta'''}.$$

Then
$$\sum_{\nu=1}^{n} a_{\mu\nu} d\xi^{(l')} x_\nu = 0 \quad (1 \leqslant \mu \leqslant m),$$

$$d\xi^{(l')} x_\nu \equiv d\xi^{(l')} x_1 \pmod{\Delta} \quad (1 \leqslant \nu \leqslant n)$$

and this, again, contradicts the definition of Δ. This completes the proof of Theorem VII.

4. We note a corollary of Theorem VII.

A system $$\sum_{\nu=1}^{n} a_{\mu\nu} x_\nu = 0 \quad (1 \leqslant \mu \leqslant m)$$

is regular in the set of all complex numbers other than 0 *if, and only if, it is regularin* $R_0 - \{0\}$, *where* R_0 *is the ring generated by all coefficients* $a_{\mu\nu}$.

(*We have to exclude the trivial case where all* $a_{\mu\nu} = 0$.)

It is, however, not true that, corresponding to every system of equations which is regular in some set, there exists a smallest ring R^* such that it is regular in $R^* - \{0\}$. For consider the equation $(m = 1;\ n = 4)$

$$\alpha x_1 + \beta x_2 - (\alpha+\beta) x_3 - \alpha\beta x_4 = 0, \tag{73}$$

where $\alpha = \sqrt{2}$; $\beta = \sqrt{3}$. The only possible sets $S_1, S_2, \ldots,$ consistent with the definition of Γ (p. 141) are

$$S_1 = \{1, 2, 3\}; \quad S_2 = \{4\}.$$

The existence of the solutions

$$(1, 1, 1, 0); \quad (0, \alpha, 0, 1); \quad (\beta, 0, 0, 1)$$

of (73) shows that (73) is regular in $R_\alpha - \{0\}$ as well as in $R_\beta - \{0\}$, where R_α, R_β are the rings generated by α, 1 and β, 1 respectively. The common numbers of R_α and R_β form the ring R_0 of all rational integers. But (73) is not regular in $R_0 - \{0\}$. For this would imply that $\Gamma(R_0)$ holds, *i.e.* that there are rational integers x_1, x_2, x_3, x_4, with $x_4 \neq 0$, such that

$$\alpha x_1 + \beta x_2 - (\alpha+\beta) x_3 - \alpha\beta x_4 = 0,$$

$$(x_1 - x_3)\sqrt{2} + (x_2 - x_3)\sqrt{3} = x_4\sqrt{6},$$

$$2(x_1 - x_3)(x_2 - x_3)\sqrt{6} = 6x_4^2 - 2(x_1 - x_3)^2 - 3(x_2 - x_4)^2,$$

which is easily seen to be impossible.

5. Let R be a ring of numbers (not consisting of zero only), and K be the field generated by R. We want to bring out more clearly the significance of the condition $\Gamma(R)$. From considerations at the beginning of the proof of Theorem VII, it is obvious that $\Gamma(R)$ and $\Gamma(K)$ are equivalent. We can easily write down a general class of systems of linear equations which satisfy $\Gamma(K)$, viz.

$$\sum_{\nu=1}^{n-m} c_{\mu\nu} x_\nu - x_{n-m+\mu} = 0 \quad (1 \leqslant \mu \leqslant m). \tag{74}$$

Here $1 \leqslant m < n$, and the $c_{\mu\nu}$ are arbitrary numbers of K subject to the two conditions:

(i) no row

$$c_{\mu 1}, c_{\mu 2}, \ldots, c_{\mu, n-m} \quad (1 \leqslant \mu \leqslant m) \tag{75}$$

contains zeros only;

(ii) the first non-zero number of every row (75) has the value 1.

Let us call systems (74) of this kind *T-systems*.

In order to prove that (74) satisfies $\Gamma(K)$, we may assume (74) to be of the form

$$\left\{\begin{array}{ll} x_{s_1} + \sum_{\nu=s_1+1}^{n-m} c_{\mu\nu} x_\nu - x_{n-m+\mu} = 0 & (1 \leqslant \mu \leqslant a_1), \\ x_{s_2} + \sum_{\nu=s_2+1}^{n-m} c_{\mu\nu} x_\nu - x_{n-m+\mu} = 0 & (a_1 < \mu \leqslant a_2), \\ \ldots \quad \ldots \quad \ldots \quad \ldots & \ldots \\ x_{s_{l-1}} + \sum_{\nu=s_{l-1}+1}^{n-m} c_{\mu\nu} x_\nu - x_{n-m+\mu} = 0 & (a_{l-2} < \mu \leqslant a_{l-1}), \end{array}\right. \tag{76}$$

where $\qquad l \geqslant 2; \quad 1 \leqslant s_1 < s_2 < \ldots < s_{l-1} \leqslant n-m;$

$$1 \leqslant a_1 < a_2 < \ldots < a_{l-1} = m.$$

Then we put

$$S_1 = \{s_1, n-m+1, n-m+2, \ldots, n-m+a_1\},$$
$$S_2 = \{s_2, n-m+a_1+1, n-m+a_1+2, \ldots, n-m+a_2\},$$
$$\ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots$$
$$S_{l-1} = \{s_{l-1}, n-m+a_{l-2}+1, n-m+a_{l-2}+2, \ldots, n-m+a_{l-1}\},$$

while the last set S_l consists of all indices out of $1, 2, \ldots, n$ which do not occur in any of the sets $S_1, S_2, \ldots, S_{l-1}$. A moment's consideration shows that we can choose numbers $\xi_\nu^{(\lambda)}$ which are consistent with the definition of $\Gamma(K)$.

In a certain sense (76) is the most general system which satisfies $\Gamma(K)$. For we have the following result.

Corresponding to every system

$$\sum_{\nu=1}^{n} a_{\mu\nu} x_\nu = 0 \quad (1 \leqslant \mu \leqslant m') \tag{77}$$

which satisfies $\Gamma(K)$, *there is a* T-*system* (74) *such that every solution of* (74) *is at the same time a solution of* (77).

In other words, by adding suitable equations to those of (77), which is supposed to satisfy $\Gamma(K)$, we can obtain a T-system (or a system equivalent to a T-system).

In proving this statement we may assume, without loss of generality, that the sets S_λ corresponding to (77) are

$$\begin{aligned} S_1 &= \{1, 2, \ldots, \beta_1\}, \\ S_2 &= \{\beta_1+1, \beta_1+2, \ldots, \beta_2\}, \\ &\ldots \quad \ldots \quad \ldots \quad \ldots \\ S_l &= \{\beta_{l-1}+1, \beta_{l-1}+2, \ldots, \beta_l\}, \end{aligned}$$

where $$l \geqslant 1; \quad 1 \leqslant \beta_1 < \beta_2 < \ldots < \beta_l = n.$$

Consequently there is a matrix

$$(78) \qquad \begin{pmatrix} 1, & 1, & \ldots, & 1, & 0, & \ldots & \ldots & \ldots & \ldots, & 0 \\ \xi_1^{(2)}, & \xi_2^{(2)}, & \ldots, & \xi_{\beta_1}^{(2)}, & 1, & \ldots, & 1, & 0, & \ldots, & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ \xi_1^{(l)}, & \xi_2^{(l)}, & \ldots & \ldots & \ldots & \ldots & \xi_{\beta_{l-1}}^{(l)}, & 1, & \ldots, & 1 \end{pmatrix}$$

every row of which is a solution of (77). The $\xi_\nu^{(\lambda)}$ belong to K. In (78) the columns with indices $\beta_1, \beta_2, \ldots, \beta_l$ are linearly independent, while every one of the remaining columns is a linear combination of these l columns. Moreover, these linear relations are expressible in the form

$$(79) \qquad \begin{cases} x_\mu = x_{\beta_1} + \sum\limits_{\lambda=2}^{l} c_{\mu\lambda} x_{\beta_\lambda} & (1 \leqslant \mu < \beta_1), \\ x_\mu = x_{\beta_2} + \sum\limits_{\lambda=3}^{l} c_{\mu\lambda} x_{\beta_\lambda} & (\beta_1 < \mu < \beta_2), \\ \ldots \quad \ldots \quad \ldots & \ldots \\ x_\mu = x_{\beta_l} & (\beta_{l-1} < \mu < \beta_l), \end{cases}$$

where $c_{\mu\lambda} \prec K$. (79) is a complete set of linearly independent relations between the columns of (78). Hence every vector $(x_1, x_2, \ldots, x_n)$ which satisfies (79) is a linear combination of the rows of (78) and is therefore a solution of (77). But (79) is a T-system, as is seen by ordering the variables as follows:

$$x_{\beta_1}, x_{\beta_2}, \ldots, x_{\beta_l}, x_1, x_2, \ldots, x_{\beta_1-1}, x_{\beta_1+1}, x_{\beta_1+2}, \ldots, x_{\beta_2-1}, x_{\beta_2+1}, \ldots, x_{\beta_l-1}.$$

This proves our statement about the connection between (77) and T-systems.

Questions about the regularity of a system of homogeneous linear equations in the set of all positive rational integers or in the set of all complex numbers $re^{i\phi}$, where $r>0$ and ϕ belongs to a given interval $\phi_0<\phi<\phi_1$, and many more, are settled by the following theorem.

THEOREM VIII. *Let R be a ring of complex numbers, and let A be a subset of $R-\{0\}$ which has the following property. There is a finite number of elements $d_1, d_2, \ldots, d_k$ of $R-\{0\}$ such that every x of $R-\{0\}$ is expressible as*

$$x = ad_\kappa, \tag{80}$$

where $a \prec A$; $1 \leqslant \kappa \leqslant k$. Then regularity of

$$\sum_{\nu=1}^{n} a_{\mu\nu} x_\nu = 0 \quad (1 \leqslant \mu \leqslant m) \tag{81}$$

in A is equivalent to regularity of (81) *in $R-\{0\}$.*

For instance, in the case of all numbers $re^{i\phi}$, where $r>0$; $\phi_0<\phi<\phi_1$, we choose an integer $k>2\pi/(\phi_1-\phi_0)$ and put

$$d_\kappa = e^{2\pi i\kappa/k} \quad (1 \leqslant \kappa \leqslant k).$$

Of course, the representation (80) need not be unique.

Proof. Suppose that (81) is regular in $R-\{0\}$. We have to show that (81) is regular even in A. Let $\Delta(x)$ be defined in A. Given any x of $R-\{0\}$, we define $\lambda=\lambda(x)$ as being the least index κ for which xd_κ^{-1} belongs to A: λ exists. Put

$$\Delta'(x) = \Delta^{(0)}\big(\lambda(x)\big)\,\Delta(xd_{\lambda(x)}^{-1}) \quad (x \prec R-\{0\}).$$

Then there are numbers x_ν satisfying (81) and

$$x_1 \equiv x_2 \equiv \ldots \equiv x_n \pmod{\Delta'}. \tag{82}$$

(82) implies that

$$\lambda(x_1) = \lambda(x_2) = \ldots = \lambda(x_n) = \lambda_0,$$

say, and
$$x_1 d_{\lambda_0}^{-1} \equiv x_2 d_{\lambda_0}^{-1} \equiv \ldots \equiv x_n d_{\lambda_0}^{-1} \pmod{\Delta};$$

while (81) yields
$$\sum_\nu a_{\mu\nu} x_\nu d_{\lambda_0}^{-1} = 0 \quad (1 \leqslant \mu \leqslant m).$$

This proves the theorem.

3. *Non-homogeneous linear equations.*

1. Let A_0 be the field of all algebraic numbers and C_0 that of all complex numbers.

THEOREM IX. *A system*

$$\sum_{\nu=1}^{n} a_{\mu\nu} x_\nu = b_\mu \quad (1 \leqslant \mu \leqslant m) \tag{83}$$

is regular in A_0 if, and only if, it is absolutely regular in A_0.

THEOREM X. *A system (83) with algebraic coefficients $a_{\mu\nu}$ (and with the b_μ arbitrary complex numbers) is regular in C_0 if, and only if, it is absolutely regular in C_0*†).

In proving the last two theorems it is sufficient to show that regularity implies absolute regularity in the sets in question.

Theorem IX follows from Theorem X. For let us assume that (83) is regular in A_0. Then there are algebraic numbers $x_\nu^{(0)}$ satisfying

$$\sum_\nu a_{\mu\nu} x_\nu^{(0)} = b_\mu \quad (1 \leqslant \mu \leqslant m).$$

This simply means that (83) is 1-regular in A_0. Therefore (83) is equivalent to

$$\sum_\nu a_{\mu\nu}(x_\nu - x_\nu^{(0)}) = 0 \quad (1 \leqslant \mu \leqslant m).$$

If we use the argument which leads from (49) to (51) we find a system

$$\sum_{\nu=1}^{n} a'_{\mu\nu} y_\nu = 0 \quad (1 \leqslant \mu \leqslant m') \tag{84}$$

with algebraic coefficients $a'_{\mu\nu}$ such that (84) and

$$\sum_{\nu=1}^{n} a_{\mu\nu} y_\nu = 0 \quad (1 \leqslant \mu \leqslant m)$$

have exactly the same solutions in A_0. Therefore

$$\sum_\nu a'_{\mu\nu}(x_\nu - x_\nu^{(0)}) = 0 \quad (1 \leqslant \mu \leqslant m') \tag{85}$$

and (83) have the same solutions in A_0. In particular, (85) is regular in A_0 and *a fortiori* regular in C_0. Now apply Theorem X to (85). It follows that (85) is absolutely regular in C_0. Since $a'_{\mu\nu}$, $x_\nu^{(0)}$ are numbers of A_0 we conclude further that (85) is absolutely regular even in A_0, and finally that (83) is absolutely regular in A_0.

---

† Absolute regularity was defined on p. 127.

It remains to prove Theorem X. We note that absolute regularity in a set M of a single equation

$$a_1 x_1 + a_2 x_2 + \ldots + a_n x_n = b$$

means that either

$$a_1 + a_2 + \ldots + a_n = 0$$

or

$$s = a_1 + \ldots + a_n \neq 0; \quad b/s \prec M.$$

LEMMA 4. *Suppose that* (83) *has the following property. For every choice of numbers t_μ belonging to the field K generated be the $a_{\mu\nu}$, the equation*

$$\sum_{\mu=1}^{m} t_\mu \sum_{\nu=1}^{n} a_{\mu\nu} x_\nu = \sum_{\mu=1}^{m} t_\mu b_\mu$$

is absolutely regular in M. Then (83) *is absolutely regular in M.*

To prove Lemma 4, put $s_\mu = \sum_\nu a_{\mu\nu}$. We may assume that

$$s_\mu \neq 0 \quad (1 \leqslant \mu \leqslant m'), \qquad s_\mu = 0 \quad (m' < \mu \leqslant m),$$

where $0 \leqslant m' \leqslant m$. By hypothesis, any numbers t_μ of K for which

$$\sum_\mu s_\mu t_\mu = 0$$

satisfy

$$\sum_\mu b_\mu t_\mu = 0.$$

Therefore, by Lemma 3 on p. 142 ($r = s = 1$: see the remark at the end of the proof of Lemma 3),

$$x' \sum_\mu s_\mu t_\mu = \sum_\mu b_\mu t_\mu,$$

identically in the t_μ. Here x' is some constant. Hence

$$\sum_\nu a_{\mu\nu} x' = b_\mu \quad (1 \leqslant \mu \leqslant m).$$

If $m' > 0$ then

$$x' = b_1/s_1 \prec M.$$

For the equation

$$\sum_\nu a_{1\nu} x_\nu = b_1$$

is absolutely regular in M. If $m' = 0$, *i.e. if*

$$s_\mu = 0 \quad (1 \leqslant \mu \leqslant m),$$

then

$$b_\mu = 0 \quad (1 \leqslant \mu \leqslant m);$$

and therefore

$$\sum_\nu a_{\mu\nu} x'' = b_\mu \quad (1 \leqslant \mu \leqslant m),$$

where x'' is any arbitrary number of M. Thus the lemma is proved.

In view of Lemma 4, it is sufficient to consider the case $m = 1$ of Theorem X. Let us therefore assume that $a_1, a_2, \ldots, a_n$ belong to A_0, that b is an arbitrary number and that

$$a_1+a_2+\ldots+a_n = 0; \quad b \neq 0. \tag{86}$$

We have to show that

$$a_1 x_1+\ldots+a_n x_n = b \tag{87}$$

is not regular in C_0. We may assume that no a_ν vanishes.

By means of transfinite induction we can find a "basis" $\xi_1, \xi_2, \ldots, \xi_\alpha, \ldots$ of all complex numbers with respect to the field K generated by $a_1, a_2, \ldots, a_n$. This means that every complex number x is uniquely representable in the form

$$x = \sum_\alpha x_\alpha' \xi_\alpha, \tag{88}$$

where α runs through certain ordinal numbers and x_α' belongs to K. For every x only a finite number of "coordinates" x_α' differ from zero. Let

$$b = \sum_\alpha b_\alpha' \xi_\alpha$$

be the representation (88) in the case $x = b$. Since $b \neq 0$, at least one of the numbers b_α' differs from zero. There is no loss of generality in assuming that $b_1' \neq 0$. The numbers $a_1, a_2, \ldots, a_n, b_1'$ are algebraic. If we multiply (87) by a suitable number of K, we can obtain a case where these $n+1$ numbers are algebraic integers of K. We may therefore assume that (87) is such that these numbers are algebraic integers. Let $\omega_1, \omega_2, \ldots, \omega_p$ be a minimal basis of K, so that every algebraic integer of K has a unique representation in the form

$$r_1 \omega_1+r_2 \omega_2+\ldots+r_p \omega_p, \tag{89}$$

where the r_λ are rational integers. Then every number of K has a unique representation (89) with rational coefficients r_λ.

Let $\mathfrak{p}$ be a prime ideal in K which is not a divisor of b_1'. Define $\Delta(x)$, for all algebraic integers of K, by means of the rule that

$$x \equiv y \pmod{\Delta}$$

if, and only if,

$$x \equiv y \pmod{\mathfrak{p}}.$$

Then no class of Δ contains a solution of

$$a_1 x_1+\ldots+a_n x_n = b_1'. \tag{90}$$

For (90) and

(91) $$x_1 \equiv x_2 \equiv \ldots \equiv x_n \pmod{\Delta}$$

would lead to the contradiction

$$b_1' = \Sigma a_\nu x_\nu = \Sigma a_\nu(x_\nu - x_1) = 0 \pmod{\mathfrak{p}}.$$

Define, for every x of K, a function $f(x)$ by means of

$$f(x) = \sum_{\lambda=1}^{p} |r_\lambda|,$$

where

(92) $$x = \sum_{\lambda=1}^{p} r_\lambda \omega_\lambda \quad (r_\lambda \text{ rational}).$$

Then, for any x_ν satisfying (91),

(93) $$f(\Sigma a_\nu x_\nu - b_1') \geqslant 1.$$

Now choose an integer N satisfying

(94) $$N > \sum_{1 \leqslant \nu \leqslant n,\ 1 \leqslant \lambda \leqslant p} f(a_\nu \omega_\lambda);$$

and put, for every x of K satisfying (92),

(95) $$\Delta'(x) = \Delta\left(\sum_\lambda [r_\lambda]\,\omega_\lambda\right) \prod_{\lambda=1}^{p} \Delta^{(N)}([Nr_\lambda])\dagger.$$

We now show that no class of Δ' contains a solution of (90). Suppose that

(96) $$x_1 \equiv x_2 \equiv \ldots \equiv x_n \pmod{\Delta'},$$

(97) $$a_1 x_1 + \ldots + a_n x_n = b_1',$$

$$x_\nu = \sum_\lambda r_{\nu\lambda}\,\omega_\lambda \quad (r_{\nu\lambda} \text{ rational}).$$

(96) and (95) imply that

(98) $$\sum_\lambda [r_{\nu\lambda}]\,\omega_\lambda \equiv \sum_\lambda [r_{1\lambda}]\,\omega_\lambda \pmod{\Delta} \quad (1 \leqslant \nu \leqslant n),$$

$$[Nr_{\nu\lambda}] \equiv [Nr_{1\lambda}] \pmod{N} \quad (1 \leqslant \nu \leqslant n;\ 1 \leqslant \lambda \leqslant p),$$

$$\frac{1}{N}[Nr_{\nu\lambda}] - [r_{\nu\lambda}] = \frac{1}{N}[Nr_{\nu\lambda}] - \left[\frac{1}{N}[Nr_{\nu\lambda}]\right]$$

$$= \frac{1}{N}[Nr_{1\lambda}] - \left[\frac{1}{N}[Nr_{1\lambda}]\right] = \frac{1}{N}[Nr_{1\lambda}] - r_{1\lambda} = d_\lambda,$$

† $\Delta^{(N)}$ was defined on p. 124.

say, for $$1 \leqslant \nu \leqslant n, \quad 1 \leqslant \lambda \leqslant p,$$

$$\sum_\nu a_\nu \sum_\lambda [r_{\nu\lambda}]\, \omega_\lambda = \sum_\nu a_\nu \sum_\lambda \frac{1}{N}\, [N r_{\nu\lambda}]\, \omega_\lambda - \sum_\nu a_\nu \sum_\lambda d_\lambda\, \omega_\lambda$$

$$= \sum_\nu a_\nu \sum_\lambda \frac{1}{N}\, [N r_{\nu\lambda}]\, \omega_\lambda \quad [\text{by } (86)]$$

$$= \sum_\nu a_\nu \sum_\lambda r_{\nu\lambda}\, \omega_\lambda - \sum_\nu a_\nu \sum_\lambda r'_{\nu\lambda}\, \omega_\lambda,$$

where $r'_{\nu\lambda}$ is rational, $0 \leqslant r'_{\nu\lambda} < 1/N$. Hence, from (98) and (93),

$$1 \leqslant f\Big(\sum_\nu a_\nu \sum_\lambda [r_{\nu\lambda}]\, \omega_\lambda - b_1'\Big) = f\Big(\sum_\nu a_\nu \sum_\lambda r_{\nu\lambda}\, \omega_\lambda - \sum_\nu a_\nu \sum_\lambda r'_{\nu\lambda}\, \omega_\lambda - b_1'\Big)$$

$$= f\Big(-\sum_\nu a_\nu \sum_\lambda r'_{\nu\lambda}\, \omega_\lambda\Big) \leqslant \sum_{\nu, \lambda} |r'_{\nu\lambda}|\, f(a_\nu\, \omega_\lambda) \leqslant \frac{1}{N} \sum_{\nu, \lambda} f(a_\nu\, \omega_\lambda).$$

The last inequality contradicts (94). Therefore, as stated above, no class of Δ' contains a solution of (90).

We now proceed to define a distribution of C_0. Put, for every complex x,

$$g(x) = x_1',$$

where x_1' is the first "coordinate" in the representation (88). $g(x)$ has the following properties:

$$g(x) \prec K, \quad g(b) = b_1' \neq 0, \quad g\Big(\sum_{\nu=1}^{n} a_\nu x_\nu\Big) = \sum_\nu a_\nu g(x_\nu)$$

for all complex x_ν.

Put $$\Delta''(x) = \Delta'\big(g(x)\big) \quad (x \prec C_0).$$

Suppose that, for some x_ν of C_0,

$$x_1 \equiv x_2 \equiv \dots \equiv x_n \pmod{\Delta''}.$$

Then $$g(x_\nu) \equiv g(x_1) \pmod{\Delta'} \quad (1 \leqslant \nu \leqslant n).$$

Hence, by the definition of Δ',

$$\sum_\nu a_\nu g(x_\nu) \neq g(b),$$

i.e., $$g(\sum_\nu a_\nu x_\nu) \neq g(b);$$

and therefore $$\sum_\nu a_\nu x_\nu \neq b.$$

This completes the proofs of Theorems IX and X.

2. The foregoing proof made use of transfinite induction in order to define a certain distribution of all complex numbers. In the case of rational coefficients $a_{\mu\nu}$ (the b_μ may be arbitrary complex numbers) it is possible to eliminate transfinite induction. In this section we give a proof of the assertion of Theorem X in the special case of rational coefficients $a_{\mu\nu}$ or even in the slightly more general case where

$$a_{\mu\nu} = a'_{\mu\nu} + i a''_{\mu\nu} \quad (a'_{\mu\nu},\ a''_{\mu\nu} \text{ rational}).$$

In view of Lemma 4 it is sufficient to prove the following proposition.

Let $$a_\nu = a_\nu' + i a_\nu'' \quad (1 \leqslant \nu \leqslant n),$$

where a_ν', a_ν'' *are rational integers. Suppose that*

$$a_1 + \ldots + a_n = 0,$$

while b *is a complex number different from zero. Then*

$$a_1 x_1 + \ldots + a_n x_n = b \tag{99}$$

is not regular in C_0.

Define, for any

$$x = x' + i x'' \quad (x',\ x'' \text{ real}),$$

$$[x] = [x'] + i[x''].$$

Choose positive integers m', m'' such that

$$m' > \sqrt{2}\sum_\nu |a_\nu| + |b|; \quad m'' > \sqrt{2}|b|^{-1}\sum_\nu |a_\nu|.$$

Put, for every such x,

$$\Delta^*(x) = \Delta^{(m'm'')}([m''x'])\,\Delta^{(m'm'')}([m''x'']).$$

Then our proof is complete when we can show that no class of Δ^* contains a solution of (99). Suppose that

$$x_1 \equiv x_2 \equiv \ldots \equiv x_n \pmod{\Delta^*}.$$

Then (using an obvious notation)

$$[m''x_\nu] \equiv [m''x_1] \pmod{m'm''},$$

$$[m''x_\nu] - [m''x_1] = k_\nu m'm'' \quad (1 \leqslant \nu \leqslant n), \tag{100}$$

where $$k_\nu = k_\nu' + i k_\nu'' \quad (k_\nu',\ k_\nu'' \text{ rational integers}).$$

(100) implies that

$$\frac{1}{m''}[m''x_\nu]-\frac{1}{m''}[m''x_1]=k_\nu m',$$

$$\left[\frac{1}{m''}[m''x_\nu]\right]-\left[\frac{1}{m''}[m''x_1]\right]=k_\nu m',$$

$$[x_\nu]-[x_1]=k_\nu m'.$$

Also,
$$m''x_\nu-m''x_1=k_\nu m'm''+r_\nu,$$
where
$$|r_\nu|<\sqrt{2}.$$

Case (1): Let
$$\Sigma a_\nu[x_\nu]\neq 0.$$
Then

$$|\Sigma a_\nu[x_\nu]|=|\Sigma a_\nu([x_\nu]-[x_1])|=|\Sigma a_\nu k_\nu m'|\geqslant m',$$

$$|\Sigma a_\nu x_\nu|\geqslant|\Sigma a_\nu[x_\nu]|-|\Sigma a_\nu(x_\nu-[x_\nu])|\geqslant m'-\Sigma|a_\nu|\sqrt{2}>|b|.$$

Case (2): Let
$$\Sigma a_\nu[x_\nu]=0.$$
Then

$$|\Sigma a_\nu x_\nu|=|\Sigma a_\nu\{(x_\nu-x_1)-([x_\nu]-[x_1])\}|$$

$$=\left|\Sigma a_\nu\left(\frac{k_\nu m'm''+r_\nu}{m''}-k_\nu m'\right)\right|=\left|\Sigma\frac{a_\nu r_\nu}{m''}\right|\leqslant\frac{1}{m''}\Sigma|a\ |\sqrt{2}<|b|.$$

Hence in either case

$$a_1x_1+a_2x_2+\ldots+a_nx_n\neq b.$$

4. *ω-regularity.*

A systems of conditions

(101) $$S(x_1,\ldots,x_n)=0$$

was called *ω-regular* in a set M if, given any distribution Δ of M into denumerably many classes, there is always a solution of (101) satisfying

$$x_1\equiv x_2\equiv\ldots\equiv x_n\pmod{\Delta}.$$

Of the three propositions:

(i) (101) is k-regular in C_0 for every $k=1, 2, 3, \ldots$;

(ii) (101) is ω-regular in C_0;

(iii) (101) is absolutely regular in C_0;

(i) is weaker than (ii), and (ii) is weaker than (iii). For the equation

$$x_1+x_2-x_3=0$$

satisfies (i) but not (ii) (as follows from Theorem XI below), and the condition

$$x_1 \neq x_2$$

satisfies (ii) but not (iii). We prove that in the case of linear equations or, more generally, of conditions whose solutions form a closed set, (ii) and (iii) are equivalent.

THEOREM XI. *A system of equations*

$$\sum_{\nu=1}^{n} a_{\mu\nu} x_\nu = b_\mu \quad (1 \leqslant \mu \leqslant m) \tag{102}$$

is ω-regular in a set M of complex numbers if, and only if, (102) *is absolutely regular in M.*

Proof. Suppose that (102) is not absolutely regular in M. We have to show that (102) is not ω-regular in M. We call a circle

$$|x-(c'+ic'')|<r \tag{103}$$

a *rational circle* if c', c'', r are rational numbers. Every point z of M is contained in a rational circle which does not contain any solution of (102). For, if this is not true for some $z=z_0$ of M, then every rational circle (103) which contains z_0 also contains a solution x_ν of (102), and making $r \to 0$ shows that

$$\sum_{\nu} a_{\mu\nu} z_0 = b_\mu \quad (1 \leqslant \mu \leqslant m),$$

i.e. that (102) is absolutely regular in M. The rational circles are enumerable. Let R_1, R_2, ... be a sequence containing every rational circle. Define, for every z of M, $f(z)$ as being the least λ such that z lies in R_λ but, at the same time, R_λ contains no solution of (102). Then, obviously no class of

$$\Delta^{*}(z) = \Delta^{(0)}\big((f(z)\big) \quad (z \prec M)\dagger$$

contains a solution of (102). Therefore (102) is not ω-regular in M.

The University,
Sheffield.

† $\Delta^{(0)}$ was defined on p. 124.