The Canonical Van Der Waerden's Theorem: An Exposition
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1 Introduction

We first recall van der Waerden's theorem.

Notation 1.1 If $m \in \mathsf{N}$ then $[m]$ is $\{1, \ldots, m\}$.

Definition 1.2 If $k \in \mathsf{N}$ then a k-AP is an arithmetic progression of length k. Henceforth we abbreviate "arithmetic progression' by AP and "arithmetic progression of length k" by k-AP.

The following statement is the original van der Waerden's Theorem. It was first proven in [4] but see also [2].

Theorem 1.3 *For every $k \geq 1$ and $c \geq 1$ there exists $W = W(k, c)$ such that for every c-coloring $COL : [W] \rightarrow [c]$ there exists a monochromatic k-AP. In other words there exists $a, d,\ d \neq 0$, such that*

- $a, a + d, a + 2d, \ldots, a + (k - 1)d \in [W]$, *and*
- $COL(a) = COL(a + d) = \cdots = COL(a + (k - 1)d)$.

Note 1.4 Formally colors are numbers; however, we will often use R, B, G, etc for colors for clarity.

What if we use an infinite number of colors instead of a finite number of colors. Then the analog of Theorem 1.3 is false as the coloring $COL(x) = x$ shows. However in this case we may get something else.

Definition 1.5 Let $k \in \mathsf{N}$. Let COL be a coloring of N (which may use a finite or infinite number of colors). A *rainbox k-AP* is an arithmetic sequence $a, a+d, a+2d, \ldots, a+(k-1)d$ such that all of these are colored differently.

The following is the *Canonical van der Waerden's theorem*. It was first proven by Erdos and Graham [1] using Szemerédi 's theorem. Rödl and Prömel [3] later came up with an elementary proof. We present their proof.

Theorem 1.6 *Let $k \in \mathsf{N}$. Let $COL : \mathsf{N} \rightarrow \mathsf{N}$ be a coloring of the naturals. One of the following two must occur.*

- *There exists a monochromatic k-AP.*
- *There exists a rainbox k-AP.*

2 Proof of theorem

We will need the following lemma to prove the canonical van der Waerden's Theorem. It is the two-diminsional case of the Gallai-Witt theorem.

Lemma 2.1 *Let* $c, M \in \mathsf{N}$. *Let* $COL^* : \mathsf{N} \times \mathsf{N} \rightarrow [c]$. *There exists* a, d, D *such that all of the following are the same color.*

$$\{(a + iD, d + jD) \mid -M \leq i, j \leq M\}.$$

Theorem 2.2 *Let* $k \in \mathsf{N}$. *Let* $COL : \mathsf{N} \rightarrow \mathsf{N}$ *be a coloring of the naturals. One of the following two must occur.*

- *There exists a monochromatic* k*-AP.*
- *There exists a rainbox* k*-AP.*

Proof:

Let COL^* be the following *finite* coloring of $\mathsf{N} \times \mathsf{N}$. Given (a, d) look at the following sequence

$$(COL(a), COL(a + d), COL(a + 2d), \ldots, COL(a + kd)).$$

(Yes- we need to look at $k + 1$ long sequences.)

This coloring partitions the numbers $\{0, \ldots, k\}$ in terms of which ones are colored the same. For example, if $k = 3$ and

$$(COL(a), COL(a + d), COL(a + 2d), COL(a + 3d)) = (R, B, R, G)$$

then the partition is $\{\{0, 2\}, \{1\}, \{3\}\}$. We map (a, d) to the partition induced on $\{0, \ldots, k\}$ by the coloring. There are only a finite number of such partitions (actually the number of them is the kth Bell Numbers).

Example 2.3

1. Let $k = 10$ and assume

   $$(COL(a), COL(a + d), \ldots, COL(a + (9d)) = (R, Y, B, I, V, Y, R, B, B, R).$$

   Then (a, d) maps to $\{\{0, 6, 9\}, \{1, 5\}, \{2, 7, 8\}, \{3\}, \{4\}, \}$.

2. Let $k = 6$ and assume

   $$(COL(a), COL(a + d) \ldots, COL(a + (5d)) = (R, Y, B, I, V, Y).$$

   Then (a, d) maps to $\{\{0\}, \{1\}, \{2\}, \{3\}, \{4\}, \{5\}\}$.

Let M be a constant to be picked later. By Lemma 2.1 There exists a, d, D such that all of the following are the same COL^*

$$\{(a+iD, d+jD) \mid -M \le i, j \le M\}.$$

There are two cases.

Case 1: $COL^*(a,d)$ is the partition where the last k elements all go into a class by themselves. (we do not care what happens to the first element). This means that there is a rainbow k-AP and we are done.

Case 2: There exists $x, y \neq 0$ such that $COL^*(a,d)$ is the partition that puts $a+xd$ and $a+yd$ in the same class. (We needed to use k instead of $k-1$ so that we would obtain, in this case, $x, y \neq 0$.) More simply, $COL(a+xd) = COL(a+yd)$. Since for all $-M \le i, j \le M$,

$$COL^*(a,d) = COL^*(a+iD, d+jD).$$

we have that, for all $-M \le i, j \le M$,

$$COL(a+iD+x(d+jD)) = COL(a+iD+y(d+jD)).$$

Assume that $COL(a+xd) = COL(a+yd) = R$. Note that we do not know what the color of $COL(a+iD+x(d+jD))$ or $COL^*(a+iD+y(d+jD))$ is, just that they are the same.

We want to find the (i,j) with $-M \le i, j \le M$ such that $COL^*(a+iD, d+jD)$ affects $COL(a+xd)$.

Note that

if

$$a+xd = a+iD+x(d+jD)$$

then

$$xd = iD + xd + xjD$$

$$0 = iD + xjD$$

$$0 = i + xj$$

$$i = -xj.$$

Hence we have that

$$a+xd = (a-xj) + x(d+jD).$$

So what does this tell us? In the equation

$$COL(a+iD+x(d+jD)) = COL(a+iD+y(d+jD)).$$

Let $i = -xj$ and you get

$$COL(a - xjD + x(d + jD)) = COL(a - xjD + y(d + jD)).$$

$$R = COL(a + xd) = COL(a + yd + j(yD - xD)).$$

This holds for $-M \le j \le M$. Looking at $j = 0, 1, \ldots, k-1$, and letting $A = a + yd$ and $D' = yD - xD$, we get

$$COL(A) = COL(A + D') = COL(A + 2D') = \cdots = COL(A + (k-1)D') = R.$$

This yields an monochromatic k-AP.

What value do we need for M? We want $j = 0, 1, \ldots, k-1$. We want $i = -xj$. We know that $1 \le x \le k$. Hence it suffices to take $M = k^2$. ∎

References

[1] P. Erödos and R. Graham. *Old and New Problems and results in Combinatorial Number Theory*. Academic Press, 1980. book 28 in a series called *L Enseignement Math*. This book seems to be out of print.

[2] R. Graham, B. Rothchild, and J. Spencer. *Ramsey Theory*. Wiley, 1990.

[3] H. J. Prömel and V. Prömel. An elementary proof of the canonizing version of Gallai-Witt's theorem. *JCTA*, 42:144–149, 1986. `http://www.cs.umd.edu/~gasarch/vdw/vdw.html`.

[4] B. van der Waerden. Beweis einer Baudetschen Vermutung. *Nieuw Arch. Wisk.*, 15:212–216, 1927.