The Book Review Column[1]

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In this column we review the following books.

1. **Erdős on Graphs: His Legacy of Unsolved Problems** by Fan Chung and Ron Graham. Review by Andy Parrish. Paul Erdős has posed an incredible number of tantalizing problems in fields ranging from number theory to geometry to combinatorics. This is a comprehensive collection of his problems.

2. **Roots to Research** by Judith D. Sally and Paul J. Sally, Jr. Review by Eowyn Cenek. This book traces the development and presentation of five mathematical problems from the early grades of elementary school to, in some cases, upper graduate level mathematics.

3. **Chromatic Graph Theory** by Gary Chartrand and Ping Zhang. Review by Vance Faber. This book is survey of Graph Theory from the point of view of colorings. Since coloring graphs has been one of the motivating forces behind the development of graph theory, it is natural that coloring can be used as a consistent theme for an entire textbook.

4. **Applied Combinatorics** by Fred S. Roberts and Barry Tesman. There are two reviews: one by Dimitris Papamichail, and one by by Miklós Bóna. This is an intro Combinatorics book but, as the title says, has more applications than most. It is also longer than most.

5. **Combinatorics, A Guided Tour** by David R. Mazur. Review by Michaël Cadilhac. This is an undergraduate text in combinatorics; however, as the title suggests, it's good for self-study as well.

6. **Famous Puzzles of Great Mathematicians** by Miodrag S. Petkovi¸ Review by Lev Reyzin. This book contains a nice collection of recreational mathematics problems and puzzles, problems whose solutions do not rely on knowledge of advanced mathematics. These problems mostly originated from great mathematicians, or had at least captured their interest, and hence the title of the book.

7. **Combinatorics – A Problem Oriented Approach** by Daniel A. Marcus. Review by Myriam Abramson. This book is a collection of problems laid out in such a way to be good for self-study— for a very motivated student.

8. **Probability: Theory and Examples** by Rick Durrett. Review by Miklós Bóna. This is the fourth edition of a classic text on probability. This is a serious text meant for advanced study.

9. **Games, Puzzles, & Computation** by Robert A. Hearn and Erik D. Demaine. Review by Daniel Apon. Given a game and a position in a game we want to determine which player will win. How hard is this problem? This book gives a framework to prove many results on the complexity of games.

BOOKS I NEED REVIEWED FOR SIGACT NEWS COLUMN

1. *Introduction to Bio-ontologies* by Robinson and Bauer.
2. *The Dots and Boxes Game: Sophisticated Child's play* By Berlekamp.
3. *New Mathematical Diversions* by Martin Gardner.
4. *The Magic Numbers of the Professor* by O'Shea and Dudley.
5. *Perspectives on Projective Geometry: A Guided Tour through Real and Complex Geometry* by Richter-Gebert.
6. *History of Mathematics: Highways and Byways* by Amy Dahan-Dalmedico and Jeanne Peilfer.
7. *Random walks and diffusion on graphs and database* by Blanchard and Volchenkov.

Review of[2]
Erdős on Graphs: His Legacy of Unsolved Problems
by Fan Chung and Ron Graham
A K Peters/CRC Press, 1998
142 pages, Softcover

Review by
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Paul Erdős has posed an incredible number of tantalizing problems in fields ranging from number theory to geometry to combinatorics. Erdős' problems have helped to shape mathematical research, but there seems to be no comprehensive collection of his problems.

Erdős on Graphs is an effort by Fan Chung and Ron Graham to collect some of his problems — those about graphs — which remain unsolved. (Full disclosure: I am a student working with both Chung and Graham, but I believe this review to be fair.) The problems are divided up into chapters by subject: Ramsey theory, extremal graph theory, coloring/packing/covering, random graphs, hypergraphs, and infinite graphs. For each chapter, the authors give relevant definitions, and then dive into the problems.

Since the book is focused on open problems, proofs are rare. However, each problem is followed by whatever progress has been made[3], by whom, and occasionally hints at the techniques used. Of course references are abundant. Since "Uncle Paul" liked to offer cash for solutions to his favorite problems, those awards are listed as well.

The book ends with three personal stories about Erdős from his longtime friend Andy Vázsonyi.

By my count, there are over 170 problems of Erdős discussed in the book, underlining his prowess in formulating them.

In this review, I will discuss the chapters, and some individual problems, with the hope of conveying the breadth and depth of the problems asked.

Ramsey Theory

Graph Ramsey theory explores edge-colorings of large graphs, looking for monochromatic subgraphs. For instance, Ramsey's theorem states that there is a least number $r(k, \ell)$ so that, whenever the edges of the complete graph K_N are colored red and blue (with $N \geq r(k, \ell)$), there is either a red k-clique or a blue ℓ-clique. It is well-known that $2^{n/2} \leq r(n, n) \leq 2^{2n}$. Some of the first questions posed in this book are about the true behavior of $r(n, n)$. Namely, does $\lim r(n, n)^{1/n}$ exist, and what is its value? These infamous questions are valued at \$100 and \$250 respectively. But Erdős also asks some perhaps more approachable questions. For example, Burr and Erdős asked about the local growth of Ramsey numbers: is $r(n+1, n) > c \cdot r(n, n)$ for some fixed $c > 1$.

Similarly, Erdős and Sós wondered about the growth of $r(3, n)$. Kim and Ajtai-Komlós-Szemerédi showed that $r(3, n) = \Theta\left(\frac{n^2}{\log n}\right)$, but it remains unknown whether the difference between

[3] Because the book came out in 1998, many problems have had further progress, some no doubt *because of* the book.

successive values $r(3,n+1)-r(3,n)$ tends to infinity. (It is easily shown that the difference is less than n.)

There is a more general Ramsey function. Given two graphs G and H, let $r(G,H)$ denote the smallest integer so that, for $N \geq r(G,H)$, every red/blue-coloring of the edges of K_N contains either a red G or a blue H. This is the graph-Ramsey function. Erdős asked about some very specific bounds to get a handle on this. For example, is $r(C_m,H) \leq 2n+\lceil (m-1)/2 \rceil$, where $m \geq 3$ and H is a connected graph with n edges and no isolated vertices? There are many questions of this form — a particular special case which may help the general problem of understanding $r(G,H)$.

There is has been some scattered progress on several of these problems since the book appeared in 1998. Markedly, Alon and Rödl proved that $r(3,3,n) = \Theta(n^3 \text{polylog } n)$, strongly affirming the conjecture of Erdős and Sós that $r(3,3,n)/r(3,n) \to \infty$. (Here $r(a,b,c)$ refers to the 3-color Ramsey number, guaranteeing either a red K_a, a blue K_b, or a yellow K_c.)

Extremal graph theory

The canonical example of a result in extremal graph theory is Turán's theorem. Let $T_{n,k}$ denote the complete k-partite graph on n vertices, whose parts each have size either $\lfloor \frac{n}{k} \rfloor$ or $\lceil \frac{n}{k} \rceil$. Turán's theorem states that any K_{k+1}-free graph G on n vertices has at most as many edges as $T_{n,k}$, with equality only when $G \cong T_{n,k}$. More generally, for a fixed graph H, $t(n,H)$ is used to denote the maximum number of edges in an H-free graph on n vertices[4].

Erdős was very fond of extremal problems, and the book gives a short account of his "near miss" of discovering the field in 1938, two years before Turán.

The Erdős-Simonovits-Stone theorem gives the asymptotics for general graphs: $t(n,H) = \left(1 - \dfrac{1}{\chi(H)-1}\right)\binom{n}{2} + o(n^2)$, essentially settling the question of $t(n,H)$ except in the bipartite case, when the bound becomes $o(n^2)$. A key conjecture then is that $t(n,K_{r,r}) > c_r n^{2-1/r}$ for $r \geq 2$, where $c_r > 0$ is some constant depending only on r. This would match the upper bound given independently by Kővári-Sós-Turán and Erdős. It has been proven for $r = 2,3$.

Erdős and Simonovits propose the behavior of the Turán number for even cycles: $t(n,C_{2k}) \geq cn^{1+1/k}$, to complement the known upper bound of $ckn^{1+1/k}$. The lower bound is known for $k = 2,3,5$.

Erdős and Simonovits also proposed a more general bound on the Turán number for a bipartite graph H: $t(n,H) = O(n^{3/2})$ if and only if every subgraph of H has some vertex of degree ≤ 2. Erdős offered \$250 for a proof, or \$100 for a counterexample.

Some further problems in this chapter, getting away from Turán numbers, ask whether all connected graphs on n vertices can be edge-partitioned into at most $\lfloor (n+1)/2 \rfloor$ paths (Erdős-Gallai), and whether every graph with $1 + \lfloor n^2/4 \rfloor$ edges has $2n^2/9$ of its edges each within some pentagon (Erdős).

Coloring, packing and covering

A graph G is k-colorable of it is possible to paint all vertices of G using up to k colors so that no neighboring vertices are the same color. The chromatic number of a graph G, denoted by $\chi(G)$, is the smallest value k so that G is k-colorable.

[4] Some texts call this number $\text{ex}(n,H)$.

Although intuition suggests that graphs with large chromatic number should have many edges within the neighborhood of each vertex, Tutte found a family of graphs with arbitrarily large chromatic number, yet no triangles. Erdős improved this result by proving that, for any k and n, there is a graph on n vertices with chromatic number k and girth[5] at least $\frac{\log n}{4 \log k}$.

Letting $g_k(n)$ be the largest girth of a graph on n vertices with chromatic number at least k, Erdős asked whether g_k grows smoothly — does $\lim \frac{g_k(n)}{\log n}$ exist, for k fixed? The above result, together with an upper bound by Erdős, show that the limit would have to be between $\frac{1}{4 \log k}$ and $\frac{2}{\log(k-2)}$.

Erdős and Lovász asked whether, if $a+b=k+1$, and G has chromatic number k but is K_k-free, does G contain two vertex-disjoint subgraphs with chromatic numbers a and b?

Beyond chromatic number, there are various problems asking about chromatic index, list-coloring, and acyclic chromatic number.

For a taste of a packing problem, I'll briefly mention the final question of this chapter, which seems particularly tantalizing. Does every graph G with $n(n+1)/2$ edges (ignoring the number of vertices) have an "ascending subgraph decomposition?" This is a decomposition into n edge-disjoint subgraphs $G_1, \ldots, G_n$ so that G_i is isomorphic to a proper subgraph of G_{i+1}. The number of edges of G requires that G_i has exactly i edges. For example, it is easy to see that K_{n-1}, which has $n(n+1)/2$ edges, may be decomposed as a sequence of star graphs.

Random graphs and graph enumeration

Erdős led the way in studying random graphs. He gave some of the first results in (deterministic) graph theory. Then, working with Rényi, he explored the evolution of the random graph. The primary model, though not what they studied, is $G_{n,p}$, where there are n labeled vertices, and each edge appears with probability p. Chung and Graham recount the fascinating details of how $G_{n,p}$ changes while p grows as a function of n. I will mention a few problems of Erdős for $p = \frac{1}{2}$. In this case, $G_{n,p}$ is simply called the "random graph," since it is uniformly chosen from all possible graphs on n vertices.

It is known that the chromatic number of the random graph is asymptotically $n/\log n$. How well concentrated is this? Shamir and Spencer showed that there is an interval of length $O(n^{1/2})$ so that, as n grows, the probability of the chromatic number being in this range tends to 1. Erdős asks us to prove that the "true" range is $\omega(1)$. In contrast to what we believe here, the range for the clique number of the random graph contains only 1 or 2 values (depending on n).

A very nice conjecture proposed by Erdős and Bollobás is that the random graph on 2^d vertices contains a d-dimensional hypercube with probability tending to 1. Alon and Füredi showed that the conjecture holds when the edge probability is greater than $\frac{1}{2}$.

This chapter also poses questions about subgraph enumeration — counting subgraphs of a certain type. For example, Erdős and Simonovits asked whether every graph on n vertices and $t(n, C_4)$ edges contains at least two copies of C_4, rather than just the one guaranteed. If we ask the same C_4 by a triangle, Rademacher showed that there are actually $\lfloor n/2 \rfloor$ triangles in such a graph.

Another question along these lines: Ramsey's theorem guarantees that any graph on n vertices contains either an empty or complete subgraph on $c \log n$ vertices. Erdős, Fajtowicz, and Staton ask whether we can find a larger induced subgraph, if we allow any regular graph.

[5] The girth of a graph is the length of its shortest cycle.

Hypergraphs

A hypergraph has the same structure as a graph: there is a set of vertices, and edges connecting those vertices. The only difference is that the edges are now allowed to be any size. Most often considered are uniform hypergraphs, where all edges have the same size. For example, graphs are exactly the 2-uniform hypergraphs.

Many questions asked about graphs can be asked just as easily for hypergraphs (though answering them may be a different story). For example, the Turán problem of maximizing edges in a graph while avoiding a particular subgraph translates directly: for an r-regular hypergraph H, let $t(n, H)$ be the maximum number of edges r-regular hypergraph avoiding H. It is easy to show that $\lim t(n, H)/\binom{n}{r}$ exists, but the limit is not known for any $r \geq 3$. Though Turán posed the question, Erdős offered $1000 for its resolution.

Relating to a problem mentioned above, Erdős conjectured that every 3-uniform hypergraph on n vertices with more than $t(n, K_k^{(3)})$ edges must contain $K_{k+1}^{(3)}$ minus an edge. Here $K_k^{(3)}$ refers to the 3-uniform hypergraph on k vertices containing all possible edges. This question is among many other extremal questions about hypergraphs posed in the book.

Much of the rest of the chapter involves hypergraph coloring. A proper coloring of a hypergraph is a vertex-coloring so that no edge is monochromatic. As for graphs, $\chi(H)$ is the chromatic number — the minimum number of colors needed to color H properly. Erdős and Lovász discovered that any 3-chromatic r-uniform hypergraph always has at least three different sizes of edge-intersections (for r large enough). They ask whether the number of sizes increases without bound as r grows, and suggested that the true number is $r - 2$.

Infinite graphs

Erdős did a lot of work on infinite graphs, and several of these results are mentioned by Chung and Graham. Many of the open problems in this section are on infinite Ramsey theory, so we just mention the one most valued by Erdős. Let ω be the smallest infinite ordinal. For which ordinals α does it hold that every 2-coloring of the complete graph on ω^α vertices must either contain a red clique on ω^α vertices, or a blue triangle. Equivalently, for which α does every triangle-free graph on ω^α vertices contain an independent set of size ω^α? Erdős offered $1000 for the solution.

Stories of Erdős

The reader who works through all the problems, rather than jumping around the book, with likely finish with the sense that Erdős was utterly devoted to these problems (nevermind his rich work in other fields). It is therefore fitting that the book closes with Andy Vázsonyi's stories of Erdős. As teenagers, the first time they met, Erdős demanded a four-digit number, which he then squared, and followed it up by proving that the reals are uncountable. As the stories go on, we learn that Erdős never held a job, owned almost no possessions, and was in fact homeless, relying on colleagues to give him a place to stay for as long as they could tolerate him. Though much of this is common knowledge, reading the stories makes it seem far more real. It makes clear just how incredible a person he must have been to live and thrive this way.

Opinion

This book has two main sources of value. Of course it is a great repository of open problems. Those listed above are a quick sampling of the many posed and described in the book. In collecting them all in one place, Chung and Graham have done a service to anyone interested in Erdős' style of graph theory who is looking for a difficult problem to work on. Though the book gives the necessary definitions, it is mostly geared toward people already familiar with the topics discussed, so it would not make a good stand-alone reference to graph theory. However, a mathematician with a light background in graph theory should be able to at least make sense of most of the problems, and begin to think about how to approach them.

For those planning to work on these problems, I should mention that three are no longer open. I already mentioned that Alon and Rödl showed that $r(3,3,n)$ grows much faster than $r(3,n)$. Additionally, Gerken showed that every large set of non-collinear points in the plane contains an empty hexagon with none of the other points in its interior. This closed a question related to the Erdős-Szekeres theorem, Erdős' first foray into graph theory. Additionally, Lu found a relatively small graph with no 4-clique, with the property that any 2-coloring of the edges creates a monochromatic triangle, earning $100 by reducing the number of vertices from 3 billion to 9697. Erdős asked whether a million vertices was enough. (Dudek-Rödl independently reduced the number to 130,000).

The other source of value of the book is that it helps give an understanding of Erdős the man. In detailing some of his results, many of his problems, and a bit about his life, I think any reader will learn a great deal about what allowed this hero of mathematics to succeed so spectacularly.

Review of[6] of
Roots to Research
by Judith D. Sally and Paul J. Sally, Jr.
American Mathematical Society, 2007
333 pages, hardcover

Review by
Eowyn Cenek (eowyn.cenek@eagles.usm.edu)

1 Introduction

"Roots to Research" traces the development and presentation of five mathematical problems from the early grades of elementary school to, in some cases, upper graduate level mathematics. In contrast to most mathematical texts, the development is vertical; each of the problems is first presented at a fairly low level, and then developed using more powerful mathematical tools. As such the authors introduce a variety of mathematical fields by showing how these specific problems can be addressed using tools from those fields.

2 Summary

Each of the five mathematical problems is developed in its own self-contained chapter, including bibliography; the chapters can be read in any order with only minor adjustments. The five problems studied include:

- **The Four Numbers Game**: start with a square whose four vertices are each marked with a non-negative integer. Mark the midpoint of each side, and label it with the absolute difference of the two endpoints. Then connect the midpoints to draw a new side. Continue doing so; the game ends when all four vertices of the last square are labeled with zeroes.

  The game is analyzed as to its length (or number of new squares drawn), whether two games are identical, and whether it is possible to construct games of arbitrary length. In the process, the student is exposed to polynomial algebra over a finite field, characteristic polynomials and eigenvalues in linear algebra, and elementary probability distributions.

- **The Congruent Number Problem**: a rational right triangle is a triangle whose sides have lengths that are positive rational numbers, and a congruent number is a positive integer that is the area of a rational right triangle.

  Beginning with a study of Pythagorean triangles and triples, number theory is introduced in the characterization of integers that can be written as the sum of two squares of non-negative integers. The theory is then extended as rational right triangles are introduced and the connection between elliptic curves and congruent numbers is traced.

- **Lattice Point Geometry**: lattice points are exactly those points in the plane that have integer coordinates.

  Readers are first introduced to the lattice in elementary school, using it to study geometric shapes. Rather than studying a specific problem, this chapter studies applications of problems that are built on a lattice. As such, the problems build from the simple "Can this polygon be constructed as a plane lattice polygon?" to results including Pick's theorem on the number of lattice points inside and on the boundary of a polygon. A further extension to Gauss' theorem on lattice points in and on a circle follows, and in turn is extended to a study of convex regions in the plane. Lastly, integer lattices in higher dimensions are discussed, and Ehrhardt's generalization of Picks' theorem to higher dimensions is proved.

- **Rational Approximation**: the approximation to within distance δ of a real number α by a rational number $\frac{p}{q}$.

  A first approach in approximating α, after observing that q grows very large as the approximation comes close to α, is to bound q by η. The problem is that no matter how large we choose η, there will be an open interval around α in which no $\frac{p}{q}$ satisfying the bound $q \leq \eta$ exists. However, if we let the bound depend on q, so that we want to satisfy $|\alpha - \frac{p}{q}| \leq \frac{1}{q^2}$, then by Dirichlet's theorem we can find such a number, and thus the rationality or irrationality of a real number can be determined by means of rational approximation.

  The bound is further expanded, by making the exponent variable and setting the bound to $\frac{1}{q^t}$. Liouville's theorem is proved three ways, and leads to a discussion of the work done by Thue, Roth, and Siegel. The discussion focuses on applications of their work, rather than proofs.

- **Dissection**: an object is cut into disjoint pieces, which are then analyzed so that conclusions can be drawn about the object as a whole. This is first introduced in elementary school, where students are encouraged to dissect geometric figures into simpler components to calculate the area of the whole. Bolyai-Gerwien's theorem, which states that for any two polygons having the same shape, one can be cut into convex polygons and rearranged to form the other, is proven, followed by a transition into three dimensions and a counter example showing that the same is not true for all pairs of polyhedra with equal volume.

  Introducing set-theoretic dissection, the Hausdorff and Banach-Tarski paradoxes are covered, including the result, implied by the Banach-Tarski paradox, that any solid ball B can be dissected set-theoretically into a finite number of pieces which can be rearranged to form two balls, each of which is congruent to B.

  Lastly, two problems are included: "squaring the circle", is introduced, although the proof is not included, and Borsuk's problem, which asks for the minimum number of pieces required so that every set of diameter d in R^n can be decomposed into that number of sets of smaller diameter.

3 Opinion

I enjoyed reading the book, and seeing how relatively simple problems could be expanded, and then solved. However, according to the foreword the book is intended for students and teachers as an illustration of how simple problems can be studied at greater depths. As a teacher's resource, the book felt somewhat lacking. The exercises read almost as an afterthought in many cases, and commonly there are only one or two per section. There is also little historical development; it is unclear, for instance, whether elliptic curves were developed whilst looking for congruent numbers, or whether their usefulness was discovered after they had been used for other topics. The introduction in each chapter does mention briefly when the problem was first studied, but not why. When teaching, I am often asked how the original researchers came to study (and prove) the results we cover in class; this book introduces interesting results but does not provide a useful answer to that question. Thus, while reading the book does suggest that from simple questions interesting research can spring, it does not provide a suggestion of a roadmap for the aspiring scientist. When I lent the book to a local high school student, I ended up lending him my copy of Polya's "How to Solve It" as a follow-up.

I would recommend the book for mathematics students at all levels, with the caveat that they read it to see how any problem can be expanded and studied in greater depth, rather than as a specific study of one subject.

3.1 Quibbles

The typesetting is sometimes disconcerting; the book is clearly written as five separate papers, each which their own bibliography, using LaTeX, but final proofreading did not resolve spacing or numbering issues. There are blank spaces in the middle of pages, as text is stretched out – presumably – to adjust to a figure that lies on the following page. The numbering is confusing, as lemmas, corollaries, observations, etc share the same counter as the exercises. Thus, when a section includes exercises, those listed do not begin with exercise 1 but with some later number. For instance, section 5 of chapter 3 includes the exercises 5.12, 5.13, and 5.14, but no problems 5.1 through 5.11. The exercises are sometimes set apart with an "Exercises" heading, but not always. These make it hard to find the exercises quickly.

Review of[7] of
Chromatic Graph Theory
by Gary Chartrand and Ping Zhang
Published by the Chapman and Hall, 2008
504 pages, Hardcover, $84.00 on amazon
Review by
Vance Faber vance.faber@gmail.com

1 Introduction

This book is survey of Graph Theory from the point of view of colorings. Since coloring graphs has been one of the motivating forces behind the development of graph theory, it is natural that coloring can be used as a consistent theme for an entire textbook.

Graph Theory is one of the most accessible topics in mathematics. A graph is nothing more than a set (the vertices) and a collection of pairs of elements (the edges). A coloring is equally simple - just a partition of the pairs. Chromatic Graph Theory is a testimony to how universal, powerful and motivating this simple concept can be.

2 Summary

The book starts with a very engaging history of the origin of graph colorings which is primarily the Four Color Problem. This problem dates back to 1852 and was not solved until 1976. The proof, when it finally came, was extremely long and complicated and involved using the computer to check that a long list of graphs have a certain property. The next five chapters lay out the fundamentals of Graph Theory including trees and connectivity, Eulerian and Hamiltonian graphs, matchings and factorizations, and graph embeddings. The last nine chapters, besides covering most if not all of the coloring theorems that are well-known to experts in the field, delve into some of the more esoteric problems connected to coloring. Unlike many other books of this nature, most of the stated theorems have proofs that are given in great detail— most likely more detail than in the original papers. In addition, the book is quite scholarly with great attention to detail in its attributions of concepts, conjectures, theorems and proofs. (There are 195 numbered references and 7 pages of unnumbered references.) There are a wealth of exercises at the end of each chapter and 14 open-ended study projects in the Appendix. For example, Study Project 9 says: "Investigate set colorings of graphs."

Here are some of the major topics that are covered in the book:

1. Planar graphs and general embedding of graphs on surfaces
2. Graph Minor Theorem
3. Perfect Graph Theorem and Strong Perfect Graph Theorem
4. Brooks's Theorem
5. Vizing's Theorem

6. Heawood map coloring problem
7. List colorings
8. Total colorings
9. Ramsey Numbers
10. Rainbow colorings
11. Road coloring problem
12. Complete colorings
13. Distinguishing colorings
14. Channel assignment problem
15. Domination

3 Expanded summary

The book has 15 chapters. Chapters 1-5 are essentially a beginning course in graph theory while the other chapters focus on graph coloring. Here is a summary of all the chapters.

3.1 Chapter 0. The Origin of Graph Coloring

This chapter outlines the history of the Four Color Theorem starting with the initial conjecture by Francis Guthrie in 1852 to the computer aided proof by Kenneth Appel and Wolfgang Haken. Along the way there were several celebrated false proofs, including an incorrect proof by Alfred Kemp in 1879. Although Percy Heawood showed, in 1890, that the proof was incorrect, Kempe's technique did allow one to show that all planar graphs are 5-colorable. This technique, recoloring chains, is a staple of many coloring theorems. In Percy Heawood's paper he also generalized the conjecture to manifolds of higher genus (see below in the summary of Chapter 8).

3.2 Chapter 1. Introduction to Graphs

This chapter sets up the terminology to be used in the remainder of the book. It gives proofs of some of the fundamental identities and concepts of graph theory such as connectivity, isomorphism and graph operations that will be necessary to discuss general coloring problems.

3.3 Chapter 2. Trees and Connectivity

This chapter covers the concepts of vertex and edge connectivity. It also gives the proof of Menger's theorem which is the basis for the min-cut max-flow algorithm.

3.4 Chapter 3. Eulerian and Hamiltonian Graphs

Here we find the concepts of edge spanning and vertex spanning paths in graphs. It also defines the de Bruijn digraphs which consist of all words of length k in n letters where two words are considered adjacent if the first word can be converted into the second by moving the initial letter to the end. I used the same relation on words in my work on cycle prefix digraphs, graphs with fixed degree and very small diameter (see [1]).

3.5 Chapter 4. Matchings and Factorizations

Chapter 4 covers matchings and independence. Matchings naturally arise in coloring because each color class in an edge coloring yields a matching on the vertex set an each color class in a vertex coloring is an independent set. A factor of a graph is a spanning subgraph. A k-factor is a factor each of whose vertices has degree k. Every complete graph with an even number of vertices is 1-factorable. Every complete graph with an odd number of vertices is 2-factorable by spanning cycles. I explored the ability to 2-factor a complete directed graph with factors consisting of cycles of fixed length in [2]. A simple open conjecture on factoring graphs is the 1-Factorization Conjecture: If every vertex of a graph of even order n has r neighbors where $2r$ is greater than or equal to n, then the edges of the graph can be partitioned into 1-factors.

3.6 Chapter 5. Graph Embeddings

Chapter 5 discusses embedding of graphs into two dimensional surfaces. Planar graphs are graphs which can be embedded into the sphere. The genus of a graph is the smallest genus of a surface which can support an embedding. The Robertson-Seymour Theorem is probably the most important theorem in the field of graph embeddings. A minor of a graph G is any graph which can be obtained by any sequence of the following three operations: edge deletion, vertex deletion or edge contraction. The Robertson-Seymour Theorem says that given any infinite sequence of graphs, there is one that is a minor of a later member of the sequence. A corollary of this theorem is the Graph Minor Theorem. A set S of graphs is minor-closed if for every graph in S, all of its minors are also in S. The Graph Minor Theorem says that for every minor-closed set S there is a finite set M of graphs such that a graph G is in S if and only if no graph in M is a minor of G. Note that the set of graphs of genus $\leq g$ is closed under minor. Hence we have the following corollary: In particular, for any fixed g, there is a finite set of graphs $\mathcal{F}$ such that for all graphs G, G is of genus $\leq g$ iff none of the graphs in $\mathcal{F}$ is a minor of G.

3.7 Chapter 6. Introduction to Vertex Colorings

In this chapter, vertex colorings are discussed. The chromatic number of a graph is the smallest number of colors required to color its vertices. The major theorem in this section is the Perfect Graph Theorem of Lovasz. The clique number of a graph is the size of the largest clique, a set of vertices all of which are connected by edges. A graph is perfect if every induced subgraph has chromatic number equal to its clique number. The Perfect Graph Theorem (originally conjectured by Berge) says that a graph is perfect if and only if its complement is perfect. An even stronger result is the Strong Perfect Graph Theorem (also conjectured by Berge) of Chudnovsky, Robertson, Seymour and Thomas: a graph is not perfect if and only if it or its complement contains an induced odd cycle of length more than 3.

3.8 Chapter 7. Bounds for the Chromatic Number

This chapter mentions that finding the chromatic number of a graph is NP complete. This justifies finding bounds for the chromatic number of classes of graphs. The prime example of a bound for the chromatic number is given by Brook's Theorem: unless a graph is an odd cycle or complete, it can be colored by the number of colors equal to its maximum degree.

3.9 Chapter 8. Coloring Graphs on Surfaces

This chapter takes us back to the setting for the Four Color Theorem. It discusses the relationship between the chromatic number of a graph and its genus. Not only did Heawood find a hole in Kempe's proof of the four color conjecture in his 1890 paper but he produced an upper bound for the chromatic number of any graph embedded in a surface of genus g greater than zero (note for g = 0, this is the four color theorem):

$$\chi \leq \left\lfloor \frac{7 + \sqrt{1 + 48g}}{2} \right\rfloor.$$

Heawood actually believed that he had proved that this bound was sharp but the next year Heffter found a hole in his proof. Proving that this bound is sharp is accomplished by finding the genus of complete graphs. The problem was settled for $g \geq 1$, in 1968 by Ringel and Youngs (The $g = 0$ case is the four color theorem, settled in 1976.) I discussed upper bounds for the genus of complete graphs when all faces are required to have at least n sides in [3] and [4]. Whether these bounds are sharp is still open.

3.10 Chapter 9. Restricted Vertex Colorings

In this chapter, more technically restrictive types of colorings are discussed. For example, uniquely colorable graphs are graphs for which colorings in the minimum number of colors result in identical color classes. A graph is k-list-colorable if given lists of colors for each vertex with more than k colors, the graph can be colored using only the colors from the lists. A number of theorems on list colorings are discussed.

3.11 Chapter 10. Edge Colorings of Graphs

The smallest number of colors required to edge color a graph is called the chromatic index. The Four Color Theorem is equivalent to the fact that any planar bridgeless cubic graph has chromatic index three. Vizing proved that the chromatic index of any graph is never more than the maximum degree plus one. I gave an algorithm for creating this coloring in [5] which leads to a very short proof of the theorem. The authors have given an unnecessarily complicated proof. (For an easier one see a proof that was originally by Lovasz which Ehrenfeucht, Kierstead, and I present here http://books.google.com/books?id=mqGeSQ6dJycC&pg=PA465&lpg=PA465&dq=vizing+theorem+proof+f
DrThW3fe4wn3PXYR8k69s&hl=en&ei=0D4nTaubC4zEsAOWl_ibBw&sa=X&oi=book_result&ct=result&res

The chapter also discusses list edge colorings and total colorings of graphs.

3.12 Chapter 11. Monochromatic and Rainbow Colorings

The main focus of this chapter is Ramsey's Theorem and specifically the classical case where there are only two colors. In this case, edges are colored without regard to whether or not edges of the same color share a vertex. The prototype Ramsey result is that given a coloring of the edges of any complete graph with at least six vertices by two colors, red and blue, there is either a red or a blue triangle. More generally, for all k there is an n such that for any 2-coloring of the edges of K_n there is a *monochromatic* (hence the name of this subsection) K_k. If $m < n$ then there is a 2-coloring such that there is no monochromatic K_k. This is called a *rainbow coloring.*

Also discussed is Turan's Theorem. A prototype Turan result is if more than $n^2/4$ of the edges of a complete graph with n vertices are colored red then there is a red triangle.

3.13 Chapter 12. Complete Colorings

A complete k-coloring of a graph is a vertex coloring with by k colors the property that every two distinct colors appear on some edge. Clearly any k coloring of a k-chromatic graph is a complete coloring and so the chromatic number is the smallest k for which there is a complete k-coloring. Some graphs can have complete k-coloring for values of k larger than the chromatic number. The largest k is called the achromatic number of the graph. This chapter also discusses graph homomorphisms and isomorphisms, specifically in the context of complete colorings.

3.14 Chapter 13. Distinguishing Colorings

This chapter discusses vertex labellings which may or may not be colorings. These labellings can be vertex-distinguishing (different vertices have different labels), edge-distinguishing (if edges have distinct pairs of labels) or neighbor-distinguishing (two adjacent vertices have distinct labels that is a coloring). There are also analogous definitions for edge colorings. Colorings which are distinguishing labellings are called distinguishing colorings. A number of theorems an open conjectures concerning distinguishing colorings and other labellings are given. The best known and most widely studied is the concept of the graceful labeling introduced by Rosa. A graceful labeling of a graph with m edges is a labeling of the vertices with numbers from 0 to m such that the absolute differences of the labels on each edge are distinct. Still open is the graceful tree conjecture that all trees have a graceful labeling.

3.15 Chapter 14. Colorings, Distance, and Domination

This chapter is motivated by the Channel Assignment Problem. In this model problem, the vertices of the graph are transmitters and the edges are pairs of transmitters that can not share the same channel frequency. A coloring of the graph assigns channels to the transmitters so that two transmitters dont interfere. However, often the channels interfere even if the frequencies are distinct if the frequencies differ by a particular amount. Thus the Channel Assignment Problem asks for a coloring so that the absolute difference of colors on the same edge avoids a given set. The last topic in the book is dominating sets. A dominating set is a set S of vertices so that every vertex is either in S or adjacent to a vertex in S. The domination number of a graph is the minimum cardinality a dominating set. The domination number is useful in a wide array of applications including coding theory and parallel computer design.

4 Opinion

The intended uses for the book as stated in the Preface are:

1. as a course in graph theory, either at the beginning level or as a follow-up course to an elementary course in graph theory,
2. a reading course on graph colorings,
3. a seminar on graph colorings,
4. as a reference book for individuals interested in graph colorings.

By and large, the book is quite successful at carrying out its stated goals. I really enjoyed reading the book but it is not easy going if you want to read every word because of the tremendous detail. I even got caught up in trying my hand at a few of the open problems that I had not thought about for a long time. The fact that most of the stated theorems have proofs in great detail is something that most advanced undergraduate or beginning graduate students should appreciate. It presents a really good opportunity to see what a thorough proof looks like, including all the boundary cases which are often ignored or left to the reader. Anyone planning to use this book for a textbook should not expect to cover the whole book in a single semester, even if the first five chapters are just a review.

Although the book is very thorough, there are a few notable omissions that one might expect to find in a recent textbook. First, there is no mention of algorithms and only one mention of computational complexity and P = NP. So although there are a myriad of esoteric theorems and conjectures about coloring, there is nothing about how to use a computer to color. Second, there is nothing about Random Graph Theory and the powerful role it plays in finding bounds for many coloring problems. Third, there is nothing about hypergraphs. Of course, one can state just about any coloring problem on hypergraphs in terms of some graph problem but still I would have liked to have seen a little space devoted to these structures. I am sure that the authors faced some tough choices for what to include and what to exclude but I think that these topics deserve at least a little more space in a book of this depth.

I believe this book would make an excellent textbook for a second course in graph theory. It will also be quite useful as a reference book for graph coloring problems, especially in electronic form.

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Review of[8] of
Applied Combinatorics
by Fred S. Roberts and Barry Tesman
CRC Press, 2009
860 pages, HARDCOVER

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1 Introduction

Combinatorics is a broad area of mathematics, considered as a branch of discrete mathematics. It could be described as the study of arranging objects according to specific rules, resulting in patterns, designs, assignments, schedules and other configurations. Combinatorial problems arise in many areas in mathematics, such as algebra, probability, geometry and topology, while applications can be found in many disciplines, including computer science, statistical physics, industrial engineering, genetics, linguistics and many others.

Problems in combinatorics can be categorized according to their objective, which can include counting objects of a given kind, deciding whether or when certain criteria in some arrangement can be met, and constructing objects meeting the criteria, or being "best" according to some criteria. Following this paradigm, this book is divided into four parts. The first is an introduction to tools and definitions necessary to explore the rest of the material. The second is concerned with problems involving counting objects. The third deals with the existence of arrangements of a particular kind, while the fourth explores combinatorial optimization techniques. The chapters in each section of the book and their contents are discussed in the following section.

2 Summary

Chapter 1 This chapter addresses the definition of combinatorics, introducing the subject through a series of simple and diverse examples. It also includes a small history of the field.

Part I - Basic Tools of Combinatorics

Chapter 2 The most basic methods of counting are discussed in this chapter. It starts with the product and sum rules, derivations and formulas for permutations and combinations, sampling with and without replacement, and continues with the balls and bins paradigm, multinomial coefficients and the binomial expansion and algorithms for generating permutations and combinations. There is also an introduction to algorithmic concepts such as asymptotic analysis and the theory of NP-completeness. The chapter ends with the pigeonhole principle and some of its generalizations, including Ramsey numbers.

Chapter 3 The third chapter is an introduction to graph theory. It starts with the definitions of isomorphism, subgraphs and connected components. Significant number of pages are devoted to graph coloring, chromatic numbers/polynomials and trees, including applications of spanning trees, searching, sorting and phylogeny. Computer graph representations and Ramsey properties reduced to graphs are also included in this chapter.

Chapter 4 This chapter deals with binary relations, which are represented throughout the chapter as digraphs, as well as different relation properties, such as reflexivity, transitivity, etc. Order relations and variants are introduced based on the minimal set of properties that define them. Linear, weak and partial orders, chains, lattices and boolean algebras are devoted their own sections.

Part II - The Counting Problem

Chapter 5 Here the reader finds an introduction to generating functions, starting with the basics of power series. Different methodologies on using generating functions are presented, followed by applications to counting and the binomial theorem. Exponential and probability generating functions, a few theorems and some of their applications can be found at the end of this chapter.

Chapter 6 Chapter six deals with recurrences and big part of it uses material from the generating functions chapter. A very nice introduction explains recurrence relations simply, with numerous examples. The method of characteristic equations and their roots is presented next, followed by using generating functions to solve recurrences and some examples of recurrences involving convolutions. The problem of derangements is introduced here and used extensively in this and several following chapters.

Chapter 7 In many counting problems, we face the case of double-counting items with specific properties, which have to be excluded using set intersections. The principle of inclusion and exclusion generalizes this procedure and is applied in this chapter in a variety of counting applications.

Chapter 8 Counting configurations under equivalence relations is the subject of chapter 8. Definitions and examples of equivalence relations and classes are initially presented, followed by permutation groups and distinct colorings. Automorphisms in graphs and switching functions consist the major examples through which many of the concepts are explained. A special case of Pólya's theorem and the actual theorem with its proof are presented at the end.

Part III - The Existence Problem

Chapter 9 Block designs, with emphasis on latin squares and variants, such as orthogonal latin squares, is the topic of chapter 9. There is discussion about complete and incomplete block designs, modular arithmetic and finite projective planes, with cryptography dominating the application part of the chapter.

Chapter 10 In this chapter the reader is introduced to coding theory, ways to encode and decode messages for transmission. Linear codes such as the Hamming code, and error correcting codes, such as the Hadamard codes, which use block designs introduced in the previous chapter, dominate the material here.

Chapter 11 The existence problems section continues with graph theory, where several problems commonly encountered in algorithms books are addressed. These include testing a graph for connectedness, with techniques such as depth first search, finding Eulerian paths and chains, with variations including the Chinese postman and sequencing with hybridization, and the Hamiltonian path problem and its applications. One interesting problem that is not usually found in algorithms texts and is presented here is the one-way street problem, which deals with efficient one-way street assignments to grids and other street topologies.

Part III - Combinatorial Optimization

Chapter 12 The combinatorial optimization part of the book starts by describing a variety of matching problems, many of which are actually existence problems, such as bipartite and perfect matching. It continues with maximum matchings and minimum coverings and their relation, describes theorems and algorithms to find maximum matchings using augmenting chains and introduces the maximum weight matching problem. Stable matchings algorithms, counting their number and observations about their structure conclude this chapter.

Chapter 13 The last chapter of the book presents a number of algorithms for solving optimization problems. These include Kruskal's and Prim's minimum spanning tree algorithms, Dijkstra's shortest path algorithm, and augmenting chains and the max-flow algorithm for the network flow problem.

3 Opinion

"Applied Combinatorics" by Roberts and Tesman is quite possibly the closest to a "bible" in the field than any other text. In an area as vast as combinatorics it is obviously impossible to cover every possible branch and theory, but this book touches most of the important subjects in sufficient detail, especially the ones having immediate connections to applications in different fields. There is little theory in this book that is presented only for "theory's sake", without being escorted by some example or application.

This book is highly instructive. The writing is simple and concise, the intuition behind most of the concepts is explained in good detail and the reader is afforded ample background coverage, before diving into more elaborate theorems and examples. Very little non-basic algebra and/or calculus is assumed, and often references are provided to texts that explain needed prerequisites in more detail. Most chapters actually start with small introductions to their content, which include relevant bibliography.

The most interesting feature of this book is the relevant examples that accompany almost every section. The "applied" part of the title is sufficiently justified with a plethora of applications drawn from an impressive array of different areas, including scheduling, industrial design, information theory, genomics, phylogeny, and a wide variety of topics in computer science, just to name a few. This is probably the most unique characteristic of this book that sets it apart from other texts in the field and makes it particularly interesting and appropriate for undergraduate instruction. Certain concepts are actually demonstrated through a large number of diverse examples, providing instructors the option to choose the ones they prefer and are more relevant to their audience, while all examples are simple enough and include all necessary information to be understood by a general audience with elementary scientific background.

Another nice feature is the inclusion of "why" questions in parentheses next to statements that the reader is urged to verify, which provides immediacy and enhances active learning. This concept is further enforced by assigning several simple theorem proofs to exercises, also used as references, which serves a double purpose, saving space and motivating students to prove these theorems in order to understand and sufficiently cover the material. The exercises vary in difficulty from simple to challenging, making them appropriate for assignments at different undergraduate and graduate levels.

Given the wide variety of topics presented, the very approachable writing and the large number of useful references at the end of each chapter, this book will also make a great addition to the library of many scientists that may find themselves dealing with combinatorial problems in their research. The material is well structured, the subject and author indices comprehensive, and several section prerequisites are mentioned explicitly, making this book great as a reference text, in addition to a wonderful course textbook.

Review of[9] of
Applied Combinatorics
by Fred S. Roberts and Barry Tesman
CRC Press, 2009
860 pages, HARDCOVER

Review by
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There are many introductory combinatorics textbooks on the market. Some core material is part of all of them, hence this review will focus on what this book does differently from the rest.

The title "Applied Combinatorics" is correct. The book indeed has more applications than more competing textbooks, and many of them come surprisingly early in the test. In order to do this, the authors had to make some unusual and risky decisions as far as the sequence of the covered topics goes.

After a short introduction, the book really starts with Chapter 2 (Basic Counting Rules). Even this earliest chapter has advanced examples, of which we will mention two. There is an entire section on the inversion distance (also called the reversal distance) of permutations. Let us take a permutation written in the one-line notation, like 25671843, and let us say that one reversal on that permutation consists of taking a string of entries in consecutive positions (the string can be of any length), and reversing them while leaving the rest of permutation unchanged. The question is how many reversals it takes to sort a given permutation, that is, how many reversals are needed to turn a given permutation into the identity permutation. Another problem is to determine the set of permutations of a given length that require the highest number of reversals in order to be sorted.

The topic has its roots in evolutionary biology, and it is a fascinating one. This reviewer strongly supports the idea of including it in this book, though not in the first real chapter. The problem with placing it so early is that by doing so, the authors cannot afford any time to spend on other, equally important, biologically motivated sorting algorithms of permutations, such as sorting by block transpositions, block interchanges, transreversals, or cut-and-paste operations. In fact, the authors do not even mention these other sorting operations. Had this section been placed later in the book, perhaps some of them could have been included, especially because of their extremely intriguing open problems.

The other advanced topic in this early chapter that we want to mention is NP-complete problems. Again, that is a supremely interesting subject, and the reviewer strongly endorses the idea of including it in introductory combinatorics textbooks, but questions its early placement. Many of the most striking examples of NP-complete problems comes from graph theory, and at this point, the reader has not yet seen graph theory. In fact, the Pigeon-hole Principle is discussed after NP-completeness, though few would disagree with the claim that the former is the simpler concept.

Chapter 3 is a relatively traditional chapter on graph theory, with one section focusing on sorting and searching. Chapter 4, new to this edition, starts out with Relations (which the majority of students taking this class should now), and then goes to the advanced topic of partially ordered sets, and even a special kind of these sets, the so-called interval orders. Lattices and Boolean algebras are also mentioned, though not as thoroughly as in the more theoretically focused books. Möbius functions and the Möbius inversion formula is not discussed.

After this first, introductory part of the book, the rest is organized in three main parts, Counting Problems, Existence Problems, and Optimization Problems.

The part on Counting Problems starts, surprisingly, *first* on a chapter on Generating Functions, and *then* a chapter on Recurrences. This is unusual since Generating functions are useful in many parts of enumerative combinatorics, and solving of recurrence relations is only one of them. The examples in these two chapters are more application-oriented than in most textbooks, but the theory does not go quite as deep. The Product Formula, the Exponential Formula, and the Compositional Formula are not covered.

The last two chapters of the counting part of the book are two thorough chapters on the Inclusion-Exclusion Principle, and on Pólya theory. For instance, the former has a section on Rook Polynomials, which is usually only in graduate level textbook. These polynomials count the ways in which non-attacking rooks can be placed on a given board, which does not have to have the shape of a square. The chapter on Pólya theory is also quite extensive, and it goes as far as the Cycle Index.

Then comes the third part, the one on Existence problems. The first two chapters here are well-written and quite traditional chapters on Combinatorial Designs and Coding Theory. The third one, Existence problems in Graph Theory, raises a few questions. Earlier, this reviewer pointed out a few topics that he thought were covered too early in the book; now is the time to pay for that. The reviewer believes that this chapter comes too *late*. The book is clearly a two-semester textbook, but there will be many students who only take the first semester. If the instructor follows the order of topics in the book, then those students will not hear about some of the most classic problems of graph theory, such as the Königsberg bridge problem of Euler, the Hamiltonian cycle problem, or about tournaments. The reviewer does agree with the authors in that graphs should come after enumeration, but he thinks that they should come before designs and coding because they are more fundamental. This, of course, is a matter of taste. Otherwise, the chapter is interesting, with applications to RNA-DNA chains, and Sequencing by Hybridization.

The last part of the book is about Combinatorial Optimization. It contains two chapters, the first of which is on Matching and Covering. A matching in a graph G is a set of pairwise vertex-disjoint edges, while a perfect matching for G is a matching that covers all vertices of G. The authors discuss existence theorems for perfect matchings in both bipartite and arbitrary graphs, and algorithms to find the largest possible matching in both cases. Then they treat the more advanced problem of finding the matching with maximum weight in a graph. In this problem, each edge has a weight, and the weight of matching is the sum of the weights of the edges that form that matching. The famous subject of stable marriages is also discussed.

The last chapter is on classic optimization problems, such as finding the cheapest spanning tree in a graph (in this case, unlike for matchings, the greedy algorithm works when done right). Both Kruskal's and Prim's algorithms are covered. Another classic problem that is discussed is finding the shortest path between two points of a graph, which is handled by the somewhat more complicated algorithm of Dijkstra. The chapter ends with a longer section on Network flows, including the classic Max Flow - Min Cut theorem and Menger's theorem.

The exercises are placed at the end of each section. There are plenty of them. More than half of them have their answers included at the end of the book, but none come with full solutions.

On the whole, the book has plenty of interesting material, and numerous exercises. The applications are well-chosen. The increased weight given to applications comes at the price of not covering some topics which are in many similar textbooks, such as discrete probability, Möbius functions, and the Product, Exponential, and Compositional formulas for generating functions. Whether you want to teach from the book depends on whether you agree with these choices, as well as with sequence in which these topics are covered.

Miklós Bóna

Review of[10]
Combinatorics, A Guided Tour
David R. Mazur
Mathematical Association of America, 2010
xviii+391 pages, Hardcover

Review by Michaël Cadilhac michael@cadilhac.name

1 Overview

This book surveys, at an undergraduate to first-year graduate level, the three keystones of combinatorics: counting, existence, and construction questions, with an emphasis on the first. The material is quite standard in the first 5 chapters, and the last 3 focus on more diverse topics, namely graphs, combinatorial designs, error-correcting codes, and partially ordered sets. The numerous examples and questions offer a great number of original applications, even of the most standard concepts, and help the reader stay both focused and entertained. The main value of this book lies in its story-telling style, thus its name "A Guided Tour" — more, in my opinion, a "Guided Adventure."

Although its format makes it perfect for self-study, this book can be used for (first- or second-year) courses in combinatorics or discrete math, and both its tone and format can provide great teaching hints. Moreover, the author asserts most of the book can be covered in a semester, and gives a list of the sections he believes form the core of such a course.

Quantitatively, the book is roughly a quarter definitions and theorems, a half examples, and a quarter exercises. The examples precede *and* follow the formal portions, first leading naturally to the theorems, then exploring all their power. The exercises are of two kinds: the questions (~ 350 of them) spread throughout the text are little challenges, usually to make sure the concepts are understood, but also to give the opportunity to try the new tools; the exercises (~ 470 of them) at the end of the sections are more involved, and strengthen the notions just acquired. They are usually preceded by a summary of the section, and followed by "Travel Notes" that give a historical perspective on the material. A book on this subject should be read with paper and pencil, and the numerous questions along the text help the reader keep this in mind.

2 Summary

Chapter 1 introduces the base principles of counting: the product principle (leading to the number of lists of n elements taken from a set of k, according to whether the order matters and repetitions are allowed), the sum principle (with the idea of overcounting then subtracting the extra), the bijection principle (i.e., counting a bijective image of a set), the equivalence principle (a simpler version of Pólya's theorem where the equivalence classes are of the same size), and the pigeonhole principle (with the Erdős-Szekeres theorem as an application).

Chapter 2 presents the concept of combinatorial proof. It answers pretty much all the questions of the form "How many distributions of k distinct/identical objects to n distinct/identical recipients exist such that each recipient can receive any amount/less than 1/more than 1/exactly 1 object?" with applications to questions like "How many one-to-one functions $f\colon [4] \to [6]$ have $5 \notin \mathrm{Rng}(f)$?".

Chapter 3 presents the algebraic tools for combinatorics. It starts with the inclusion-exclusion formula, and mathematical induction. Then generating functions are studied (including the convolution formula and Euler's famous result stating that the number of partitions of n into distinct parts is equal to the number of partitions of n into odd parts). The chapter is concluded with techniques for solving recurrences, with an emphasis on linear recurrence relations.

Chapter 4 uses the preceding tools to derive "nice" formulas for famous number families. These include binomial and multinomial coefficients, Fibonacci and Lucas numbers, Stirling numbers (first and second kinds), and the integer partition numbers (using Ferrers diagrams).

Chapter 5's goal is to present Pólya's enumeration theorem. Basic group theory (including the theory of symmetric, dihedral, and cyclic groups) is presented before giving a proof of the Cauchy-Frobenius-Burnside theorem (a.k.a. Burnside's lemma), which gives a way to count the number of orbits. This chapter includes a great number of applications of this theorem. The last section comes short to proving Pólya's enumeration theorem but focuses instead on its applications.

Chapter 6 focuses on combinatorics on graphs. The counting portion of this chapter concentrates on labeled trees (with two proofs of Cayley's formula), binary search trees, and proper colorings. The last section introduces Ramsey theory, proving that $R(3,4) = 9$, i.e., that the smallest value of n so that every red-blue coloring of K_n contains either a red K_3 or a blue K_4 is 9. Finally, a lower bound for $R(a,a)$ is proved.

Chapter 7 describes combinatorial designs and error-correcting codes, and links the two. A combinatorial design is a pair $(V, \mathcal{B})$ such that V is a finite set of points and $\mathcal{B}$ is a multiset of nonempty subsets of V called blocks. It is said to be a (v,k,λ)-design if $v = |V|$, k is the number of points in each block, and each pair of distinct points appears together in exactly λ blocks. The study of such designs includes necessary conditions for a design to exist, given v, k, and λ, and construction methods for specific designs. In particular, Steiner triple systems (where λ has a specific value) are investigated. Then perfect binary codes are introduced, and, answering questions about those, a close relationship between designs and codes is unveiled.

Chapter 8 presents the contents of the first chapters in a new light: partially ordered sets. In a unifying effort, Sperner's theorem (giving the size of a Sperner family) is presented, together with Dilworth's theorem (partitioning the ground set of a partially ordered set into chains optimally). The grand finale of the book introduces the theory and applications of Möbius inversion, thus unifying a great deal of the topics of the preceding chapters (including the inclusion-exclusion formula and some Pólya-type problems).

3 Opinion

Allow me to give a feel of this book. Imagine a friend gives you a book which, he says, follows a quest-driven hero in a great story of humor, action, and adventures. He even warns you that you may spend whole nights reading it, unable to wait for the next day to see what happens next page. In short, this friend is thrilled.

Now you open the book, and find out that the world the hero lives in is... combinatorics — but wait! That doesn't mean the qualities of the novel just vanished! Read on, they didn't; this is what is so peculiar about this book.

This is one of the feelings I get from “Combinatorics, A Guided Tour.” The excitement of a great prose, where the numerous examples and the questions during the text give a pace rarely seen in math books.

It should be obvious at this point that I liked this book — I really enjoyed it. I feel this is a great book for independent study and self-teaching, from undergrad students to grad ones. It may be of great help to a student who is not sure to grasp the contents of their combinatorics class, and of great interest to anyone who wants to learn at a leisurely pace. Even though the book starts from the very ground, I am convinced that anyone can learn something from it — if not from its contents, at least from its style. On the other end, I would love to see a course given with the help of this book, especially as its bonhomie has a good chance to influence the teacher in a very positive way.

Review[11] of
Famous Puzzles of Great Mathematicians
by Miodrag S. Petkoviç
American Mathematical Society, 2009
325 pages, softcover

Review by
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1 Introduction

Famous Puzzles of Great Mathematicians contains a nice collection of recreational mathematics problems and puzzles, problems whose solutions do not rely on knowledge of advanced mathematics. These problems mostly originated from great mathematicians, or had at least captured their interest, and hence the title of the book. Despite its recreational nature, this book does not give up on being rigorous in its arguments, nor does it shy away from presenting some difficult problems, albeit solvable by elementary methods. And even though the math in this book will not be new to most readers of this column, this is not the book's purpose. Among other reasons Petkoviç states the book was written to attempt "to bring the reader closer to the distinguished mathematicians" and to "show that famous mathematicians have all communicated brilliant ideas ... by using recreational mathematics." My review is written with this in mind.

2 Summary

After giving an overview of recreational mathematics, Petkoviç divides the majority of the book's puzzles into 9 topics, each getting its own chapter, each of which I summarize herein. Each chapter not only contains a collection of problems, but is also replete with quips, anecdotes, and short biographies. Petkoviç's book follows the pattern of first giving an introduction to and history of an area, then giving a couple challenge problems whose solutions he works out, and finally leaving the reader with some challenge problems, the solutions to which appear at the end of their respective chapters.

Following the topics chapters is an extra chapter that has additional challenge problems from various mathematicians – the answers to these problems are not given in the book. Then, there is an appendix elaborating on certain topics appearing in the book. The book concludes with short biographies of the mathematicians mentioned throughout.

2.1 Arithmetics

This chapter contains many puzzles focused on numerically computing a given result – from familiar problems involving finding a person's age, to problems involving Fibonacci numbers and other series, to problems asking the reader to do "reverse computations" such as finding the sides of a shape given its area. Most of these problems require either seeing some trick and then setting up an equation or a recurrence to get the solution. These arithmetic problems are some of the most accessible to a layperson. In this chapter, Isaac Newton's biography receives the most (well-deserved) space. He is credited with solving puzzles of the form "If x people can do y tasks in z hours, how many people would it take to do . . . ?" Some challenge problems of this form are presented at the end of the chapter.

2.2 Number Theory

This chapter covers a large variety of problems from the work of many important mathematicians, from Archimedes to Euler, Dirac, and Ramanujan. This is unsurprising as number theory is known for producing many interesting problems that are both immediately enticing and easy to understand. The number theory problems in this chapter also give an opportunity to demonstrate the usefulness of computers for solving mathematical problems. Petkoviç explains how the solution to a seemingly simple problem posed by Archimedes counting a farmer's cattle wasn't solved until 1965 by a clever use of IBM digital computers. Problems in this chapter all have a similar feel – of finding numbers that satisfy certain properties, and solutions to them often call for heavy use of computation.

2.3 Geometry

The geometry chapter is the longest in this book and contains questions that at first seem like they belong in a high school geometry class, but end up being harder and more interesting. Two compelling examples from this chapter are the problem of finding the best latitude for looking at Saturn's rings and the problem of determining the maximum number of parts a space can be divided by n planes. Of more relevance to computer scientists, Petkoviç shows the solution of the problem of connecting 4 villages on the corners of a square by the smallest n road system – an instance of the Euclidean Steiner tree problem.

2.4 Tiling and Packing

This chapter, as its name suggests, focuses on problems in filling a plane or 3-dimensional space with objects. Here, Petkoviç gives an extensive history of progress in tiling problems. One example is the progress in finding a non-periodic tiling of the plane – at first it was thought impossible to do, then a solution was given using 2000 shapes, and the number of shapes needed for an aperiodic tiling kept being reduced until Roger Penrose got the number down to only 2. John Conway and Donald Knuth also expectedly make an appearance. The tiling and packing problems in this chapter all have a similar feel, and the questions are mostly challenge questions.

2.5 Physics

In a somewhat nonstandard turn for recreational mathematics books, Petkoviç includes a short chapter of problems arising from physics. His definition of physics is quite broad, and many problems inspired by real-world situations count, including the famous puzzle of calculating the distance a fast object travels while bouncing between a wall and a slower object moving towards the wall – included with this puzzle is the well-known story of von Neumann not being able to see the trick, yet solving it quickly nonetheless. This chapter also contains an interesting problem of a man being trapped inside a ring with a lion, both of whom can run at the same speed. The question is whether the man can outrun the lion forever, and I won't spoil it by putting the answer here.

2.6 Combinatorics

The combinatorics chapter is full of various counting problems, so it is hard do it justice with a summary. Here, Petkoviç puts even the most familiar problems into context. For example, he points out that the problem of counting the number of ways of choosing k objects from a set of n was posed and calculated by Mahāvira, an Indian mathematician, in 850 A.D. Then, he moves on to a variety of topics: finding good arrangements, Gray-codes, Eulerian squares, Cayley's recurrence for counting trees, the towers of Hanoi, and many other problems. And he includes a short and entertaining biography of Paul Erdös.

2.7 Probability

This short chapter contains some famous mathematical gems. As Petkoviç points out, "probability is full of surprising results and paradoxes, more than any other branch of mathematics." He includes problems on how to divide stakes when a game is interrupted (considered by both Pascal and Fermat), the gambler's ruin problem, St. Petersburg paradox (a game with low winnings but infinite expectation), among others.

2.8 Graphs

Petkoviç gives graph theory problems a chapter of their own. He begins by covering traditional problems, including the Köningsberg bridge problem and Eulerian and Hamiltonian tours, making a nice connection between the latter and the towers of Hanoi, which are covered earlier in the book. He then moves onto less traditional graph problems, (but which are traditional recreational mathematics problems,) including river-crossing problems, in which a person has to get a wolf, goat, and cabbage across a river without anything being eaten and measurement problems, where you have to measure some quantity of water with seemingly ill-suited beakers for the task.

2.9 Chess

Chess problems at first seem out of place in this book, but Petkoviç explains how they captured the attention of some famous mathematicians, from Euler to Knuth. These problems are versions of puzzles familiar to many chess players and recreational mathematicians and involve knights tours and attacking queens. Gauss's biography surprisingly makes an appearance in this, of all, chapters, but his apparent interest in the well-known eight queens problem explains the choice.

3 Opinion

This book contains a nice representative set of recreational problems spanning many different areas of math. What makes this book especially compelling is Petkoviç's efforts in putting the problems into context. He makes it clear that math is a human subject, with its own stories and history.

The biographies of the mathematicians involved are witty and perhaps the best part of the book – this is by no means a criticism of the main content but a testament to Petkoviç's extensive research and good storytelling. I also think that the problems selected are quite appropriate. This book is very easy to follow, and the problems are of varying difficulty.

However, there is room for improvement. Some of the puzzles are stated in their original form (as posed centuries ago) and are hard to understand. This can be a positive for historical reasons, but also a bit confusing – I sometimes had to look at the answers and discussion to understand what a question is asking. Furthermore, how the problems were divided among the chapters seemed arbitrary at times.

But most disappointing to me is that Petkoviç avoids discussing most core theoretical computer science ideas (which surely belong within mathematics), even when some problems beg for it. To give just a few examples – he presents the traveling salesperson and Hamiltonian path problems without mentioning NP-completeness, gives an example Euclidean Steiner tree problem without talking about its computational complexity, and has an entire chapter on tiling problems without mentioning undecidability. By the end of reading the book, a reader may be left with the impression that computers are only interesting insofar as they help search through many possibilities or can compute the solution to an otherwise intractable equation. And even though computers and algorithms are often mentioned, it is unfortunate that computer science theory is not properly included in this otherwise encyclopedic collection, and I am left with the feeling a good opportunity to introduce readers its mathematical beauty was lost.

Of course, this is not a computer science book, and it is unfair to judge this book solely from that lens, so I still wholeheartedly recommend it to a wide variety of audiences. While professional mathematicians (and theoretical computer scientists) would know the math behind the problems this book, I suspect most would discover a few new interesting puzzles and some history, especially outside their main fields. To this audience, this book could be a fun and easy read. It might even give some inspiration for problems to present in lecture, though I doubt that is the best use of this book.

Mainly, I would recommend this book to anyone interested in recreational math or anyone who likes math puzzles. Petkoviç aims to bring his readers closer to the ideas of brilliant mathematicians, and I believe he succeeds. This book would be especially appropriate for undergraduates or even high school students with aptitude in mathematics. They should find *Famous Puzzles of Great Mathematicians* both very informative and fun, and might even become inspired to explore a career in math.

Review [12] of
Combinatorics – A Problem Oriented Approach

by Daniel A. Marcus
The Mathematical Association of America, 1998

136 pages, SOFTCOVER

Review by
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4 Introduction

Daniel A. Marcus wrote two mathematics books with a problem-oriented approach. A more recent book on Graph Theory and this much earlier one on Combinatorics as the study of arrangements of objects. The organization of the books is somewhat similar but this one seems to be more radical in its effort to integrate problems and concepts. I found this somewhat unorthodox presentation refreshing to motivate computer science students to understand combinatorics from first principles. It is from this perspective that the organization of book is structured. Instead of going from theory to practice, the book goes from practice to theory with a minimal number of succinct definitions even to the risk of introducing problems for which a generalized solution will appear much later in the book. In general, a "standard problem" is first introduced and then the theory behind it to solve the problem follows. The complete list of standard problems can be found in the back of the book. The chapters are referenced as Sections in lexicographic order so that the problems can be easily identified. Each chapter includes a short summary at the end followed by still "more problems."

5 Summary

Section A is about strings as a list of elements of a particular order. The concepts of rearrangement and derangement of a string are immediately introduced. Strings as a data structure makes sense in combinatorics. Most discrete math textbooks for computer science majors where combinatorics are covered introduce sets and leave it at that, only to confuse the students in problems where order matters. Other textbooks use the concept of lists instead of strings which is even more confusing since sets are also a list of elements. The product rule is defined here but nowhere could I find the sum rule in the book. This chapter also introduces discrete probabilities.

Section B addresses combinations as unordered arrangements of the elements of sets with numerous examples. Explanations and derivations of formulas for counting the number of arrangements follow the statement of standard problems. Several variants of combinations are similarly explained around standard problems. Pascal's triangle and binomial expansions are introduced in this chapter.

Section C presents distributions as counting functions from one set to another. Here again, the concepts are very well related to the combinatorics concepts of combinations and permutations progressing from examples, standard problems and theory accompanied with explanations. Multinomial expansions where the coefficients are distributions are rightfully introduced here. The technique of T-number triangles, with reference to Pascal's triangle, is presented as a general pattern that will be revisited in later chapters. The pigeon-hole principle could have been explicitly presented here as a distribution problem but its reference is missing entirely from the book which I found to be another glaring omission (the other one being the sum rule in Section A).

Section D is about partitions which are related nicely to the concept of distribution of the previous section. Stirling numbers are introduced with a standard problem.

The next sections are grouped into Part II on "Special Counting Methods" to solve combinatoric problems from a different point of view. A few words of motivation on those advanced counting methods would have been useful. The organization around standard problems is abandoned understandably but references are made to those earlier standard problems.

Section E introduces the principle of inclusion and exclusion. I particularly liked Problem E5 here concerning the application of this principle to a number theory problem.

Section F, about recurrence relations, nicely transitions from the combinatorial problems of earlier sections with a string arrangement problem. This chapter goes into solving linear recurrence relations but does not explain exactly what it means to solve a recurrence relation (i.e. finding a closed form solution). The restriction of the book to combinatorics prevents the presentation of other aspects of recurrence relations and its important tie to mathematical induction that would be of interest to computer science majors.

Section G covers generating functions to solve combinatoric problems subject to a variety of constraints. The presentation is extremely clear and somebody learning about it for the first time could follow it.

In contrast, the last Section, Section F, on the Polya-Redfield method is confusing. Problems and text alternate freely and the presentation is hard to follow.

6 Opinion

This book is a good supplemental resource for an instructor to a more substantial textbook at the mathematics graduate level. There are a lot of problems but only a few with solutions unfortunately and that's the big drawback of this book for a student. Some of the problems are actually questions that highlight important aspects of the theory rather than application problems just like an instructor would ask students in class. For example, many problems start with "Explain why ..." What this book does extremely well is to connect the problems together and consequently tie the concepts together. A list of problem dependencies can even be found in the back of the book. There are a few typos in the book and the organization is somewhat uneven but the teaching value more than make up for that.

Review of[13]
Probability: Theory and Examples, 4th edition
Author of Book: Rick Durrett
Cambridge University Press, Hardcover, 438 pages, 2010
$70.00 list price

Review by Miklós Bóna bona@math.ufl.edu

7 Introduction

This is the fourth edition of the classic textbook. Readers familiar with earlier editions will notice one difference very soon. The book has a new first chapter, on measure theory. Most readers and instructors will skip this introductory chapter and return to it only when they need it. It has basic definitions as those of random variables and their expected values. By moving these elementary topics to this introductory chapter, the author made the rest of the book a faster read.

8 Summary

The real course on probability starts in Chapter 2, that is devoted to laws of large numbers. These laws all state some kind of convergence, hence the notion of convergence has to be defined first. After defining and discussing what it means for random variables to be independent, the author defines convergence in probability. If X_n is a sequence of random variables, we say that X_n *converges* to X in *probability*, if for all $\epsilon > 0$, we have that $P(|X_n - X| > \epsilon) \rightarrow 0$ as $n \rightarrow \infty$.

The author introduces a few theorems here that will be with us for the rest of the book, so we spend some time discussing them.

The *weak law of large numbers* is that if $X_1, X_2, \cdots$ are independent and identically distributed random variables, with $E(|X_i|) < \infty$. Then

$$\frac{S_n}{n} - \mu \rightarrow 0$$

in probability, where $S_n = \sum_{i=1}^n X_i$, and $\mu = E(X_1)$. This theorem is a central result in this chapter, together with some of its versions, for instance one that does not require that the expectation of the X_i be finite.

The first Borel-Cantelli lemma says that if $\sum_{n\geq 0} P(A_i) < \infty$, then we have $\limsup P(A_n) = 0$. The second Borel-Cantelli lemma says that if $\sum_{n\geq 0} P(A_i) = \infty$, then $\limsup P(A_n) = 1$.

The *strong law of large numbers* states that if the X_i are pairwise independent, identically distributed random variables, then $S_n/n \rightarrow \mu$ almost surely as $n \rightarrow \infty$, where S_n and μ are defined as in the weak law of large numbers.

The last two sections of this chapter discuss methods to measure the *rate of convergence* for the convergence theorems proved earlier in the chapter.

Chapter 3 is focused on Central limit theorems. Just like the various versions of the Law of Large Numbers, these theorems also show that the sum of n identically distributed independent random variables behaves, in some sense, similarly to the individual variables. The similarities end here. First, the relevant notion of convergence here is convergence in *distribution*, also called *weak convergence.* We say that the sequence of variables $X_1, X_2, \cdots$ converges to the variable X in distribution if $F_n(y) \to F(y)$ in all continuity points of $F(y)$. Here F_n (resp. F) is the distribution function of X_n (resp. X).

The Central Limit Theorem, in its best-known form, states the following. Let $X_1, X_2, \cdots$ be a sequence of identically distributed independent variables. with expectation μ and variance σ^2. Set $S_n = X_1 + X_2 + \cdots + X_n$. Then

$$\frac{S_n - n\mu}{\sqrt{n}\sigma}$$

converges in distribution to the Standard Normal Distribution.

So, while the notion of convergence is weaker here than in the Law of Large numbers, the fact that we compare S_n to a well-known, specific distribution enables us to estimate the rate of convergence more precisely. As an example, we are shown that if we play roulette 361 times (one dollar is at stake in each round), we should expect to lose 19 dollars, but we will have about 16 percent chance to be ahead.

One particularly interesting example explains that the condition that the X_i are mutually independent *cannot* be replaced by the weaker condition that they be simply pairwise independent. This is in contrast to the law of large numbers, where that weaker condition is sufficient.

Another strong application is the Erdős-Kac theorem that shows that if we randomly select an integer from the set $\{1, 2, \cdots, n\}$, then that number will have about $\log\log n + \chi(\log\log n)^{1/2}$ prime divisors. Here χ denotes the standard normal distribution.

The main topic of applications at the end of the chapter are sequences of variables that weakly converge to a Poisson distribution.

Chapter 4 is about Random Walks, a topic that is the subject of several textbooks on its own. A *simple* random walk is a sequence of moves on a d-dimensional integer lattice, in which every step is of length 1, and each of the $2d$ possible steps are equally likely to occur in any moment of time. Classic questions on random walks include the determination of the expected time it takes for the walk to return to its starting point, to reach a given point, or the determination of the probability that the walk will *ever* return to its starting point, or reach a given point.

A classic result is that a simple random walk in d dimensions will almost surely return to its starting point if and only if $d \leq 2$. The author illustrates this fact by saying that *a drunk man will eventually find his house, but a drunk bird may not.*

However, the author does not limit his coverage to *simple* random walks. He extends the definition by saying that if $X_1, X_2, \cdots, X_n$ are independent, identically distributed variables taking values in $\mathbf{R}^d$, then $S_n = X_1 + X_2 + \cdots + X_n$ is a random walk. He then proves theorems on whether random walks (defined in this more general way) will return to their starting point, in other words, whether they are *recurrent* or *transient.* He proves three results, as follows.

1. If $d = 1$, then S_n is recurrent if S_n/n converges to 0 in probability.
2. If $d = 2$, then S_n is recurrent if $S_n/\sqrt{n}$ converges in distribution to a (non-degenerate) normal distribution.
3. If $d = 3$, then S_n is recurrent if it is "truly three-dimensional", meaning that $P(aX_1 \neq 0) > 0$ for all nonzero real numbers a.

Chapter 5 is about *martingales.* A martingale is a sequence $X_1, X_2, \cdots$ of random variables so that $E(|X_i|) < \infty$ for all i, and $E(X_{n+1}|X_1, X_2, \cdots, X_n) = X_n$. In other words, a martingale is a sequence of observations so that the expected value of the next observation, given all past observations, is equal to the last observation. In order for the formal definition to make sense, the author first introduces the concept of conditional expectations. Then he proves basic properties of martingales, discusses the classic examples, and introduces submartingales, supermartingales and backward martingales. (The first two are easy to define by replacing the equality sign in the definition of martingales by a $\geq$ or a $\leq$ sign, respectively. On the whole, this chapter is pretty much what one would expect to find in any book targeted at the given audience.

The subject of Chapter 6 is *Markov chains.* A Markov chain is a sequence $X_1, X_2, \cdots$ of random variables so that

$$P(X_{n+1} = x | X_1 = x_1 \cap X_2 = x_2 \cap \cdots \cap X_n = x_n) = P(X_{n+1} = x | X_n = x_n).$$

In other words, X_{n+1} only depends on X_n. This is in contrast to martingales, discussed in the last section, where X_{n+1} depends on all of the preceding X_i. An interesting example that gets discussed is shuffling a deck of n cards. It is proved that $n \log n$ moves will completely shuffle the deck. Here one move consists of removing the top card and inserting it under one of the remaining $n-1$ cards. Several examples connect the chapter to random walks.

We stay with sequences of variables in Chapter 7 as well. The main result of the chapter is the *Ergodic Theorem.* The sequence X_n of random variables is called a *stationary sequence* if for all $k \geq 1$, the the sequence X_n and the shifted sequence X_{n+k} has the same distribution. The ergodic theorem states that if $E(|f(X_0)|) < \infty$, then

$$\lim_{n \to \infty} \frac{1}{n} \sum_{m=0}^{n-1} f(X_m)$$

exists almost surely. Two sections are spent on proving this theorem. Then we turn to applications, first to random walks, and then to the interesting and difficult question of determining the distribution of the length of the longest increasing subsequence of a random permutation of length n. (This is sometimes called *Ulam's problem.*) The result is that on average, this length is $2\sqrt{n}$, though more precise results have recently been discovered by Baik, Deift, and Johansson.

Many results, and some of the structure, of Chapter 5 have a similar version in Chapter 8, where *Brownian motion* is discussed, which can be deduced from a model that is obtained by taking the limit of a random walk as the time and space increments go to zero. It goes without saying that formalizing this concept is far from an obvious task.

9 Opinion

The best feature of the book is its selection of examples. The author has done an extraordinary job in showing not simply what the presented theorems can be used for, but also what they *cannot* be used for. Basically, every example showing an application is followed by one showing a non-application. This is very educational, since it teaches the student what the precise power of these theorems is. There are not many textbooks that do this. Another nice feature is that the chapters build on each other very closely; the theory that we learn in one chapter is used in the examples of the next chapters.

A weak feature of the book is that the definitions are often difficult to understand; they are stated in strong forms, even when a weaker form would be much easier to understand. This reviewer often consulted alternative sources to understand a definition quicker, then returned to this book to enjoy the examples.

The author sometimes forgets that he did not define something. For example, the notion of $\mathbf{L^2}$ appears from page 53, but this reviewer could not find a definition for it in the book, or the item in the index. It is, of course, very easy to look this notion up elsewhere, but it would be even easier if the book contained it.

On the whole, if you have really strong and motivated students, teach a course from the book. If your students are not quite that strong, use another book, but consult this one for the examples.

Miklós Bóna

Review of[14]

Games, Puzzles, & Computation
by Robert A. Hearn and Erik D. Demaine
A K Peters, Ltd., 2009
237 pages, Hardcover, $45.00 ($28.80 on Kindle)

Review by
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"It took about half an hour to show Konane PSPACE-complete.
[Despite previous efforts,] no prior complexity results about Konane were known.
It is this kind of result that demonstrates the utility of constraint logic."
—*Games, Puzzles, & Computation*

1 Introduction

By way of introduction, an important note is in order: The word "game" as used in the title of *Games, Puzzles, & Computation* refers to the *normal, everyday notion* of a game, like Sudoku, Go, or Bridge! This book studies the *computational complexity* of solving puzzles and games. Indeed, it generally serves a three-fold purpose: (i) a survey of known complexity results regarding games and puzzles of every kind, (ii) an exposition of a series of new complexity results in the area, and perhaps most importantly (iii) an effort to provide a *unifying framework* (via a very visually-appealing form of constraint logic) in which to analyze the complexity of games and puzzles.

In reading *Games, Puzzles, & Computation*, there are inevitable comparisons to Garey and Johnson's famous *Computers and Intractability: A Guide to the Theory of* NP*-Completeness.* Both books are of a similar length and are organized roughly the same. They begin by building a theoretical foundation for the relevant subject matter a piece at a time, then guide the reader through a sequence of completeness and hardness results to demonstrate how various components of the theory are applied, and finally include a large appendix that surveys a greater list of known results in the area.

As such, *Games, Puzzles, & Computation* will serve well in roles similar to that of Garey and Johnson's book. In particular, the text would work exceedingly well as a reference for what's known in the subfield of game/puzzle complexity or for self-study by someone familiar with basic computational complexity principles who is interested in learning more about the complexity of games and puzzles. It would also serve well as supplementary material to an upper-level undergraduate or entry-level graduate special topics course in game/puzzle complexity. It *could* also be used as the primary text for such a course (in principle) given extra preparation by the instructor, as there are not explicit exercises laid out in the book proper.

2 Summary

The book is organized into two parts, composed of multiple chapters each, plus a sizable set of appendices. Part I, entitled "Games in General" and spanning 100 pages, introduces the authors' theory of constraints graphs and constraint-logic games, which form the bulk of the proof machinery used throughout the text. It then goes into detail across four broad categories of games – zero-player games (resp. *simulations*), one-player games (resp. *puzzles*), two-player games, and team games – in order to develop the precise systems used to analyze each. Part II, entitled "Games in Particular" and spanning 60 pages, is an exposition of numerous, new complexity results for one-player and two-player games via application of the constraint logic of Part I. Finally, there are approximately 60 more pages devoted to appendices, including the Garey/Johnson-esque survey of known complexity results for games and puzzles as well as other, useful references. Here is a brief description of each chapter.

2.1 Part I: Games in General

2.1.1 Chapter 2: The Constraint-Logic Formalism

In this chapter, the authors introduce *constraint logic* on *constraint graphs*, which are directed graphs in the usual sense such that every edge is colored either red (a weight 1 edge) or blue (a weight 2 edge) and each vertex is assigned a nonnegative *minimum inflow* (where *inflow of a vertex* is the sum of the weights on inward-directed edges). A *legal configuration* of a constraint graph then becomes an arrangement where all vertices have inflow at least their minimum inflow, and a *legal move* is a reversal of an edge's orientation transforming one legal configuration into another legal configuration. *Multiplayer games* assign control of edges to certain players and alternate moves, *deterministic games* are when a unique sequence of reversals is forced, and *bounded games* are when each edge may only reverse once. Additionally, various fundamental gadgets on constraint graphs are introduced, including `AND`-, `OR`-, and `FANOUT`-gadgets, inviting a comparison to circuits and other logic networks.

The goal in all of this is to define a minimalistic version of *monotone* logic in order to simplify the resulting reductions to individual games and puzzles. And as the authors write, "One of the more surprising results about constraint logic is that monotone logic is sufficient to produce computation, even in the deterministic case."

2.1.2 Chapter 3: Constraint-Logic Games

In Chapter 3, resource and rule restrictions on constraint logic are considered (single-player, multi-player, bounded, unbounded, etc.), which enable the definition of computational problems complete for a host of classes. The question asked in the computational problems are generally of the form, "Given a target edge e, is e ever reversed?" Thus defined, the following chapters expand on each of these problems, and their completeness proofs, in greater detail.

2.1.3 Chapter 4: Zero-Player Games (Simulations)

Chapter 4 introduces the full definitions and completeness proofs for Deterministic Constraint Logic for both bounded and unbounded numbers of allowed moves. The bounded variety is shown to be P-complete, and the unbounded variety is shown to be PSPACE-complete. Various restrictions are given under which the constraint logic problems are still complete for their respective classes. These constraint logics are shown to be comparable in principle to a wide array of zero-player games.

In particular, the theorems in this chapter (and through the remainder of Part I) generally follow the *form*, "Unbounded Deterministic Constraint Logic is PSPACE-complete, even for planar graphs using *only* AND- and OR-vertex types." While the specifics of each theorem vary, the intent is to enable reductions from certain families of restricted constraint graphs (known to be C-hard or -complete, for some complexity class C) to specially-crafted *positions* of individual games. Then, given that it is (for example) PSPACE-hard to decide if the first player has a forced win from a given position in a given puzzle or game, we say the puzzle or game itself is PSPACE-hard in general.

2.1.4 Chapter 5: One-Player Games (Puzzles)

This chapter fully introduces Nondeterministic Constraint Logic, which is NP-complete in its bounded variety and PSPACE-complete in its unbounded form. This version is useful for reductions to one-player games of bounded and unbounded numbers of moves. Again, various restrictions are given under which the problems remain complete for their respective classes, and various technical issues are covered regarding the exact gadgets involved.

2.1.5 Chapter 6: Two-Player Games

This chapter introduces Two-Player Constraint Logic generally corresponding to two-player games, though the situation is a bit more involved than in previous chapters. The bounded variety is shown to be PSPACE-complete, and the unbounded variety is shown to be EXPTIME-complete (the first provably intractable type of game, under zero complexity assumptions).

However, there is an additional property that, in principle, *should* push the complexity of certain unbounded two-player games even higher: the "no-repeat" condition. For example, the *superko* rule in Go[15] states, "A play is illegal if it would have the effect of creating a position that has occurred previously in the game." And as the authors write, "a no-repeat version of Two-Player Constraint Logic *ought* to be EXPSPACE-complete... but we do not yet have one."

2.1.6 Chapter 7: Team Games

Chapter 7 begins by observing that any team game with *perfect information* is really just a two-player game in disguise (the joint behavior of one side could just as easily be simulated by a single player). Instead, a version of team games with imperfect information is considered – that is, scenarios where all players know an initial configuration, and then private information arises as a result of all moves not being visible to all players. The resulting formulation, Team Private Constraint Logic, is shown to be NEXPTIME-complete is its bounded form and *undecidable* in its unbounded form. Further, an intermediate construction is mentioned in passing (i.e., *two-player*, unbounded games with imperfect information) that is complete in doubly-exponential time.

[15] See: http://en.wikipedia.org/wiki/Rules_of_Go#Ko_and_Superko

2.2 Part II: Games in Particular

2.2.1 Chapter 9: One-Player Games (Puzzles)

In Chapter 9, the complexities of a number of one-player games are resolved, many for the first time. The proofs are surprisingly succinct, relying heavily on the theorems developed in Part I. In particular, they generally involve the construction of an `AND`-gadget and an `OR`-gadget for the game in question (sometimes a little more), and then some brief logic tying everything together. Here is a list of some of the puzzles in question: TipOver, Hitori, Sliding Block puzzles, and Rush Hour.

2.2.2 Chapter 10: Two-Player Games

In similar fashion to the previous chapter, numerous complexity results are proven for two-player games. In particular, Konane, an ancient Hawaiian game first documented by Captain Cook[16] in 1778, is shown to be PSPACE-complete in general.

2.3 Appendices

2.3.1 Survey of Games and Their Complexities

The section of the appendices surveying known complexity results for games and puzzles is particularly worthwhile. Given its length, instead here is a list of categories of such games for the interested reader's sake: Cellular Automata, Games of Block Manipulation, Games of Tokens on Graphs, Peg-Jumping Games, Connection Games, Other Board Games, Pencil Puzzles, Formula Games, Other Games, Constraint Logic Games, and Open Problems.

One appealing open problem: Is *Rengo Kriegspiel* decidable? Rengo Kriegspiel is a blindfolded, team variant of Go. Four players sit facing away from one another, with two on the White team and two on the Black team. Play proceeds in a round-robin fashion with each player *attempting* to make moves by privately communicating their intended move to a referee. The referee, who is the only one who can see the actual board, then announces whether the attempted move was legal or illegal and whether any captures were made. The moves themselves remain hidden, and players only discover the state of the board by attempting to make moves and finding stones already in a given position.

It's easy to see that Rengo Kriegspiel fits the general description of an unbounded team game with imperfect information and, by a direct generalization of the principles in Chapter 7, *could* be undecidable in principle. Such a proof is not known but would be extremely interesting, as it would make Rengo Kriegspiel the only game humans actually play that is known to be undecidable.

3 Opinion

I really enjoyed reading *Games, Puzzles, & Computation*, and there is no doubt in my mind that if you're either interested in the complexity of puzzles and games or are interested in proving new complexity results for your favorite puzzle, then this book is a *must-have*. In particular, there are numerous illustrations of all of the relevant gadgets (in color!) throughout the book, which greatly add to its value. It's also worth pointing out that the bulk of the book is derived from the author

[16] See: http://en.wikipedia.org/wiki/James_Cook

Robert Hearn's PhD thesis[17], and you can get a (very close) idea of the contents of the book, sans the book's excellent survey appendix, by skimming that.

Before reading this book, my general impression of puzzle complexity, perhaps from a vague recollection of some folklore argument, was that puzzles typically have certain types of complexities (e.g. NP-complete or EXPTIME-complete) depending on a small list of "obvious" properties (e.g. Is it a one-player game, or a two-player game?) though the actual proofs tended to be much more complicated. *Games, Puzzles, & Computation* is especially excellent, then, for two reasons: (i) it reinforces this general trend while making the connections more precise *and* more thorough, and (ii) it dramatically simplifies the proofs *and* does it in a unified fashion.

That said, one of my objections is that the number of examples given for polynomially-solvable games is extremely small (only *two* appear buried in the appendix, and none in the text proper). Intuitively, one would expect that most games and puzzles *invented by humans* are at least NP-hard, but more examples, if for nothing more than reference purposes, would be useful. Another concern that is not explicitly addressed in the text insofar as I am aware is how one should interpret certain complexity results with respect to the *actual way* games are played in practice. For instance, Chess is EXPTIME-complete on an $N \times N$ board, but what does that say about standard 8×8 Chess, specifically with respect to, say, Sudoku which is NP-complete on an $N \times N$ grid but which is *actually played* on a 9×9 grid? This, of course, is a routine issue when comparing complexity results and specific, finite instances of two problems, but I would be interested to hear what the authors have to say about the concrete puzzles and games on hand.

Finally, admittedly, you will have to invest in learning the inner workings of constraint logic in order to use it yourself, but once you do, you will have a powerful set of techniques for proving completeness and hardness results for a large swath of games and puzzles. Further – and this can be either a pro or a con depending on your tastes – the techniques involve a great deal of *visual*, combinatorial matter. For example, how do you show Sliding Block puzzles are PSPACE-hard using constraint logic? You build an `AND`-gadget[18] and an `OR`-gadget out of sliding blocks, you show how to connect them together to form arbitrary planar graphs (i.e. just a handful of "wiring" gadgets), and... you're done! For many people, this book will be a truly enlightening.

[17] Currently available at: http://groups.csail.mit.edu/mac/users/bob/hearn-thesis-final.pdf

[18] Just to emphasize how simple these can sometimes be, the `AND`-gadget in the PSPACE-completeness proof for sliding block puzzles is the yellow-and-grey, square picture on the book's front cover. Check it out!