Joint Review by
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of
Rudiments of Ramsey Theory (2nd Ed.) by Graham & Butler
AMS, 2015, 83 pages, Softcover, $63

Fundamentals of Ramsey Theory by Robertson
CRC, 2021, 207 pages, Softcover, $63

Elementary Methods of Graph Ramsey Theory by Li and Lin
Springer, 2022, 349 pages, Hardcover, $86

Basics of Ramsey Theory by Jungic
CRC, 2023, 238 pages, Hardcover, $120

1 Introduction

If I want to start an argument on my blog I might post

When I teach Automata Theory I do not cover Context Free Grammars!

There is a large group of people who teach Automata theory, and a large set of textbooks on the topic. Hence there are many different ideas of what should be in the course. Debate is possible and desirable.

If I were to post

When I teach Ramsey Theory I do not cover the Hales-Jewitt Theorem!

I doubt I would get an argument going. There are very few people who teach a course in Ramsey Theory. There are a few textbooks on it. In this column I review four of them. This could be the starting point for an argument.

I reviewed three other books on Ramsey theory in the past:

- Ramsey Theory for Discrete Structures by by Prömel. The review is in the column:

  `https://mathcs.clarku.edu/~fgreen/SIGACTReviews/bookrev/48-4.pdf`

- An Introduction to Ramsey Theory: Fast Functions, Infinity, and Metamathematics, by Katz and Reimann. The review is in the column:

  `https://mathcs.clarku.edu/~fgreen/SIGACTReviews/bookrev/50-2.pdf`

- Ramsey Theory over the Integers (Second Edition), by Landman and Robertson. The review is in the column:

  `https://mathcs.clarku.edu/~fgreen/SIGACTReviews/bookrev/47-2.pdf`

Of the four books we review, three of them (the books by Robertson, Jungic, Li-Lin) are textbooks. The fourth one (by Graham & Butler) is notes for a short course aimed at mathematicians, so not a textbook.

2 Definitions and Basic Theorems

Def 2.1

1. N is the naturals, $\{1, 2, 3, \ldots\}$.
2. If $n \in \mathsf{N}$ then $[n] = \{1, \ldots, n\}$.
3. If A is a set and $a \in \mathsf{N}$ then $\binom{A}{a}$ is the set of a-sized subsets of A. Note that $\binom{n}{a}$ is the edges of the complete a-hypergraph.
4. Let $A \subseteq \mathsf{N}$ (it will either be $[n]$ or N).
5. In Ramsey Theory we are often given, as a premise of a theorem, a *coloring* which is just a map to colors, often c of them. For notation we usually use $[n]$ rather than (say) $\{\mathrm{RED}, \mathrm{BLUE}\}$. We usually denote the coloring function COL. This is all a prelude to the next definition:

   Let $a, c \in \mathsf{N}$ and let $\mathrm{COL}\colon \binom{A}{a} \to [c]$. Let $H \subseteq A$. A is *homogeneous* (henceforth *homog*) if COL restricted to $\binom{H}{a}$ is constant.

We describe several theorems from Ramsey Theory.

1. *Ramsey's Theorem [14]:* For all a, k, c there exists $n = R_a(k, c)$ such that for all $\mathrm{COL}\colon \binom{[n]}{a} \to [c]$ there exist a homog set of size k. There has been much work on upper and lower bounds on $R_a(k, c)$ especially

$R_2(k, 2)$ which we denote $R(k)$. Using elementary methods (hence suitable for a course) one can show

$$(1 + o(1))\frac{k}{\sqrt{2}e}2^{k/2} \le R(k) \le (1 + o(1))\frac{4^{s-1}}{\sqrt{\pi k}}$$

The upper bounds is due to Erdös & Szekeres [3]. The lower bound was obtained by Erdös [4] using the the probabilistic method. That last sentence is true but odd. Erdös invented the probabilistic method in order to get this lower bound on $R(k)$.

For both the upper and lower bounds, better results are known.

2. *Van Der Warden's Theorem (henceforth VDW's theorem) [17]:* For all k, c there exists $W = W(k, c)$ such that for all COL: $[W] \to [c]$ there is a monochromatic arithmetic sequence of length k (henceforth a k-AP). The *Gallai-Witt Theorem* is a generalization to more dimensions. (There is no publication by Gallai that contains it; however, Rado [12],[13] proved it and credits Gallai. Witt [18] proved it independently.) The *Hales-Jewitt Theorem (HJ Theorem)* [8] is a generalization about coloring sequences. The initial proof of VDW's Theorem yielded enormous bounds on $W(k, c)$ that were Ackerman-like. Shelah [15] obtained primitive recursive (though still quite large) bounds on the HJ numbers and hence on $W(k, c)$. His proof is elementary (though difficult). Gowers [7] later obtained a non-elementary proof of bounds you can actually write down.

3. *Szemerédi's Theorem [16]* is a density version of VDW's Theorem. This proof is elementary in that it only uses combinatorial methods, but stretches the definition of *elementary* to the breaking point since it is quite difficult. There is also a density version of the HJ Theorem [10] with an elementary proof.

   There are also non-elementary proofs of Szemerédi 's Theorem. They may be easier to understand than the elementary proof.

4. *Rado's Theorem* Let $a_1, \ldots, a_n \in \mathsf{Z}$. The following are equivalent:

   - For all finite colorings of N there exists $x_1, \ldots, x_n \in \mathsf{N}$ that are the same color such that $a_1x_1 + \cdots + a_nx_n = 0$.

- Some subset of $\{a_1, \ldots, a_n\}$ sums to 0.

There is a generalization of this theorem to systems of linear equations. Some nonlinear equations have also been studied.

3 How the Books are Similar

All of the theorems states in the last section are covered very well in all of the books. (except Szemerédi's theorem and Gower's theorem which are stated by not proven). All of the textbooks have good exercises and good explanations. The non-textbook (by Graham & Butler) was not intended as a textbook—it was intended to introduce Ramsey Theory to Mathematicians. It does a fine job at that.

That takes care of the similarities. For each book we discuss something that is in them that is not in the other four.

4 Rudiments of Ramsey Theory, Second Edition, Graham and Butler. 1979, 2015

In 1979 Ronald Graham gave a series of lectures at a Regional Math conference (at St. Olaf College) on Ramsey Theory. A set of notes came out of that which formed the first edition of this book. Over the next 35 years there was much progress in Ramsey theory, perhaps inspired by these notes and the book: *Ramsey Theory* by Graham, Rothchild, and Spencer. The book under review is a second edition of these notes. A lot of updating was needed and was done.

These notes are from talks given to other mathematicians to introduce them to the subject. Many of the proofs are informal. This is both good and bad.

- Because of the informality much ground can be covered in a mere 80 pages. Indeed, there are some topics that are in this book that are not in the other, longer books, reviewed in this column.
- Some of the proofs are incomplete. Some of the references are to articles that were never published.

- There is no index.

I mention a lines of research that I learned existed from this book and was so inspired I made up slides on it which I will present to my Ramsey Theory class.

Note that the book only had proofs of the first few results; but it pointed me in the right direction for more material.

Euclidean Ramsey Theory Aside from the standard material on coloring the plane (also in Robertson's book) we look at another line of research.

1. The following result is due to Burr; however, he did not publish it. The result appeared in Erdos et al. [5].

   For all COL: $\mathsf{R}^6 \rightarrow [2]$ there is a monochromatic unit square (all four corners the same color). The complete proof is in the book. I blogged about the statement and proof and learned what else was known from commenters.

2. For all COL: $\mathsf{R}^5 \rightarrow [2]$ there is a monochromatic unit square. This can be done with a small trick on top of the R^6 result.

3. Kent Cantwell [1] showed the following: For all COL: $\mathsf{R}^4 \rightarrow [2]$ there is a monochromatic unit square. This is a completely different proof.

4. There exists COL: $\mathsf{R}^2 \rightarrow [2]$ with no monochromatic unit square. This is easy.

5. The case of R^3 is open. Thats to bad since we live in R^3.

5 Fundamentals of Ramsey Theory by Aaron Robertson. 2021

Most of the results in the early history of Ramsey Theory were proven using combinatorial techniques. This was great for having a low barrier to entry. I have seen High School Students, and one 10 year old, learn Ramsey Theory. By contrast, very few high school students do projects on Functional Analysis.

While it was great to have a modern field of math that was accessible to high school students, eventually non-combinatorial techniques were needed to solve some problems.

So should these techniques be put into a textbook for undergraduates? Can they be?

Robertson's book takes up that challenge. Chapter 2, section 2 is titled *Density Theorems.* Chapter 5 is titled *Other Approaches to Ramsey Theorem.* Both contain proofs that use non-combinatorial techniques.

1. Section 2.2: *Roth's Theorem* If $A \subseteq \mathsf{N}$ has positive upper density then A has a 3-AP. (Szemerédi later proved that, for all k, A has a k-AP.) This is proven with techniques from analysis, namely Fourier transforms over finite fields. A complete proof is given. This is the way Roth proved it. This Roth's Theorem via Fourier transforms is accessible and the starting point for later results in Ramsey Theory.

   A combinatorial proof for the $k = 3$ case is also known. It is not in this book. The only account I've ever seen of that is in Graham-Rothchild-Spencer's book *Ramsey Theory.*

2. Section 5.1: A topological proof of VDW's Theorem. Note that VDW's theorem has a combinatorial proof (which is in all four books); however, it's good to see a topological proof to get the readers feet wet with those techniques.

3. Hindman's theorem is as follows: *For all* r*, for all* COL: $\mathsf{Z} \rightarrow [r]$*, there exists an infinite* $A \subseteq \mathsf{Z}$ *and a color* c *such that, for all finite non-empty* $B \subseteq A$, $\mathrm{COL}(\sum_{x\in B} x) = c$. Earlier in the book (Section 2.4.2) there is a complicated combinatorial proof of Hindman's Theorem. In Section 5.1.3 there is a complicated topological proof of Hindman's Theorem. In Section 5.3.1 there is a (debatably) simpler topological proof that uses Stone-Cech compactification.

4. In Section 5.4.1. The circle method, a technique from analysis, is used to show that the primes contain an infinite number of 3-term arithmetic progressions. The complete proof is given.

This book is an excellent place to see well written (aimed at undergraduates) expositions of proofs involving non-combinatorial methods in Ramsey Theory.

6 Elementary Graph Ramsey Theory by Li and Lin. 2022

This book begins with the standard topics but then has several chapters on non-standard topics that are not in any of the other books. Note that this is the longest of the books reviewed. Here are just some of the non-standard topics:

1. *Constructive Lower Bounds* Recall that the lower bound $R(k) \geq \Omega(k2^k)$ was proven by the probabilistic method. Hence the proof does not give a way to construct the needed colorings of $\binom{[n]}{2}$. This chapter investigates constructive lower bounds including the results of Nagy's [9] $R(k) \geq \Omega(k^3)$ (the original paper is in Hungarian) and the Frankl-Wilson [6] constructive bound: $R(k) \geq 2^{\Omega(\log^2 k/\log\log k)}$. As a bonus this chapter contains a disproof of Borsuk's conjecture (for all d every $X \subseteq \mathsf{R}^d$ can be partitioned into $d+1$ sets of smaller dimension) which uses ideas from constructive Ramsey Theory.

2. *Communication Channels* We do an example. Alice and Bob want to communicate with each other. The set of messages they want to send is $\{0,1\}^3$. They are communicating over a noisy channel where 1 bit might get flipped. Hence, if Bob receives 000, the real message is one of $\{000, 001, 010, 100\}$. The *confusion graph* has as vertices the set of message that can be send, and as edges, an edge between two messages that might be confused for each other. The *Shannon Capacity* of such a graph is, roughly speaking, the amount of information that can be transmitted. This 14 page chapter introduces the topic, gives some result, and has 2 pages on how Ramsey Theory can be used in the study of Shannon Capacity.

3. *Quasi-Random Graphs* Informally, *Quasi-Random Graphs* (defined by Chung-Graham-Wilson [2]) satisfy many properties that random graphs do. Indeed, the book presents a theorem that has 7 properties of random graphs that are also satisfied by Quasi-Random Graphs. They differ from random graphs in that they can be generated deterministically and some of their properties are easier to verify. In this chapter they are defined, many theorems are proven about them, and they are used to get better lower bounds on the multicolor Ramsey numbers.

For example, $R(C_4, C_4, K_n)$ is the least N such that for all 3-colorings of the edges of K_N with Red, Blue, and Green there is either a Red C_4 or a Blue C_4, or a Green K_n

4. *Regularity Lemma and Van Der Waerden Numbers* VDW's theorem and the HJ theorem are standard topics and they are in this chapter. Szemerédi's regularity lemma is important but difficult so it is not in most other textbook. This lemma is a key ingredient in the proof of Szemerédi's theorem (every $A \subseteq \mathsf{N}$ of positive upper density has arbitrarily long arithmetic sequences). Surprisingly, this book does not mention Szemerédi's theorem.

This book has much material of interest that is not in any other book.

7 Basics of Ramsey Theory by Jungic. 2023

This book covers the standard material very well. The pace is leisurely and there are many good exercises. We point out three things that this book has that are not standard.

1. There are short biographies of Ramsey, Erdös, van der Waerden, Schur, and Rado. The biographies are interesting and have material the reader probably does not know.

2. The book has a proof of the *Canonical Van der Waerden Theorem* from *Szemerédi's theorem.* We state both theorems

   Can. VDW For all k there exist $W = W(k)$ such that, for all finite coloring of $[W]$ there is *either* a monochromatic k-AP or a rainbow k-AP (all colors different).

   Szemerédi's Theorem If A is a set of positive upper density then A has arbitrarily long arithmetic progressions.

   Surprisingly, the book does not mention that there is an elementary proof of the Can. VDW theorem by Prömel & Rodl [11].

3. There is an extensive treatment of the *Happy Ending Theorem*, which we state.

Happy Ending Theorem For all $n \geq 3$ there exists $f(n)$ such that for all arrangement of $f(n)$ points in the plane, no three colinear, there is a subset of n of them that form a convex n-gon.

The first proof of this theorem was by Ramsey Theory. That proof is in the other books. There were proofs that yielded lower values of $f(n)$ that are also included (the cups-and-caps method). This is unusual.

8 Opinion

We will discuss the four books reviewed here and the three books we reviewed in prior columns. All of the books are good and serve their purpose. We will classify the seven books into two categories.
Books that Mostly Use Combinatorial Techniques

1. *An Introduction to Ramsey Theory: Fast Functions, Infinity, and Metamathematics*, by Matthew Katz and Jan Reimann. Bonus: has a complete proof that the Paris-Harrington Theorem is independent of Peano Arithmetic.

2. *Ramsey Theory over the Integers (Second Edition)*, by Bruce M. Landman and Aaron Robertson. Caveat and Bonus: This book does not contain anything about Ramsey Theory on graphs, but has many variants of VDW's theorem that are not in any other book.

3. *Basics of Ramsey Theory*, by Jungic. Bonus: This book has biographies of some Ramsey Theorists and an extensive treatment of the Happy Ending Theorem.

4. *Rudiments of Ramsey Theory (2nd Ed.)* by Graham & Butler. Bonus and Caveat: Lots of material here that is not elsewhere, but it is somewhat terse.

Books that Use Non-Combinatorial Techniques

1. *Fundamentals of Ramsey Theory* by Robertson. The point of the book is to present non-combinatorial techniques in a way that can be understood. Bonus: The book succeeds.

2. *Elementary Methods of Graph Ramsey Theory* by Li and Lin. Covers more material than all of the books here except Prömel's. Rough going but its worth it.

3. *Ramsey Theory for Discrete Structures* by Prömel. This covers a lot of very abstract material and also the elementary proof of the Density HJ theorem. A good place to read rather advanced material.

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