Funky Dice: An Exposition

William Gasarch - University of MD
If you roll two standard 6-sided dice then

1. 2: (1,1). ONE way. Prob \( \frac{1}{36} \).
2. 3: (1,2), (2,1). TWO ways. Prob \( \frac{1}{18} \).
3. 4: (1,3), (2,2), (3,1). THREE ways. Prob \( \frac{1}{12} \).
4. 5: (1,4), (2,3), (3,2), (4,1). FOUR ways. Prob \( \frac{1}{9} \).
5. 6: (1,5), (2,4), (3,3), (4,2), (5,1) FIVE ways. Prob \( \frac{5}{36} \).
6. 7: (1,6), (2,5), (3,4), (4,3), (5,2), (6,1) SIX ways. Prob \( \frac{1}{6} \).
7. 8: (2,6), (3,5), (4,4), (5,3), (6,2) FIVE ways. Prob \( \frac{5}{36} \).
8. 9: (3,6), (4,5), (5,4), (6,3) FOUR ways. Prob \( \frac{1}{9} \).
9. 10: (4,6), (5,5), (6,4) THREE ways. Prob \( \frac{1}{12} \).
10. 11: (5,6), (6,5) TWO ways. Prob \( \frac{1}{18} \).
11. 12: (6,6) ONE way. Prob \( \frac{1}{36} \).
Questions about Dice

1. Can we load two 6-sided dice so that every number from 2 to 12 has the **same** probability. Called **fair sums**.
Questions about Dice

1. Can we load two 6-sided dice so that every number from 2 to 12 has the same probability. Called fair sums.

2. Can you label the dice something other than \{1, \ldots, 6\} and \{1, \ldots, 6\} and get the same probabilities you get with standard dice?
Loaded Dice

William Gasarch - University of MD
Fair Dice Yield Unfair Sums

Fair Die:

\[ \Pr(1) = \Pr(2) = \Pr(3) = \Pr(4) = \Pr(5) = \Pr(6) = \frac{1}{6} \approx 0.167 \]

Roll TWO of them.

\[ \Pr(\text{Sum}=2) = \frac{1}{36} \text{ (This is Min } \Pr(\text{Sum})\text{)} \]
\[ \Pr(\text{Sum}=7) = \frac{1}{6}. \text{ (This is Max } \Pr(\text{Sum})\text{)} \]

Sums are Unfair!

How Unfair?: \[ \frac{1}{6} - \frac{1}{36} \approx 0.139 \text{ unfair.} \]
What Are Loaded Dice?

**Definition:** A Die is a 6-tuple \((p_1, p_2, p_3, p_4, p_5, p_6)\) such that \(0 \leq p_i \leq 1\) and \(\sum_{i=1}^{6} p_i = 1\).
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**Our Questions:**

1. Does there exist a pair of loaded dice such that the sums all have equal probability \(1/11\)?

VOTE

NO or YES or UNKNOWN TO BILL!
What Are Loaded Dice?

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**Our Questions:**

1. Does there exist a pair of loaded dice such that the sums all have equal probability \(1/11\)?
2. What Do You Think? **VOTE** YES or NO or UNKNOWN TO BILL!
3. NO, no such dice can exist! (We prove on next few slides.)
Polynomials are our Friends!

Let $(p_1, \ldots, p_6)$ and $(q_1, \ldots, q_6)$ be dice.

Key

The coefficient of $x^5$ is $p_1q_4 + p_2q_3 + p_3q_2 + p_4q_1$.

But note that $p_1q_4 + p_2q_3 + p_3q_2 + p_4q_1 = \text{Prob}(\text{sum} = 5)$

More generally, the coefficient of $x^i$ is $\text{Prob}(\text{sum} = i)$. 
Polynomials are our Friends!

Let \((p_1, \ldots, p_6)\) and \((q_1, \ldots, q_6)\) be dice. Form polynomials based on the dice.

\[(p_6 x^6 + p_5 x^5 + \cdots + p_1 x^1)\text{ and } (q_6 x^6 + q_5 x^5 + \cdots + q_1 x^1)\.]
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\[(p_6x^6 + p_5x^5 + \cdots + p_1x^1)(q_6x^6 + q_5x^5 + \cdots + q_1x^1)\]
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\( (p_6x^6 + p_5x^5 + \cdots + p_1x^1)(q_6x^6 + q_5x^5 + \cdots + q_1x^1) \)

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Let \((p_1, \ldots, p_6)\) and \((q_1, \ldots, q_6)\) be dice. Form polynomials based on the dice. 
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More generally, the coefficient of \(x^i\) is \(\text{Prob}(\text{sum} = i)\).
No Dice!

Let \((p_1, \ldots, p_6)\) and \((q_1, \ldots, q_6)\) be dice. **Assume** they yield **fair sums**, all sums have prob \(1/11\). Then

\[
(p_6 x^6 + \cdots + p_1 x^1)(q_6 x^6 + \cdots + q_1 x^1) = \frac{1}{11}(x^{12} + x^{11} + \cdots + x^2)
\]

So

\[
(p_6 x^5 + \cdots + p_1)(q_6 x^5 + \cdots + q_1) = \frac{1}{11}(x^{10} + x^{9} + \cdots + x + 1)
\]

Continued on Next Slide.
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Continued on Next Slide.
From last slide: If there are two loaded dice that give fair sums then there exist reals \((p_1, \ldots, p_6), (q_1, \ldots, q_6)\) such that

\[
(p_6x^5 + \cdots + p_1)(q_6x^5 + \cdots + q_1) = 1.
\]

1. \(p_6x^5 + \cdots + p_1\): odd-degree poly, so has \(\geq 1\) real root.
2. \(q_6x^5 + \cdots + q_1\): odd-degree poly, so has \(\geq 1\) real root.

Does \(x^{10} + x^9 + \cdots + x + 1\) have any real roots?

\[
x^{11} - 1 = (x - 1)(x^{10} + \cdots + x + 1)
\]

1. \(r\) root of \(x^{10} + \cdots + x + 1 = \Rightarrow r\) root of \(x^{11} - 1 \& r \neq 1\).
2. \(r\) root of \(x^{11} - 1 \& r \neq 1 = \Rightarrow r\) root of \(x^{10} + \cdots + x + 1\).

The roots of \(x^{11} - 1\) are on the complex unit circle. See Next Slide.
From last slide: If there are two loaded dice that give fair sums then there exist reals $(p_1, \ldots, p_6)$, $(q_1, \ldots, q_6)$ such that

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1. \(p_6x^5 + \cdots + p_1\): odd-degree poly, so has \(\geq 1\) real root.
No Dice (cont)

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The 11th Roots of Unity: Only Real one is 1

1 is only real 11th root of unity.
The 11th Roots of Unity: Only Real one is 1

\[ \omega = \cos \left( \frac{2\pi}{11} \right) + \sin \left( \frac{2\pi}{11} \right) i \]

1 is only real 11th root of unity. \( x^{10} + \cdots + 1 = 0 \): **no** real roots.
Recap
If there exists two 6-sided dice that give fair sums then there exists reals $p_1, \ldots, p_6, q_1, \ldots, q_6$ such that

\[
\left( p_6 x^5 + \cdots + p_1 \right) \left( q_6 x^5 + \cdots + q_1 \right) = 11 \left( x^{10} + \cdots + 1 \right)
\]

The Left Hand Side has $\geq 2$ real roots.
The Right Hand Side has 0 real roots.
Contradiction.
Recap
If there exists two 6-sided dice that give fair sums then there exists reals $p_1, \ldots, p_6, q_1, \ldots, q_6$ such that

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The Right Hand Side has 0 real roots.

Contradiction
What About Two $d$-Sided Dice?

For which $d \geq 2$ can you load two $d$-sided dice to get fair sums?

**VOTE:**

1. No $d$.
2. All odd $d$.
3. All prime $d$.
4. UNKNOWN TO BILL!
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**Answer** No $d$. 
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**Answer** No $d$.

1. The proof that for even $d$ you cannot load two $d$-sided dice to get fair sums is similar to what we did for two 6-sided dice.
What About Two $d$-Sided Dice?

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**Answer** No $d$.

1. The proof that for even $d$ you cannot load two $d$-sided dice to get fair sums is similar to what we did for two 6-sided dice.

2. The proof that for odd $d$ you cannot load two $d$-sided dice to get fair sums requires new techniques.
Is there a $d_1, d_2 \geq 2$ such that there are $d_1$-sided and $d_2$-sided dice that give fair sums. **VOTE:** YES or NO or UNKNOWN TO BILL.
Can You Ever Load Dice to Get Fair Sums?

Is there a $d_1, d_2 \geq 2$ such that there are $d_1$-sided and $d_2$-sided dice that give fair sums. **VOTE:** YES or NO or UNKNOWN TO BILL. YES.

2 sided die: $(\frac{1}{2}, \frac{1}{2})$.
3 sided die: $(\frac{1}{2}, 0, \frac{1}{2})$.
Prob of a 2 is $\frac{1}{2} \times \frac{1}{2} = \frac{1}{4}$.
Prob of a 3 is $\frac{1}{2} \times 0 + \frac{1}{2} \times \frac{1}{2} = \frac{1}{4}$.
Prob of a 4 is $\frac{1}{2} \times 0 + \frac{1}{2} \times \frac{1}{2} = \frac{1}{4}$.
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Is there a $d_1, d_2 \geq 2$ such that there are $d_1$-sided and $d_2$-sided dice that give fair sums. **VOTE:** YES or NO or UNKNOWN TO BILL.

YES.
a 2-sided die and a 3-sided die can be loaded to get fair sums:

- 2 sided die: \((\frac{1}{2}, \frac{1}{2})\).
- 3 sided die: \((\frac{1}{2}, 0, \frac{1}{2})\).

Prob of a 2 is \(\frac{1}{2} \times \frac{1}{2} = \frac{1}{4}\).

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Prob of a 5 is $\frac{1}{2} \times \frac{1}{2} = \frac{1}{4}$.  

Can You Ever Load Dice to Get Fair Sums?
Can We Get Fair Sums Without Using 0 Prob?

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Rather than ponder the moral implications, let's ask a math question:
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Is there a $d_1, d_2 \geq 2$ such that $d_1$-sided and $d_2$-sided dice that give fair sums, with all the probs on the dice $> 0$?

**VOTE:** YES or NO or UNKNOWN TO BILL!
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**VOTE:** YES or NO or UNKNOWN TO BILL!

NO.
Exactly which sets of dice can be loaded to get fair sums? Gasarch and Kruskal
https://www.cs.umd.edu/~gasarch/papers/dice.pdf proved the following
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**Definition** A die \((p_1, \ldots, p_n)\) is **nice** if it is symmetric and, for all \(i\), \(p_i = 0\) or \(p_i = p_1\).
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**Definition** A die \((p_1, \ldots, p_n)\) is **nice** if it is symmetric and, for all \(i\), \(p_i = 0\) or \(p_i = p_1\).

**Theorem** Dice \(D_1, \ldots, D_m\) have fair sums iff (1) each \(D_i\) is nice, and (2) every sum can be rolled in exactly one way.
When Can you? When Can’t You?

Exactly which sets of dice can be loaded to get fair sums? Gasarch and Kruskal
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**Definition** A die \((p_1, \ldots, p_n)\) is **nice** if it is symmetric and, for all \(i\), \(p_i = 0\) or \(p_i = p_1\).

**Theorem** Dice \(D_1, \ldots, D_m\) have fair sums iff (1) each \(D_i\) is nice, and (2) every sum can be rolled in exactly one way.

**Note** The Theorem can be used to determine, given \(m_1, \ldots, m_L\), is there a set of dice, one \(m_1\)-sided, one \(m_2\)-sided, \(\ldots,\) one \(m_L\)-sided that gives fair sums.
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**Fame!** One paper refers to the **Gasarch-Kruskal Theorem**.
How Close To Uniform Can You Get?

Asgarli, Hartclass, Ostrov, Walden showed the following:
How Close To Uniform Can You Get?

Asgarli, Hartclass, Ostrov, Walden showed the following: https://arxiv.org/pdf/2304.08501.pdf

**Definition** Let \((p_1, \ldots, p_n)\) and \((q_1, \ldots, q_n)\) be two prob dist. The **distance between them** is \(\sum_i (p_i - q_i)^2\). A pair of loaded \(n\)-sided dice is **optimal** if the distance between its prob of sums and \((\frac{1}{2n-1}, \ldots, \frac{1}{2n-1})\) is minimum over all pairs of loaded dice.
How Close To Uniform Can You Get?

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How far are normal dice from uniform?

\[
2(1/11-1/36)^2 + 2(1/11-1/18)^2 + 2(1/11-1/12)^2 + 2(1/9-1/11)^2 + 2(5/36 - 1/11)^2 + (1/6 - 1/11)^2 \sim 0.0217
\]
Theorem  The optimal pair of 6-sided dice is
\((\frac{1}{2}, 0, 0, 0, 0, \frac{1}{2})\) and \((\frac{1}{8}, \frac{3}{16}, \frac{3}{16}, \frac{3}{16}, \frac{3}{16}, \frac{1}{8})\).
Theorem The optimal pair of 6-sided dice is 
\[(\frac{1}{2}, 0, 0, 0, 0, \frac{1}{2}) \text{ and } (\frac{1}{8}, \frac{3}{16}, \frac{3}{16}, \frac{3}{16}, \frac{3}{16}, \frac{1}{8})\].

\[\text{Prob}(2) = \frac{1}{16}\]
Theorem The optimal pair of 6-sided dice is $(\frac{1}{2}, 0, 0, 0, 0, \frac{1}{2})$ and $(\frac{1}{8}, \frac{3}{16}, \frac{3}{16}, \frac{3}{16}, \frac{3}{16}, \frac{1}{8})$.

$\text{Prob}(2) = \frac{1}{16}$

$\text{Prob}(3) = \text{Prob}(4) = \text{Prob}(5) = \text{Prob}(6) = \frac{3}{32}$
Theorem The optimal pair of 6-sided dice is $(\frac{1}{2}, 0, 0, 0, 0, \frac{1}{2})$ and $(\frac{1}{8}, \frac{3}{16}, \frac{3}{16}, \frac{3}{16}, \frac{3}{16}, \frac{1}{8})$.

\[
\begin{align*}
\text{Prob}(2) &= \frac{1}{16} \\
\text{Prob}(3) &= \text{Prob}(4) = \text{Prob}(5) = \text{Prob}(6) = \frac{3}{32} \\
\text{Prob}(7) &= \frac{1}{8}
\end{align*}
\]
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Distance from Uniform is $\frac{1}{352} \approx 0.0028$. 
Theorem  The optimal pair of 6-sided dice is 
\((\frac{1}{2}, 0, 0, 0, 0, \frac{1}{2})\) and \((\frac{1}{8}, \frac{3}{16}, \frac{3}{16}, \frac{3}{16}, \frac{3}{16}, \frac{1}{8})\).

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\text{Prob}(12) &= \frac{1}{16}
\end{align*}\]
Theorem  The optimal pair of 6-sided dice is
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Distance from Uniform is \(\frac{1}{352} \sim 0.0028\).
Measuring Unfairness

One measure is distance from uniform which is what Asgarli, Hartclass, Ostrov, Walden used.

1. Normal Dice
   0.0217 away from uniform

2. Optimal Dice
   0.0028 away from uniform

Another measure is the distance between the max prob of a sum and the min prob of a sum.

1. Normal Dice
   They were $\frac{1}{6} - \frac{1}{36} \sim 0.139$ unfair.

2. Optimal Dice
   They are $\frac{1}{8} - \frac{1}{16} = 0.0625$ unfair.
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One measure is distance from uniform which is what Asgarli, Hartclass, Ostrov, Walden used.

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Measuring Unfairness

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2. Optimal Dice $0.0028$ away from uniform.

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1. Normal Dice They were $1/6 - 1/36 \sim 0.139$ unfair.
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What About $n$-sided Dice?

The optimal pair of $n$-sided dice is $\left(\frac{1}{2}, \ldots, \frac{1}{2}\right)$ and $\left(\frac{2}{3n-2}, \frac{3}{3n-2}, \ldots, \frac{2}{3n-2}\right)$. The distance from uniform is $\frac{1}{2}(2n-1)(3n-2)$. 
What About \( n \)-sided Dice?

The optimal pair of \( n \)-sided dice is 
\[
\left( \frac{1}{2}, 0, \ldots, 0, \frac{1}{2} \right)
\]
and
\[
\left( \frac{2}{3n-2}, \frac{3}{3n-2}, \ldots, \frac{3}{3n-2}, \frac{2}{3n-2} \right).
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What About $n$-sided Dice?

The optimal pair of $n$-sided dice is
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The distance from uniform is \(\frac{1}{2(2n-1)(3n-2)}\).
Different Labels on Dice

William Gasarch - University of MD
A labeling of a 6-sided die has any positive natural numbers as labels. We allow using a number twice. We allow using numbers higher than 6. So \((1, 2, 2, 3, 5, 8)\) would be allowed.
Can You Label Dice To Get Same Probs?

A **labeling** of a 6-sided die has any positive natural numbers as labels. We allow using a number twice. We allow using numbers higher than 6. So \((1, 2, 2, 3, 5, 8)\) would be allowed.

A **non-standard labeling** is a labeling that is not \((1, 2, 3, 4, 5, 6)\).
Can You Label Dice To Get Same Probs?

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Is there a non-standard labeling of a pair of 6-sided dice so that the dice yield the SAME probabilities as the standard dice?
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**VOTE:** YES or NO or UNKNOWN TO BILL!
A labeling of a 6-sided die has any positive natural numbers as labels. We allow using a number twice. We allow using numbers higher than 6. So (1, 2, 2, 3, 5, 8) would be allowed.

A non-standard labeling is a labeling that is not (1, 2, 3, 4, 5, 6).

Is there a non-standard labeling of a pair of 6-sided dice so that the dice yield the SAME probabilities as the standard dice? **VOTE:** YES or NO or UNKNOWN TO BILL!

YES. We prove this.
Let Polynomials Do The Work For You!

\[(x^6 + x^5 + x^4 + x^3 + x^2 + x)(x^6 + x^5 + x^4 + x^3 + x^2 + x)\]
Let Polynomials Do The Work For You!

\[(x^6 + x^5 + x^4 + x^3 + x^2 + x)(x^6 + x^5 + x^4 + x^3 + x^2 + x)\]

Look at coefficient of \(x^6\)
Let Polynomials Do The Work For You!

$$(x^6 + x^5 + x^4 + x^3 + x^2 + x)(x^6 + x^5 + x^4 + x^3 + x^2 + x)$$

Look at coefficient of $x^6$

$$x^1x^5 + x^2x^4 + x^3x^3 + x^4x^2 + x^5x^1 = 5x^6$$
Let Polynomials Do The Work For You!

\[(x^6 + x^5 + x^4 + x^3 + x^2 + x)(x^6 + x^5 + x^4 + x^3 + x^2 + x)\]

Look at coefficient of \(x^6\)

\[x^1x^5 + x^2x^4 + x^3x^3 + x^4x^2 + x^5x^1 = 5x^6\]

\[= (\text{Number of ways to get 6})x^6\]
Let Polynomials Do The Work For You!

\[
(x^6 + x^5 + x^4 + x^3 + x^2 + x)(x^6 + x^5 + x^4 + x^3 + x^2 + x)
\]

Look at coefficient of \(x^6\)

\[
x^1 x^5 + x^2 x^4 + x^3 x^3 + x^4 x^2 + x^5 x^1 = 5x^6
\]

\[= (\text{Number of ways to get 6})x^6\]

Coefficient of \(x^n\) is number of ways to get \(n\).
Example of Non-Standard Labelings

What if we label the dice \((1, 2, 2, 2, 5, 5)\) and \((1, 3, 3, 3, 3, 7)\)?
Example of Non-Standard Labelings

What if we label the dice (1, 2, 2, 2, 5, 5) and (1, 3, 3, 3, 3, 7)?

\[(2x^5+3x^2+x)(x^7+4x^3+x) = 2x^{12}+3x^9+9x^8+2x^6+12x^5+4x^4+3x^3+x^2\]
Example of Non-Standard Labelings

What if we label the dice \((1, 2, 2, 2, 5, 5)\) and \((1, 3, 3, 3, 3, 7)\)?

\[
(2x^5 + 3x^2 + x)(x^7 + 4x^3 + x) = 2x^{12} + 3x^9 + 9x^8 + 2x^6 + 12x^5 + 4x^4 + 3x^3 + x^2
\]

1. 12: TWO ways. Prob \(\frac{1}{18}\).
2. 9: THREE ways. Prob \(\frac{1}{12}\).
3. 8: NINE ways. Prob \(\frac{1}{4}\).
4. 6: TWO ways. Prob \(\frac{1}{18}\).
5. 5: TWELVE ways. Prob \(\frac{1}{3}\).
6. 4: FOUR ways. Prob \(\frac{1}{9}\).
7. 3: THREE ways. Prob \(\frac{1}{12}\).
8. 2: ONE ways. Prob \(\frac{1}{36}\).
**Question** Is there a nonstandard labeling of two 6-sided dice that gives the same probabilities as the standard dice?
Is there a Non-Standard Labeling That...

**Question** Is there a nonstandard labeling of two 6-sided dice that gives the same probabilities as the standard dice?

**Question Phrased In Terms of Polynomials** Does there exist $a_1 \geq \cdots \geq a_6$ and $b_1 \geq \cdots \geq b_6$ such that

$$(x^{a_1} + x^{a_2} + x^{a_3} + x^{a_4} + x^{a_5} + x^{a_6})(x^{b_1} + x^{b_2} + x^{b_3} + x^{b_4} + x^{b_5} + x^{b_6}) = (x^6 + x^5 + x^4 + x^3 + x^2 + x)^2.$$
Is there a Non-Standard Labeling That...

**Question** Is there a nonstandard labeling of two 6-sided dice that gives the same probabilities as the standard dice?

**Question Phrased In Terms of Polynomials** Does there exist $a_1 \geq \cdots \geq a_6$ and $b_1 \geq \cdots \geq b_6$ such that

\[
(x^{a_1} + x^{a_2} + x^{a_3} + x^{a_4} + x^{a_5} + x^{a_6})(x^{b_1} + x^{b_2} + x^{b_3} + x^{b_4} + x^{b_5} + x^{b_6}) =
\]

\[
(x^6 + x^5 + x^4 + x^3 + x^2 + x)^2.
\]
Is there a Non-Standard Labeling That... Cont.

\((x^{a_1} + x^{a_2} + x^{a_3} + x^{a_4} + x^{a_5} + x^{a_6})(x^{b_1} + x^{b_2} + x^{b_3} + x^{b_4} + x^{b_5} + x^{b_6}) =\)

\((x^6 + x^5 + x^4 + x^3 + x^2 + x)^2 = x^2(x^5 + x^4 + x^3 + x^2 + x + 1)^2 =\)

\(x^2(x + 1)^2(x^2 - x + 1)^2(x^2 + x + 1)^2.\)
Need to factor

\[ x^2(x + 1)^2(x^2 - x + 1)^2(x^2 + x + 1)^2. \]

into two polynomials, each of which represents a 6-sided die.
Need to Factor... 

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\[ x^2(x + 1)^2(x^2 - x + 1)^2(x^2 + x + 1)^2. \]

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Finite Number of cases.
Need to Factor...

Need to factor

\[ x^2(x + 1)^2(x^2 - x + 1)^2(x^2 + x + 1)^2. \]

into two polynomials, each of which represents a 6-sided die.

Finite Number of cases.

\[ x(x + 1)(x^2 + x + 1) \times x(x + 1)(x^2 - x + 1)^2(x^2 + x + 1) \]
Need to Factor...

Need to factor

\[ x^2(x + 1)^2(x^2 - x + 1)^2(x^2 + x + 1)^2. \]

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Finite Number of cases.

\[ x(x + 1)(x^2 + x + 1) \times x(x + 1)(x^2 - x + 1)^2(x^2 + x + 1) \]

\[ x(x + 1)(x^2 + x + 1) = x^4 + 2x^3 + 2x^2 + x. \]

DIE: (1, 2, 2, 3, 3, 4)
Need to Factor... 

Need to factor 

\[ x^2(x + 1)^2(x^2 - x + 1)^2(x^2 + x + 1)^2. \]

into two polynomials, each of which represents a 6-sided die.

Finite Number of cases.

\[ x(x + 1)(x^2 + x + 1) * x(x + 1)(x^2 - x + 1)^2(x^2 + x + 1) \]

\[ x(x + 1)(x^2 + x + 1) = x^4 + 2x^3 + 2x^2 + x. \]

DIE: (1, 2, 2, 3, 3, 4)

\[ x(x + 1)(x^2 - x + 1)^2(x^2 + x + 1) = x^8 + x^6 + x^5 + x^4 + x^3 + x. \]

DIE: (1, 3, 4, 5, 6, 8).
Need to Factor...

Need to factor

\[ x^2(x + 1)^2(x^2 - x + 1)^2(x^2 + x + 1)^2. \]

into two polynomials, each of which represents a 6-sided die. Finite Number of cases.

\[ x(x + 1)(x^2 + x + 1) \ast x(x + 1)(x^2 - x + 1)^2(x^2 + x + 1) \]

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DIE: (1, 2, 2, 3, 3, 4)

\[ x(x + 1)(x^2 - x + 1)^2(x^2 + x + 1) = x^8 + x^6 + x^5 + x^4 + x^3 + x. \]

DIE: (1, 3, 4, 5, 6, 8).

So desired dice are (1, 2, 2, 3, 3, 4) and (1, 3, 4, 5, 6, 8).
What About Two $d$-Sided Dice?

For which $d \geq 2$ are there two non-standard $d$-sided dice that have the same prob as standard dice? **VOTE:**

1. All even $d$.
2. All non-prime $d$
3. Something Else
4. UNKNOWN TO BILL!
For which \( d \geq 2 \) are there two non-standard \( d \)-sided dice that have the same prob as standard dice? **VOTE:**

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**Answer** All non-prime \( d \).
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**Answer** All non-prime $d$.

1. The proof is similar to what we did, though requires some thought.
What About $d_1, d_2$-Sided Dice?

For which $d_1, d_2 \geq 2$ are there non-standard $d_1$-sided die and $d_2$-sided die that have the same prob as standard dice? VOTE:

1. One of $d_1, d_2$ has to be non-prime.
2. Both $d_1, d_2$ have to be non-prime.
3. Something Else
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**Answer** UNKNOWN TO BILL. Will say why on next slide.
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**Answer** UNKNOWN TO BILL. Will say why on next slide.
More is Known

1. George Sicherman first posed the problem and solved it in 1978. The dice produced are sometimes called Sicherman Dice. You can buy these dice on the web!

2. Gasarch has an exposition on this material: https://www.cs.umd.edu/~gasarch/BLOGPAPERS/billdice.pdf

3. Gallian and Rusin’s paper exactly characterizes when this is possible: https://www.cs.umd.edu/~gasarch/BLOGPAPERS/nonstandarddice.pdf The paper only looked at $n d$-sided dice. I do not know if someone else later did the case of $d_1$, $d_2$. 
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Parting Thoughts

William Gasarch - University of MD
Parting Thoughts

1. Easy to state problems about dice lead to math of interest.
2. Polynomials are useful for problems with dice since multiplication gives information.
3. It is remarkable that a problem about dice lead to looking at complex roots of polynomials!
4. Congratulations for doing well on the UMCP HS Math Competition!
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