How big & how small can the set of all subset-sums be for a set of size \( n \).

1. \( \{2^0, 2^1, \ldots, 2^{n-1}\} \) has all \( 2^n \) subset-sums. This is of course max.

2. \( \{1, \ldots, n\} \) has \( \frac{n(n+1)}{2} + 1 \) subset-sums. Lance proved that this is min.

Are there any other subsets of \( \{1, \ldots, n\} \) that have \( \frac{n(n+1)}{2} + 1 \) subset-sums?

\textbf{Stupid Answer} Yes, take \( \{x, 2x, \ldots, nx\} \).

\((\exists A)[A \neq \{x, \ldots, nx\}] \) that has \( \frac{n(n+1)}{2} \) subset-sums?

\( n = 3 \): \( \{a, b, a+b\} \) sums \( \{0, a, b, a+b, 2a+b, a+2b, 2a+2b\} \). 7 of them.

For \( n \geq 4 \) we will show that the answer is NO.

Suppose a set \( A = \{a_1 < \cdots < a_n\} \) has \( \frac{n(n+1)}{2} + 1 \) subset-sums. Now add a larger number \( b \) to the set. Suppose the new set has \( \frac{(n+1)(n+2)}{2} + 1 \) subset-sums.

\( \frac{n(n+1)}{2} + 1 \) of them are all the subsetsums of the \( A \) with \( b \) added to them. So, we can have at most \( n + 1 \) other subsetsums that do not contain \( b \). Since the subsets \( \emptyset \), \( \{a_1\}, \{a_2\}, \ldots \{a_n\} \) are \( n + 1 \) subsetsums which are less than \( b \), there must be no other subsetsums. This implies that \( b \) is the smallest number greater than \( a_n \) which is a subsetsum of \( A \). This also implies that if \( A \cup \{b\} \) has minimal subsetsums, then so does \( A \). So, we only need to find all the 4-element sets that have 11 subsetsums.

We know that any candidate set will look like \( \{a, b, a+b, 2a+b\} \) where \( a < b \).

That set will have the following 11 subsetsums:

0
\( a \)
\( b \)
\( a+b \)
\( 2a+b \)
\( 3a+b \)
\( 2a+2b \)
\( 3a+2b \)
\( 4a+2b \)
\( 3a+3b \)
\( 4a+3b \)
as well as \( a+2b \).

So, \( a + 2b \) must equal one of the above numbers. The only possibility is \( 3a+b \). So, \( a + 2b = 3a + b \), which implies \( b = 2a \). So, our set had to be \( \{a, 2a, 3a, 4a\} \).