

Rectangle Free Coloring of Grids

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Credit Where Credit is Due

This Work Grew Out of a Project In the UMCP SPIRAL (Summer Program in Research and Learning) Program for College Math Majors at HBCU's.

One of the students, **Brett Jefferson** has his own paper on this subject.

ALSO: Multidim version has been worked on by Cooper, Fenner, Purewal (submitted)

ALSO: Zarankiewics [7] asked similar questions.

Square Theorem:

Theorem

*For all c , there exists G such that
for every c -coloring of $G \times G$ there exists a monochromatic square.*

...
...	R	...	R	...
...	\vdots	...	\vdots	...
...	R	...	R	...
...

Proving the Square Theorem and Bounding $G(c)$

How to prove Square Theorem?

1. Corollary of Hales-Jewitt Theorem [1]. Bounds on G HUGE!
2. Corollary of Gallai's theorem [3,4,6]. Bounds on G HUGE!
3. From VDW directly (folklore). Bounds on G HUGE!
4. Directly (folklore?). Bounds on G HUGE!
5. Graham and Solymosi [2]. $G \leq 2^{2^{81}}$. Better but still HUGE.

Best known upper and lower bounds:

1. $G(2) \leq 2^{2^{81}}$.
2. $\Omega(c^{4/3}) \leq G(c)$. (Upper bound not writable-downable.)

What If We Only Care About Rectangles?

Definition

$G_{n,m}$ is the grid $[n] \times [m]$.

1. $G_{n,m}$ is **c -colorable** if there is a c -colorings of $G_{n,m}$ such that no rectangle has all four corners the same color.
2. $\chi(G_{n,m})$ is the least c such that $G_{n,m}$ is c -colorable.

Our Main Question

Fix c

Exactly which $G_{n,m}$ are c -colorable?

Two Motivations!

1. Relaxed version of Square Theorem- hope for better bounds.
2. Coloring $G_{n,m}$ without rectangles is equivalent to coloring edges of $K_{n,m}$ without getting monochromatic $K_{2,2}$.

Our results yield **Bipartite Ramsey Numbers!**

Obstruction Sets

Definition

$G_{n,m}$ contains $G_{a,b}$ if $a \leq n$ and $b \leq m$.

Theorem

For all c there exists a unique finite set of grids OBS_c such that

$G_{n,m}$ is c -colorable iff

$G_{n,m}$ does not contain any element of OBS_c .

1. Can prove using well-quasi-orderings. No bound on $|\text{OBS}_c|$.
2. Our tools yield alternative proof and show

$$2\sqrt{c}(1 - o(1)) \leq |\text{OBS}_c| \leq 2c^2.$$

Rephrase Main Question

Fix c

What is OBS_c

Rectangle Free Sets and Density

Definition

$G_{n,m}$ is the grid $[n] \times [m]$.

1. $X \subseteq G_{n,m}$ is **Rectangle Free** if there are NOT four vertices in X that form a rectangle.
2. $\text{rfree}(G_{n,m})$ is the size of the largest Rect Free subset of $G_{n,m}$.

Rectangle Free subset of $G_{21,12}$ of size $63 = \left\lceil \frac{21 \cdot 12}{4} \right\rceil$

	01	02	03	04	05	06	07	08	09	10	11	12
1	•	•										
2	•		•									
3		•	•									
4			•	•	•							
5		•		•		•						
6	•				•	•						
7						•	•	•				
8					•		•		•			
9				•				•	•			
10						•				•	•	
11					•					•		•
12				•							•	•
13			•			•			•			•
14			•					•		•		
15			•				•				•	
16		•							•	•		
17		•			•			•			•	
18		•					•					•
19	•								•		•	
20	•							•				•
21	•			•			•			•		

Colorings Imply Rectangle Free Sets

Lemma

Let $n, m, c \in \mathbb{N}$. If $\chi(G_{n,m}) \leq c$ then $\text{rfree}(G_{n,m}) \geq \lceil mn/c \rceil$.

Note: We use to get non-col results as density results!!

Zarankiewics's Problem

Definition

$Z_{a,b}(m, n)$ is the largest subset of $G_{n,m}$ that has no $[a] \times [b]$ submatrix.

Zarankiewics [7] asked for exact values for $Z_{a,b}(m, n)$.
We care about $Z_{2,2}(m, n)$.

PART I: 2-COLORABILITY

We will **EXACTLY** Characterize which $G_{n,m}$ are 2-colorable!

$G_{5,5}$ IS NOT 2-Colorable!

Theorem

$G_{5,5}$ *is not* 2-Colorable.

Proof:

$$\begin{aligned}\chi(G_{5,5}) = 2 &\implies \text{rfree}(G_{5,5}) \geq \lceil 25/2 \rceil = 13 \\ &\implies \text{there exists a column with } \geq \lceil 13/5 \rceil = 3 \text{ } R\text{'s}\end{aligned}$$

Case 1: There is a column with 5 R 's

Case 1: There is a column with 5 R 's.

R	○	○	○	○
R	○	○	○	○
R	○	○	○	○
R	○	○	○	○
R	○	○	○	○

Remaining columns have ≤ 1 R so

$$\text{Number of } R\text{'s} \leq 5 + 1 + 1 + 1 + 1 = 9 < 13.$$

Case 2: There is a column with 4 R 's

Case 2: There is a column with 4 R 's.

R	○	○	○	○
R	○	○	○	○
R	○	○	○	○
R	○	○	○	○
○	○	○	○	○

Remaining columns have ≤ 2 R 's

$$\text{Number of } R\text{'s} \leq 4 + 2 + 2 + 2 + 2 \leq 12 < 13$$

Case 3: Max in a column is 3 R 's

Case 3: Max in a column is 3 R 's.

Case 3a: There are ≤ 2 columns with 3 R 's.

Number of R 's $\leq 3 + 3 + 2 + 2 + 2 \leq 12 < 13$.

Case 3b: There are ≥ 3 columns with 3 R 's.

R	○	○	○	○
R	○	○	○	○
R	R	○	○	○
○	R	○	○	○
○	R	○	○	○

Can't put in a third column with 3 R 's!

Case 4: Max in a column is $\leq 2R$'s

Case 4: Max in a column is $\leq 2R$'s.

Number of R 's $\leq 2 + 2 + 2 + 2 + 2 \leq 10 < 13$.

No more cases. We are Done! Q.E.D.

$G_{4,6}$ IS 2-Colorable

Theorem

$G_{4,6}$ *is* 2-Colorable.

Proof.

R	R	R	B	B	B
R	B	B	R	R	B
B	R	B	R	B	R
B	B	R	B	R	R



$G_{3,7}$ IS NOT 2-Colorable

Theorem

$G_{3,7}$ *is not* 2-Colorable.

Proof.

$$\begin{aligned}\chi(G_{3,7}) = 2 &\implies \text{rfree}(G_{3,7}) \geq (\lceil 21/2 \rceil = 11 \\ &\implies \text{there is a row with } \geq \lceil 11/3 \rceil = 4 \text{ } R\text{'s}\end{aligned}$$

Proof similar to $G_{5,5}$ — lots of cases.



Complete Char of 2-Colorability

Theorem

$$\text{OBS}_2 = \{G_{3,7}, G_{5,5}, G_{7,3}\}.$$

Proof.

Follows from results $G_{5,5}, G_{7,3}$ not 2-colorable and $G_{4,6}$ is 2-colorable. □

PART II: TOOLS TO SHOW $G_{n,m}$ NOT c -COLORABLE

We show that if A is a Rectangle Free subset of $G_{n,m}$ then there is a relation between $|A|$ and n and m .

Bound on Size of Rectangle Free Sets

Theorem

Let $n, m \in \mathbb{N}$. If there exists rectangle-free $A \subseteq G_{n,m}$ then

$$|A| \leq \frac{m + \sqrt{m^2 + 4m(n^2 - n)}}{2}$$

Note: Proved by Reiman [5] while working on Zarankiewicz's problem.

Proof of Theorem

$A \subseteq G_{n,m}$, rectangle free.

x_i is number of points in i^{th} column.

	1	\dots	m
1		\dots	
\vdots		\vdots	
n		\dots	
	x_1 points $\binom{x_1}{2}$ pairs of points	\dots \dots	x_m points $\binom{x_m}{2}$ pairs of points

Proof of Theorem

$A \subseteq G_{n,m}$, rectangle free.

x_i is number of points in i^{th} column.

	1	...	m
1		...	
\vdots		\vdots	
n		...	
	x_1 points $\binom{x_1}{2}$ pairs of points	...	x_m points $\binom{x_m}{2}$ pairs of points

$$\sum_{i=1}^m \binom{x_i}{2} \leq \binom{n}{2}.$$

Proof of Theorem (cont)

$$\sum_{i=1}^m \binom{x_i}{2} \leq \binom{n}{2}.$$

Sum minimized when $x_1 = \dots = x_m = x$

$$m \binom{x}{2} \leq \binom{n}{2}.$$

$$x \leq \frac{m + \sqrt{m^2 + 4m(n^2 - n)}}{2m}$$

$$|A| \leq xm \leq \frac{m + \sqrt{m^2 + 4m(n^2 - n)}}{2}$$

Bound on Size of Rectangle Free Sets (new)

Theorem

Let $a, n, m \in \mathbb{N}$. Let q, r be such that $a = qn + r$ with $0 \leq r \leq n$. Assume that there exists $A \subseteq G_{m,n}$ such that $|A| = a$ and A is rectangle-free.

1. If $q \geq 2$ then

$$n \leq \left\lfloor \frac{m(m-1) - 2rq}{q(q-1)} \right\rfloor.$$

2. If $q = 1$ then

$$r \leq \frac{m(m-1)}{2}.$$

Refined ideas from proof above.

PART III: TOOLS TO SHOW $G_{n,m}$ IS c -COLORABLE

We define and use Strong c -Colorings to get c -Colorings

Strong c -Colorings

Definition

Let $c, n, m \in \mathbb{N}$. $\chi : G_{n,m} \rightarrow [c]$. χ is a **strong c -coloring** if the following holds: CANNOT have a rectangle with the two right most corners are same color and the two left most corners the same color.

Example: A strong 3-coloring of $G_{4,6}$.

R	R	G	R	G	G
B	G	R	G	R	G
G	B	B	G	G	R
G	G	G	B	B	B

Strong Coloring Lemma

Let $c, n, m \in \mathbb{N}$. If $G_{n,m}$ is strongly c -colorable then $G_{n,cm}$ is c -colorable.

Example:

R	R	G	R	G	G	B	B	R	B	R	R	G	G	B	G	B	B
B	G	R	G	R	G	G	R	B	R	B	R	R	B	G	B	G	B
G	B	B	G	G	R	R	G	G	R	R	B	B	R	R	B	B	G
G	G	G	B	B	B	R	R	R	G	G	G	B	B	B	R	R	R

Combinatorial Coloring Theorem

Let $c \geq 2$.

1. There is a strong c -coloring of $G_{c+1, \binom{c+1}{2}}$.
2. There is a c -coloring of $G_{c+1, m}$ where $m = c \binom{c+1}{2}$.

Example: Strong 5-coloring of $G_{6,15}$.

O	O	O	O	O	R	R	R	R	R	R	R	R	R	R
O	R	R	R	R	O	O	O	O	B	B	B	B	B	B
R	O	B	B	B	O	B	B	B	O	O	O	G	G	G
B	B	O	G	G	B	O	G	G	O	G	G	O	O	P
G	G	G	O	P	G	G	O	P	G	O	P	O	P	O
P	P	P	P	O	P	P	P	O	P	P	O	P	O	O

Coloring Using Primes!

Theorem

Let p be a prime.

1. There is a strong p -coloring of $G_{p^2, p+1}$.
2. There is a p -coloring of G_{p^2, p^2+p} .

Proof.

Uses geometry over finite fields.



Note: Have more general theorem.

Generalization of of Strong Colorings

Definition

Let $c, c', n, m \in \mathbb{N}$. $\chi : G_{n,m} \rightarrow [c]$. χ is a *strongly* (c, c') -coloring if the following holds: If have rectangles where two right most corners same and two left most corners same, then diff colors, and both colors in $[c']$.

Generalization of of Strong Colorings

Definition

Let $c, c', n, m \in \mathbb{N}$. $\chi : G_{n,m} \rightarrow [c]$. χ is a *strongly* (c, c') -coloring if the following holds: If have rectangles where two right most corners same and two left most corners same, then diff colors, and both colors in $[c']$.

Strong $(4, 2)$ -coloring of $G_{6,15}$. ($R = 1, B = 2$)

R	R	R	R	R	G	G	G	B	G	G	B	B	B	B
R	B	B	B	B	R	R	R	R	P	P	G	G	G	B
B	R	G	G	B	R	B	B	B	R	R	R	P	P	G
B	B	R	P	G	B	R	P	G	R	B	B	R	R	P
G	G	B	R	P	B	B	R	P	B	R	P	R	B	R
P	P	P	B	R	P	P	B	R	B	B	R	B	R	R

Lemma: Generalized Strong Colorings Yield Colorings

Lemma

Let $c, c', n, m \in \mathbb{N}$. Let $x = \lfloor c/c' \rfloor$. If $G_{n,m}$ is strongly (c, c') -colorable then $G_{n, xm}$ is c -colorable.

Proof is similar to proof of strong coloring Lemma.

Using a Generalization of Strong Coloring

Theorem

Let $c \geq 2$.

1. There is a c -coloring of $G_{c+2, m'}$ where $m' = \lfloor c/2 \rfloor \binom{c+2}{2}$.
2. There is a c -coloring of $G_{2c, 2c^2 - c}$.

Another Combinatorial Coloring Theorem

Theorem

Let $c \geq 2$.

1. There is a strong $(c, 2)$ -coloring of $G_{c+2, m}$ where $m = \binom{c+2}{2}$.
2. There is a c -coloring of $G_{c+2, m'}$ where $m' = \lfloor c/2 \rfloor \binom{c+2}{2}$.

Another Combinatorial Coloring Theorem

Theorem

Let $c \geq 2$.

1. There is a strong $(c, 2)$ -coloring of $G_{c+2, m}$ where $m = \binom{c+2}{2}$.
2. There is a c -coloring of $G_{c+2, m'}$ where $m' = \lfloor c/2 \rfloor \binom{c+2}{2}$.

Similar to proof of Combinatorial Coloring Theorem.

Tournament Graph Coloring Theorem

Let $c \geq 2$.

1. There is a strong c -coloring of $G_{2c, 2c-1}$.
2. There is a c -coloring of $G_{2c, 2c^2-c}$.

Proof.

Uses tournament graphs.



PART IV: 3-COLORABILITY

We will EXACTLY Characterize which $G_{n,m}$ are 3-colorable!

Easy 3-Colorable Results

Theorem

1. *The following grids **are not** 3-colorable.*

$G_{4,19}$, $G_{19,4}$, $G_{5,16}$, $G_{16,5}$, $G_{7,13}$, $G_{13,7}$, $G_{10,12}$, $G_{12,10}$, $G_{11,11}$.

2. *The following grids **are** 3-colorable.*

$G_{3,19}$, $G_{19,3}$, $G_{4,18}$, $G_{18,4}$, $G_{6,15}$, $G_{15,6}$, $G_{9,12}$, $G_{12,9}$.

Proof.

Follows from tools.



$G_{10,10}$ is 3-colorable

Theorem

$G_{10,10}$ is 3-colorable.

Proof.

UGLY! TOOLS DID NOT HELP AT ALL!!

R	R	R	R	B	B	G	G	B	G
R	B	B	G	R	R	R	G	G	B
G	R	B	G	R	B	B	R	R	G
G	B	R	B	B	R	G	R	G	R
R	B	G	G	G	B	G	B	R	R
G	R	B	B	G	G	R	B	B	R
B	G	R	B	G	B	R	G	R	B
B	B	G	R	R	G	B	G	B	R
G	G	G	R	B	R	B	B	R	B
B	G	B	R	B	G	R	R	G	G

$G_{10,11}$ is not 3-colorable

Theorem

$G_{10,11}$ is not 3-colorable.

Proof.

You don't want to see this. UGLY case hacking. □

Complete Char of 3-colorability

Theorem

$\text{OBS}_3 =$

$$\{G_{4,19}, G_{5,16}, G_{7,13}, G_{10,11}, G_{11,10}, G_{13,7}, G_{16,5}, G_{19,4}\}.$$

Proof.

Follows from above results on grids being or not being 3-colorable. □

PART V: 4-COLORABILITY

We will **MAKE PROGRESS ON** Characterizing which $G_{n,m}$ are 4-colorable.

Easy NOT 4-Colorable Results

Theorem

The following grids **are** NOT 4-colorable:

1. $G_{5,41}$ and $G_{41,5}$
2. $G_{6,31}$ and $G_{31,6}$
3. $G_{7,29}$ and $G_{29,7}$
4. $G_{9,25}$ and $G_{25,9}$
5. $G_{10,23}$ and $G_{23,10}$
6. $G_{11,22}$ and $G_{22,11}$
7. $G_{13,21}$ and $G_{21,13}$
8. $G_{17,20}$ and $G_{20,17}$
9. $G_{18,19}$ and $G_{19,18}$

Follows from tools for proving grids **are** NOT colorable.

Easy IS 4-Colorable Results

Theorem

The following grids *are* 4-colorable:

1. $G_{4,41}$ and $G_{41,4}$.
2. $G_{5,40}$ and $G_{40,5}$.
3. $G_{6,30}$ and $G_{30,6}$.
4. $G_{8,28}$ and $G_{28,8}$.
5. $G_{16,20}$ and $G_{20,16}$.

Follows from tools for proving grids *are* colorable.

Theorems with UGLY Proofs

Theorem

1. $G_{17,19}$ *is NOT 4-colorable: Used some tools.*
2. $G_{24,9}$ *is 4-colorable: Used strong coloring of $G_{9,6}$.*

Theorems with UGLY Proofs

Theorem

1. $G_{17,19}$ *is NOT 4-colorable*: Used some tools.
2. $G_{24,9}$ *is 4-colorable*: Used strong coloring of $G_{9,6}$.

P	R	R	P	R	R
P	B	B	R	P	B
P	G	G	B	B	P
R	P	G	P	G	B
B	P	R	B	P	G
G	P	B	G	R	P
G	B	P	P	B	G
R	G	P	G	P	R
B	R	P	R	G	P

4-coloring of $G_{21,11}$ Due to Brad Loren

	1	2	3	4	5	6	7	8	9	10	11
1	G	B	B	G	R	P	R	G	P	B	P
2	B	G	G	P	B	G	P	R	R	B	R
3	R	R	B	P	B	P	B	P	G	G	R
4	P	R	P	G	B	B	R	P	R	G	B
5	R	P	G	B	B	P	P	B	R	G	G
6	B	R	P	R	G	P	B	R	G	P	B
7	P	G	B	R	G	B	R	G	P	P	R
8	P	P	G	B	R	B	G	R	G	B	P
9	R	B	R	B	G	G	R	P	P	G	B
10	R	P	P	R	G	R	B	B	P	B	G
11	B	P	R	R	P	B	G	G	R	P	G
12	R	B	P	P	P	B	B	R	G	R	G
13	G	G	B	B	R	R	P	P	R	P	G
14	G	B	R	P	B	G	G	R	B	P	P
15	G	P	G	P	G	R	R	R	B	B	B
16	B	B	R	G	P	G	P	B	P	R	G
17	P	G	B	G	P	P	R	B	G	R	B
18	B	P	B	G	G	R	G	P	B	R	R
19	P	G	R	P	R	B	G	B	B	G	R
20	B	R	P	B	R	G	P	G	G	R	P
21	G	R	R	B	P	R	B	P	B	G	P

4-coloring of $G_{22,10}$ Due to Brad Loren

	1	2	3	4	5	6	7	8	9	10
1	P	G	R	R	G	G	P	P	B	B
2	G	P	B	G	B	B	P	R	P	R
3	B	G	B	R	P	P	G	R	P	B
4	P	P	G	G	R	R	B	B	G	P
5	P	B	P	P	G	R	R	G	G	R
6	P	B	R	B	R	P	G	R	G	G
7	G	P	G	P	B	P	R	B	R	G
8	P	R	R	B	P	B	G	G	B	R
9	P	B	B	R	R	G	R	G	P	G
10	R	R	B	B	P	G	R	B	G	P
11	R	G	G	P	R	B	B	G	P	R
12	R	B	R	G	G	P	P	B	B	G
13	B	R	G	B	G	R	B	R	P	P
14	G	G	P	B	B	P	R	R	G	B
15	R	G	P	R	B	R	B	P	P	G
16	B	B	P	G	P	B	P	G	R	R
17	G	P	B	R	P	G	B	P	B	R
18	R	B	G	P	B	G	P	R	R	P
19	G	B	R	P	P	R	B	G	R	B
20	B	R	P	G	R	G	G	B	R	P
21	B	R	G	R	B	P	G	P	B	P
22	G	P	P	R	G	B	G	B	R	B

Absolute Results

Theorem

1. *The following grids are in OBS_4 : $G_{5,41}$, $G_{6,31}$, $G_{7,29}$, $G_{9,25}$, $G_{10,23}$, $G_{11,22}$, $G_{22,11}$, $G_{23,10}$, $G_{25,9}$, $G_{29,7}$, $G_{31,6}$, $G_{41,5}$.*
2. *For each of the following grids it is not known if it is 4-colorable. These are the only such. $G_{17,17}$, $G_{17,18}$, $G_{18,17}$, $G_{18,18}$. $G_{21,12}$, $G_{12,21}$.*
3. *Exactly one of these is in OBS_4 : $G_{21,12}$, $G_{21,13}$.*
4. *Exactly one of these is in OBS_4 : $G_{17,19}$, $G_{17,18}$, $G_{17,17}$.*

Rectangle Free Conjecture

Recall the following lemma:

Lemma

Let $n, m, c \in \mathbb{N}$. If $\chi(G_{n,m}) \leq c$ then $\text{rfree}(G_{n,m}) \geq \lceil nm/c \rceil$.

Rectangle Free Conjecture

Recall the following lemma:

Lemma

Let $n, m, c \in \mathbb{N}$. If $\chi(G_{n,m}) \leq c$ then $\text{rfree}(G_{n,m}) \geq \lceil nm/c \rceil$.

Rectangle-Free Conjecture (RFC) is the converse:

Let $n, m, c \geq 2$. If $\text{rfree}(G_{n,m}) \geq \lceil nm/c \rceil$ then $G_{n,m}$ is c -colorable.

Rectangle Free Subset of $G_{22,10}$ of Size of size $55 = \lceil \frac{22 \cdot 10}{4} \rceil$

	01	02	03	04	05	06	07	08	09	10
1	•						•			
2		•					•			
3			•				•			
4				•			•			
5					•		•			
6						•	•			
7	•	•						•		
8			•	•				•		
9					•	•		•		
10		•	•						•	
11				•	•				•	
12	•					•			•	
13	•			•						•
14		•				•				•
15			•		•					•
16		•			•					
17	•		•							
18				•		•				
19			•			•				
20		•		•						
21	•				•					
22							•	•	•	•

If RFC is true then $G_{22,10}$ is 4-colorable.

Rectangle Free subset of $G_{21,12}$ of size $63 = \left\lceil \frac{21 \cdot 12}{4} \right\rceil$

	01	02	03	04	05	06	07	08	09	10	11	12
1	•	•										
2	•		•									
3		•	•									
4			•	•	•							
5		•		•		•						
6	•				•	•						
7						•	•	•				
8					•		•		•			
9				•				•	•			
10						•				•	•	
11					•					•		•
12				•							•	•
13			•			•			•			•
14			•					•		•		
15			•				•				•	
16		•							•	•		
17		•			•			•			•	
18		•					•					•
19	•								•		•	
20	•							•				•
21	•			•			•			•		

If RFC is true then $G_{21,12}$ is 4-colorable.

Rectangle Free subset of $G_{18,18}$ of size $81 = \left\lceil \frac{18 \cdot 18}{4} \right\rceil$

	01	02	03	04	05	06	07	08	09	10	11	12	13	14	15	16	17	18
1		•		•										•		•	•	
2	•	•								•	•		•					
3	•								•						•	•		•
4						•			•			•	•	•				
5		•	•			•												•
6	•			•		•	•											
7							•	•		•				•				•
8			•				•		•		•						•	
9		•			•		•					•			•			
10				•							•	•						•
11	•		•		•									•				
12			•	•				•					•		•			
13					•	•		•			•					•		
14	•							•				•					•	
15				•	•				•	•								
16						•				•					•		•	
17			•							•		•				•		
18					•								•				•	•

If RFC is true then $G_{18,18}$ is 4-colorable. NOTE: If delete 2nd column and 5th Row have 74-sized RFC of $G_{17,17}$.

Assuming RFC...

Theorem

If RFC then

$$\text{OBS}_4 = \{G_{41,5}, G_{31,6}, G_{29,7}, G_{25,9}, G_{23,10}, G_{22,11}, G_{21,13}, G_{19,17}\} \cup \\ \{G_{13,21}, G_{11,22}, G_{10,23}, G_{9,25}, G_{7,29}, G_{6,31}, G_{5,41}\}.$$

Proof.

Follows from known 4-colorability, non-4-colorability results, and Rect Free Sets above. □

PART VI: BIPARTITE RAMSEY THEORY

Theorem

(Bipartite Ramsey Theorem) For all a, c there exists $n = BR(a, c)$ such that for all c -colorings of the edges of $K_{n,n}$ there will be a monochromatic $K_{a,a}$. (See Graham-Rothchild-Spencer [1] for history and refs.)

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Equivalent to:

Theorem

For all a, c there exists $n = BR(a, c)$ so that for all c -colorings of $G_{n,n}$ there will be a monochromatic $a \times a$ submatrix.

APPLICATION TO BIPARTITE RAMSEY NUMBERS

Theorem

1. $BR(2, 2) = 5$. (*Already known.*)
2. $BR(2, 3) = 11$.
3. $17 \leq BR(2, 4) \leq 19$.
4. $BR(2, c) \leq c^2 + c$.
5. *If p is a prime and $s \in \mathbb{N}$ then $BR(2, p^s) \geq p^{2s}$.*
6. *For almost all c , $BR(2, c) \geq c^2 - c^{1.525}$.*

PART VII: OPEN QUESTIONS

1. Is $G_{17,17}$ 4-colorable? We have a Rectangle Free Set of size $\lceil (17 \times 17)/4 \rceil + 1 = 74$.
2. What is OBS_4 ? OBS_5 ?
3. Prove or disprove **Rectangle Free Conjecture**.
4. Have $\Omega(\sqrt{c}) \leq |\text{OBS}_c| \leq O(c^2)$. Get better bounds!
5. Refine tools so can prove **ugly** results **cleanly**.

CASH PRIZE!

The first person to email me both (1) plaintext, and (2) LaTeX, of a 4-coloring of the 17×17 grid that has no monochromatic rectangles will receive \$289.00.

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