Rectangle Free Coloring of Grids

Stephen Fenner- U of SC William Gasarch- U of MD
Charles Glover- U of MD Semmy Purewal- Col. of Charleston

Credit Where Credit is Due

This Work Grew Out of a Project In the UMCP SPIRAL (Summer Program in Research and Learning) Program. Program was for College Math Majors at HBCU's.

One of the students, Brett Jefferson has his own paper on this subject.

ALSO: Multidim version has been worked on by Cooper, Fenner, Purewal (submitted)

Square Theorem:

Theorem

For all c, there exists G such that for every c-coloring of $G \times G$ there exists a monochromatic square.

$$\begin{array}{ccccc} \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & R & \cdots & R & \cdots \\ \cdots & \vdots & \cdots & \vdots & \cdots \\ \cdots & R & \cdots & R & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \end{array}$$

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$$\begin{array}{ccccc}
\cdots & \cdots & \cdots & \cdots & \cdots \\
\cdots & R & \cdots & R & \cdots \\
\cdots & \vdots & \cdots & \vdots & \cdots \\
\cdots & R & \cdots & R & \cdots \\
\cdots & \cdots & \cdots & \cdots & \cdots
\end{array}$$

How to prove?

1. Corollary of Gallai's theorem [3,4,6]. Bounds on G HUGE!
2. From VDW directly (folklore). Bounds on G HUGE!
3. Directly (folklore?). Bounds on G HUGE!
4. Graham and Solymosi [2]. Bounds on G huge! But smaller.

What If We Only Care About Rectangles?

Definition

$G_{n,m}$ is the grid $[n] \times [m]$.

1. $G_{n,m}$ is c-colorable if there is a c-colorings of $G_{n,m}$ such that no rectangle has all four corners the same color.
2. $\chi(G_{n,m})$ is the least c such that $G_{n,m}$ is c-colorable.

Our Main Question

Fix c

Exactly which $G_{n,m}$ are c-colorable?

Two Motivations!

1. Relaxed version of Square Theorem- hope for better bounds.
2. Coloring $G_{n,m}$ without rectangles is equivalent to coloring edges of $K_{n,m}$ without getting monochromatic $K_{2,2}$.

   Our results yield Bipartite Ramsey Numbers!

Obstruction Sets

Theorem

For all c there exists a unique finite set of grids OBS_c such that

$G_{n,m}$ is c-colorable iff
$G_{n,m}$ does not contain any element of OBS_c.

1. Can prove using well-quasi-orderings. No bound on $|\mathrm{OBS}_c|$.
2. Our tools yield alternative proof and show

$$2\sqrt{c}(1 - o(1)) \leq |\mathrm{OBS}_c| \leq 2c^2.$$

Rephrase Main Question

Fix c

What is OBS_c

Rectangle Free Sets and Density

Definition

$G_{n,m}$ is the grid $[n] \times [m]$.

1. $X \subseteq G_{n,m}$ is Rectangle Free if there are NOT four vertices in X that form a rectangle.
2. $\mathrm{rfree}(G_{n,m})$ is the size of the largest Rect Free subset of $G_{n,m}$.

Rectangle Free subset of $G_{21,12}$ of size $63 = \left\lceil \frac{21 \cdot 12}{4} \right\rceil$

	01	02	03	04	05	06	07	08	09	10	11	12
1	●	●										
2	●		●									
3		●	●									
4			●	●	●							
5		●		●		●						
6	●				●	●						
7						●	●	●				
8					●		●		●			
9				●				●	●			
10						●				●	●	
11					●					●		●
12				●							●	●
13			●			●			●			●
14			●					●		●		
15			●				●				●	
16		●							●	●		
17		●			●			●			●	
18		●					●					●
19	●								●		●	
20	●							●				●
21	●			●			●			●		

Lemma

Let $n, m, c \in \mathbb{N}$. If $\chi(G_{n,m}) \le c$ then $\mathrm{rfree}(G_{n,m}) \ge \lceil mn/c \rceil$.

Note: We use to get non-col results as density results!!

Zarankiewics's Problem

Definition

$Z_{a,b}(m,n)$ is the largest subset of $G_{n,m}$ that has no $[a] \times [b]$ submatrix.

Zarankiewics [7] asked for exact values for $Z_{a,b}(m,n)$.
We care about $Z_{2,2}(m,n)$.

PART I: 2-COLORABILITY

We will EXACTLY Characterize which $G_{n,m}$ are 2-colorable!

$G_{5,5}$ IS NOT 2-Colorable!

Theorem

$G_{5,5}$ is not 2-Colorable.

Proof:

$$\chi(G_{5,5}) = 2 \implies \text{rfree}(G_{5,5}) \geq \lceil 25/2 \rceil = 13$$
$$\implies \text{there exists a column with } \geq \lceil 13/5 \rceil = 3 \; R\text{'s}$$

$G_{5,5}$ IS NOT 2-Colorable (Continued)

Case 1: Max in a column is 3 R's.
Case 1a: There are ≤ 2 columns with 3 R's.

$$\text{Number of } R\text{'s} \leq 3+3+2+2+2 \leq 12 < 13.$$

Case 1b: There are ≥ 3 columns with 3 R's.

$$\begin{array}{ccccc} R & \circ & \circ & \circ & \circ \\ R & \circ & \circ & \circ & \circ \\ R & R & \circ & \circ & \circ \\ \circ & R & \circ & \circ & \circ \\ \circ & R & \circ & \circ & \circ \end{array}$$

Can't put in a third column with 3 R's!

$G_{5,5}$ IS NOT 2-Colorable (Continued)

Case 2: There is a column with ≥ 4.
Easy exercise to show can't have 13.

$G_{4,6}$ IS 2-Colorable

Theorem

$G_{4,6}$ is 2-Colorable.

Proof.

R	R	R	B	B	B
R	B	B	R	R	B
B	R	B	R	B	R
B	B	R	B	R	R

□

$G_{3,7}$ IS NOT 2-Colorable

Theorem

$G_{3,7}$ is not 2-Colorable.

Proof.

$$\chi(G_{3,7}) = 2 \implies \text{rfree}(G_{3,7}) \geq (\lceil 21/2 \rceil = 11$$
$$\implies \text{there is a row with} \geq \lceil 11/3 \rceil = 4\ R\text{'s}$$

Proof similar to $G_{5,5}$— lots of cases.

□

Complete Char of 2-Colorability

Theorem

$$\mathrm{OBS}_2 = \{G_{3,7}, G_{5,5}, G_{7,3}\}.$$

Proof.
Follows from results $G_{5,5}, G_{7,3}$ not 2-colorable and $G_{4,6}$ is 2-colorable. □

PART II: TOOLS TO SHOW $G_{n,m}$ NOT c-COLORABLE

We show that if A is a Rectangle Free subset of $G_{n,m}$ then there is a relation between $|A|$ and n and m.

Theorem

Let $n, m \in \mathsf{N}$*. If there exists rectangle-free* $A \subseteq G_{n,m}$ *then*

$$|A| \le \frac{m + \sqrt{m^2 + 4m(n^2 - n)}}{2}$$

Note: Proved by Reiman [5] while working on Zarankiewicz's problem.

Bound on Size of Rectangle Free Sets (new)

Theorem

Let $a, n, m \in \mathbb{N}$. Let q, r be such that $a = qn + r$ with $0 \le r \le n$. Assume that there exists $A \subseteq G_{m,n}$ such that $|A| = a$ and A is rectangle-free.

1. *If $q \ge 2$ then*
$$n \le \left\lfloor \frac{m(m-1) - 2rq}{q(q-1)} \right\rfloor.$$
2. *If $q = 1$ then*
$$r \le \frac{m(m-1)}{2}.$$

Ideas Used in Proof

$A \subseteq G_{n,m}$, rectangle free.
x_i is number of points in i^{th} column.

	1	$\cdots$	m
1		$\cdots$	
$\vdots$		$\vdots$	
n		$\cdots$	
	x_1 points	$\cdots$	x_m points
	$\binom{x_1}{2}$ pairs of points	$\cdots$	$\binom{x_m}{2}$ pairs of points

Ideas Used in Proof

$A \subseteq G_{n,m}$, rectangle free.
x_i is number of points in i^{th} column.

	1	$\cdots$	m
1		$\cdots$	
$\vdots$		$\vdots$	
n		$\cdots$	
	x_1 points	$\cdots$	x_m points
	$\binom{x_1}{2}$ pairs of points	$\cdots$	$\binom{x_m}{2}$ pairs of points

$$\sum_{i=1}^{m} \binom{x_i}{2} \le \binom{n}{2}.$$

PART III: TOOLS TO SHOW $G_{n,m}$ IS c-COLORABLE

We define and use Strong c-Colorings to get c-Colorings

Strong c-Colorings

Definition

Let $c, n, m \in \mathbb{N}$. $\chi : G_{n,m} \rightarrow [c]$. χ is a strong c-coloring if the following holds: CANNOT have a rectangle with the two right most corners are same color and the two left most corners the same color.

Example: A strong 3-coloring of $G_{4,6}$.

R	R	G	R	G	G
B	G	R	G	R	G
G	B	B	G	G	R
G	G	G	B	B	B

Strong Coloring Lemma

Let $c, n, m \in \mathbb{N}$. If $G_{n,m}$ is strongly c-colorable then $G_{n,cm}$ is c-colorable.

Example:

R	R	G	R	G	G	B	B	R	B	R	R	G	G	B	G	B	B
B	G	R	G	R	G	G	R	B	R	B	R	R	B	G	B	G	B
G	B	B	G	G	R	R	G	G	R	R	B	B	R	R	B	B	G
G	G	G	B	B	B	R	R	R	G	G	G	B	B	B	R	R	R

Combinatorial Coloring Theorem

Let $c \ge 2$.

1. There is a strong c-coloring of $G_{c+1,\binom{c+1}{2}}$.
2. There is a c-coloring of $G_{c+1,m}$ where $m = c\binom{c+1}{2}$.

Example: Strong 5-coloring of $G_{6,15}$.

O	O	O	O	O	R	R	R	R	R	R	R	R	R	R
O	R	R	R	R	O	O	O	O	B	B	B	B	B	B
R	O	B	B	B	O	B	B	B	O	O	O	G	G	G
B	B	O	G	G	B	O	G	G	O	G	G	O	O	P
G	G	G	O	P	G	G	O	P	G	O	P	O	P	O
P	P	P	P	O	P	P	P	O	P	P	O	P	O	O

Coloring Using Primes!

Theorem

Let p be a prime.

1. *There is a strong p-coloring of $G_{p^2,p+1}$.*
2. *There is a p-coloring of G_{p^2,p^2+p}.*

Proof.

Uses geometry over finite fields. □

Note: Have more general theorem.

Using a Generalization of Strong Coloring

Theorem

Let $c \geq 2$.

1. *There is a c-coloring of $G_{c+2,m'}$ where $m' = \lfloor c/2 \rfloor \binom{c+2}{2}$.*
2. *There is a c-coloring of $G_{2c,2c^2-c}$.*

PART IV: 3-COLORABILITY

We will EXACTLY Characterize which $G_{n,m}$ are 3-colorable!

Easy 3-Colorable Results

Theorem

1. *The following grids are not 3-colorable.* $G_{4,19}$, $G_{19,4}$, $G_{5,16}$, $G_{16,5}$, $G_{7,13}$, $G_{13,7}$, $G_{10,12}$, $G_{12,10}$, $G_{11,11}$.
2. *The following grids are 3-colorable.* $G_{3,19}$, $G_{19,3}$, $G_{4,18}$, $G_{18,4}$, $G_{6,15}$, $G_{15,6}$, $G_{9,12}$, $G_{12,9}$.

Proof.

Follows from tools. □

Theorem
$G_{10,10}$ is 3-colorable.

Proof.
UGLY! TOOLS DID NOT HELP AT ALL!!

R	R	R	R	B	B	G	G	B	G
R	B	B	G	R	R	R	G	G	B
G	R	B	G	R	B	B	R	R	G
G	B	R	B	B	R	G	R	G	R
R	B	G	G	G	B	G	B	R	R
G	R	B	B	G	G	R	B	B	R
B	G	R	B	G	B	R	G	R	B
B	B	G	R	R	G	B	G	B	R
G	G	G	R	B	R	B	B	R	B
B	G	B	R	B	G	R	R	G	G

$G_{10,11}$ is not 3-colorable

Theorem

$G_{10,11}$ is not 3-colorable.

Proof.

You don't want to see this. UGLY case hacking. □

Complete Char of 3-colorability

Theorem

$\mathrm{OBS}_3 =$

$$\{G_{4,19}, G_{5,16}, G_{7,13}, G_{10,11}, G_{11,10}, G_{13,7}, G_{16,5}, G_{19,4}\}.$$

Proof.

Follows from above results on grids being or not being 3-colorable. □

PART V: 4-COLORABILITY

We will MAKE PROGRESS ON Characterizing which $G_{n,m}$ are 4-colorable.

Easy 4-Colorable Results

Theorem

1. *The following grids are NOT 4-colorable:* $G_{5,41}$, $G_{41,5}$, $G_{6,31}$, $G_{31,6}$, $G_{7,29}$, $G_{29,7}$, $G_{9,25}$, $G_{25,9}$, $G_{10,23}$, $G_{23,10}$, $G_{11,22}$, $G_{22,11}$, $G_{13,21}$, $G_{21,13}$, $G_{17,20}$, $G_{20,17}$, $G_{18,19}$, $G_{19,18}$.
2. *The following grids are 4-colorable:* $G_{4,41}$, $G_{41,4}$, $G_{5,40}$, $G_{40,5}$, $G_{6,30}$, $G_{30,6}$, $G_{8,28}$, $G_{28,8}$, $G_{16,20}$, $G_{20,16}$.

Proof.
Follows from tools. □

Theorems with UGLY Proofs

Theorem

1. *$G_{17,19}$ is NOT 4-colorable: Used some tools.*
2. *$G_{24,9}$ is 4-colorable: Used strong coloring of $G_{9,6}$.*

Theorem

1. *$G_{17,19}$ is NOT 4-colorable: Used some tools.*
2. *$G_{24,9}$ is 4-colorable: Used strong coloring of $G_{9,6}$.*

P	R	R	P	R	R
P	B	B	R	P	B
P	G	G	B	B	P
R	P	G	P	G	R
B	P	R	B	P	G
G	P	B	G	R	P
G	B	P	P	B	G
R	G	P	G	P	R
B	R	P	R	G	P

Rectangle Free Conjecture

Recall the following lemma:

Lemma

Let $n, m, c \in \mathbb{N}$*. If* $\chi(G_{n,m}) \leq c$ *then* $\mathrm{rfree}(G_{n,m}) \geq \lceil nm/c \rceil$*.*

Rectangle Free Conjecture

Recall the following lemma:

Lemma

Let $n, m, c \in \mathrm{N}$. If $\chi(G_{n,m}) \leq c$ then $\mathrm{rfree}(G_{n,m}) \geq \lceil nm/c \rceil$.

Rectangle-Free Conjecture (RFC) is the converse:

Let $n, m, c \geq 2$. If $\mathrm{rfree}(G_{n,m}) \geq \lceil nm/c \rceil$ then $G_{n,m}$ is c-colorable.

Assuming RFC...

Theorem

If RFC then

$$\mathrm{OBS}_4 = \{G_{41,5}, G_{31,6}, G_{29,7}, G_{25,9}, G_{23,10}, G_{22,11}, G_{21,13}, G_{19,17}\} \bigcup$$

$$\{G_{13,21}, G_{11,22}, G_{10,23}, G_{9,25}, G_{7,29}, G_{6,31}, G_{5,41}\}.$$

Proof.

Follows from known 4-colorability and non-4-colorability results, and from some Rect Free Sets we found by computer search. □

PART VI: BIPARTITE RAMSEY THEORY

Theorem

(Bipartite Ramsey Theorem) For all a, c there exists $n = BR(a, c)$ such that for all c-colorings of the edges of $K_{n,n}$ there will be a monochromatic $K_{a,a}$. (See Graham-Rothchild-Spencer [1] for history and refs.)

Theorem

(Bipartite Ramsey Theorem) For all a, c *there exists* $n = BR(a, c)$ *such that for all c-colorings of the edges of* $K_{n,n}$ *there will be a monochromatic* $K_{a,a}$*. (See Graham-Rothchild-Spencer [1] for history and refs.)*

Equivalent to:

Theorem

For all a, c *there exists* $n = BR(a, c)$ *so that for all c-colorings of* $G_{n,n}$ *there will be a monochromatic* $a \times a$ *submatrix.*

APPLICATION TO BIPARTITE RAMSEY NUMBERS

Theorem

1. $BR(2,2) = 5$. *(Already known.)*
2. $BR(2,3) = 11$.
3. $17 \leq BR(2,4) \leq 19$.
4. $BR(2,c) \leq c^2 + c$.
5. *If p is a prime and* $s \in \mathbb{N}$ *then* $BR(2,p^s) \geq p^{2s}$.
6. *For almost all c,* $BR(2,c) \geq c^2 - c^{1.525}$.

PART VII: OPEN QUESTIONS

1. Is $G_{17,17}$ 4-colorable? (Other 4-col also open.)
2. What is OBS_4? OBS_5?
3. Prove or disprove Rectangle Free Conjecture.
4. Have $\Omega(\sqrt{c}) \le |\mathrm{OBS}_c| \le O(c^2)$. Get better bounds!
5. Refine tools so can prove ugly results cleanly.

Bibliography

1. R. Graham, B. Rothchild, and J. Spencer. *Ramsey Theory*. Wiley, 1990.
2. R. Graham and J. Solymosi. Monochromatic equilateral right triangles on the integer grid. *Topics in Discrete Mathematics, Algorithms and Combinatorics*, 2006. www.math.ucsd.edu/~/ron/06_03_righttriangles.pdf or www.cs.umd.edu/~/vdw/graham-solymosi.pdf.
3. R. Rado. Studien zur kombinatorik. *Mathematische Zeitschrift*, pages 424–480, 1933. http://www.cs.umd.edu/~gasarch/vdw/vdw.html.
4. R. Rado. Notes on combinatorial analysis. *Proceedings of the London Mathematical Society*, pages 122–160, 1943. http://www.cs.umd.edu/~gasarch/vdw/vdw.html.

Bibliography (continued)

5 I. Reiman. Uber ein problem von k.zarankiewicz. *Acta. Math. Acad. Soc. Hung.*, 9:269–279, 1958.

6 Witt. Ein kombinatorischer satz de elementargeometrie. *Mathematische Nachrichten*, pages 261–262, 1951. `http://www.cs.umd.edu/~gasarch/vdw/vdw.html`.

7 K. Zarankiewicz. Problem P 101. *Colloq. Math.*