Midterm Two Solutions
Today’s Lecture we

1. **WILL** give some stats about the exam (one slide).

2. **WILL** go over the exam, the answers, and some enlightening MATH issues that arise. THIS is the main POINT of the talk.

3. **WILL NOT** get into If I did BLAH how much partial credit will I get?
1. 100: 38 students
2. 90-99: 56 students
3. 80-89: 77 students
4. 70-79: 77 students (what is the prob of that!)
5. 60-69: 76 students
6. 50-59: 70 students
7. 40-49: 47 students
8. 30-39: 24 student
9. 20-29: 17 students
10. 10-19: 3 students, 1-9: 3 students, 0: 2 students
11. Legit makeups being dealt with: 4

Mean: 68.5, Standard Deviation (square root of Variance): 21.5
1. Showed up at Jason’s exam with 30 seconds left and wanted to take it and do as best as he can: 1.
2. Showed up at the wrong room and hence got a 0: 2
3. People that didn’t show up, didn’t ask for a makeup: 6
PROBLEM:
If you rearrange the letters in the word $abcdeee$e randomly what is the probability that the first 5 letters will be $abcde$?
Problem 1a

PROBLEM:
If you rearrange the letters in the word \textit{abcdeeeeee} randomly what is the probability that the first 5 letters will be \textit{abcde}?

SOLUTION ONE: We rearrange ALL of the letters so that really is 8! ways to do that. The number where the first four are \textit{abcd} (which forces the next one to be e) is 4!. So the answer is 4!/8!.
PROBLEM:
If you rearrange the letters in the word *abcdeeeeee* randomly what is the probability that the first 5 letters will be *abcde*?

**SOLUTION ONE:** We rearrange ALL of the letters so that really is $8!$ ways to do that. The number where the first four are *abcd* (which forces the next one to be e) is $4!$. So the answer is $4!/8!$

**SOLUTION TWO:** The number of strings is $8!/4!$ but only 1 of them is *abcdeeeeee* So the answer is $1/(8!/4!) = 4!/8!$
PROBLEM 1b

PROBLEM What is the coefficient of $x^2y^2z^4$ in the expansion of $(x + y + z)^8$?
Problem 1b

**PROBLEM** What is the coefficient of $x^2y^2z^4$ in the expansion of $(x + y + z)^8$?

**SOLUTION ONE**
This is the same as the number of perms of $xxyyyyzzzz$ which is

$$\frac{8!}{2!2!4!}$$

**SOLUTION TWO**
The formula for the coefficient of $x^ay^bz^c$ in $(x + y + z)^n$ is $\frac{n!}{a!b!c!}$. Since $a = 2$, $b = 2$, $c = 4$ and $n = 8$ this is

$$\frac{8!}{2!2!4!}$$
Generalized $L$-nomial Theorem

Coefficient of $x_1^{a_1} \cdots x_L^{a_L}$ in

$$(x_1 + x_2 + \cdots + x_L)^n$$

(where $a_1 + \cdots + a_L = n$)

is
Generalized $L$-omial Theorem

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$$(x_1 + x_2 + \cdots + x_L)^n$$

(where $a_1 + \cdots + a_L = n$)

is

$$\frac{n!}{a_1!a_2!\cdots a_L!}$$
Problem 1c

PROBLEM
In the year 2020 ART HISTORY 450 will have 2 teachers and 5 TA’s. A team of 4 people are picked to make up the final. What is the probability that there are exactly 2 TA’s on this team?
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SOLUTION
There are \( \binom{7}{4} \) ways to make the team. Of those there are \( \binom{5}{2} \times \binom{2}{2} = \binom{5}{2} \) ways to have 2 TA’s on the team. Hence prob is

\[
\frac{\binom{5}{2}}{\binom{7}{4}} = \frac{\frac{5!}{2!3!} \times \frac{4!3!}{7!}}{\frac{2!7!}{2!7!}} = \frac{\frac{4!}{2!7!}}{\frac{2!}{7!} \times 6} = \frac{4 \times 3 \times 2}{2 \times 7 \times 6} = \frac{4}{2} \times \frac{6}{7 \times 6} = 2 \times \frac{1}{7} = \frac{2}{7}
\]
Problem 2a

There are THREE coins on the table.
2 biased: \( P(H) = \frac{2}{3}, \; P(T) = \frac{1}{3} \). 1 fair: \( P(H) = \frac{1}{2}, \; P(T) = \frac{1}{2} \).
Jason picks one at random. Probability he picks a biased coin is \( \frac{2}{3} \).
\( B \) is Jason picked biased. \( F \) is Jason picked fair.
What is \( P(B \mid H) \)?
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\( B \) is Jason picked biased. \( F \) is Jason picked fair.
What is \( P(B \mid H) \)?

SOLUTION

\[
P(B \mid H) = \frac{P(B)P(H \mid B)}{P(H)}
\]

\[
P(B) = \frac{2}{3}.
\]
\[
P(H \mid B) = \frac{2}{3}
\]
\[
P(H) = \frac{1}{3} \times \frac{1}{2} + \frac{2}{3} \times \frac{2}{3} = \frac{1}{6} + \frac{4}{9} = \frac{11}{18}
\]
Putting this all together we get

\[
P(B \mid H) = \frac{(2/3)(2/3)}{11/18} = \frac{4}{9} \times \frac{18}{11} = \frac{8}{11}
\]
Problem 2a- Does the Answer Make Sense?

There are THREE coins on the table.
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Jason picks one at random. Probability he picks a biased coin is \( \frac{2}{3} \).
\( B \) is Jason picked biased. \( F \) is Jason picked fair.
What is \( P(B \mid H) \)?
We Found It Was \( \frac{8}{11} \)
Why So Large?

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Jason picks one at random. Probability he picks a biased coin is $\frac{2}{3}$.
$B$ is Jason picked biased. $F$ is Jason picked fair.

What is $P(B \mid H)$?

We Found It Was $\frac{8}{11}$

Why So Large?

$\frac{8}{11}$ seems large for just seeing ONE $H$.

BUT- there is $\frac{2}{3}$ prob of bias.
Problem 2a- Does the Answer Make Sense?

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SO if you see ZERO coin flips, $P(B) = \frac{2}{3}$. 
Problem 2a- Does the Answer Make Sense?

There are THREE coins on the table.  
2 biased:  \( P(H) = \frac{2}{3}, \ P(T) = \frac{1}{3} \).  1 fair:  \( P(H) = \frac{1}{2}, \ P(T) = \frac{1}{2} \).  
Jason picks one at random. Probability he picks a biased coin is \( \frac{2}{3} \).  
\( B \) is Jason picked biased.  \( F \) is Jason picked fair.  
What is \( P(B \mid H) \)?

We Found It Was \( \frac{8}{11} \)  
Why So Large?

\( \frac{8}{11} \) seems large for just seeing ONE \( H \).  
BUT- there is \( \frac{2}{3} \) prob of bias.  
SO if you see ZERO coin flips, \( P(B) = \frac{2}{3} \).  
So we START OFF with \( \frac{2}{3} \) and then seeing a \( H \) increases the prob of bias.
Problem 2b

There are THREE coins on the table.
2 biased: \( P(H) = \frac{2}{3}, \ P(T) = \frac{1}{3} \). 1 fair: \( P(H) = \frac{1}{2}, \ P(T) = \frac{1}{2} \).
Jason picks one at random. Probability he picks a biased coin is \( \frac{2}{3} \).
\( B \) is Jason picked biased. \( F \) is Jason picked fair.
What is \( P(B \mid T) \)?
Problem 2b

There are THREE coins on the table.
2 biased: \( P(H) = \frac{2}{3}, \ P(T) = \frac{1}{3} \). 1 fair: \( P(H) = \frac{1}{2}, \ P(T) = \frac{1}{2} \).
Jason picks one at random. Probability he picks a biased coin is \( \frac{2}{3} \).
B is Jason picked biased. F is Jason picked fair.
What is \( P(B \mid T) \)?

SOLUTIONS By Bayes’s Theorem

\[
P(B \mid T) = \frac{P(B)P(T \mid B)}{P(T)}
\]

\( P(B) = \frac{2}{3} \).
\( P(T \mid B) = \frac{1}{3} \)
\( P(T) = \frac{1}{3} \times \frac{1}{2} + \frac{2}{3} \times \frac{1}{3} = \frac{1}{6} + \frac{2}{9} = \frac{7}{18} \)
Putting this all together we get

\[
P(B \mid T) = \frac{(2/3)(1/3)}{7/18} = \frac{2}{9} \times \frac{18}{7} = \frac{4}{7}
\]
Problem 3

The Vorlons play a card game called Tongo. They do NOT use the usual cards we do on planet earth.

▶ Each card has a *number* from 1 to 100 and a suite which is one of \{\Gamma, \Delta, \Lambda, \Theta, \Omega, \Pi, \Sigma\} (there are 7 suits).

▶ A hand is THREE cards.

Problems on the next two slides

But NOTE:
The number of cards is \(100 \times 7 = 700\).
The number of hands is \(\binom{700}{3}\)
PROBLEM A *SUPERTONGO* is when all of the cards are in a row and have the same suite AND we DO allow wrap-around.

SUPERTONGO’s: (1) \{4\Gamma, 5\Gamma, 6\Gamma\}, (2) \{99\Omega, 100\Omega, 1\Omega\}.
What is the probability of getting dealt a SUPERTONGO?
PROBLEM A SUPERTONGO is when all of the cards are in a row and have the same suite AND we DO allow wrap-around. SUPERTONGO’s: (1) \{4\Gamma, 5\Gamma, 6\Gamma\}, (2) \{99\Omega, 100\Omega, 1\Omega\}. What is the probability of getting dealt a SUPERTONGO?

SOLUTION Number of SUPERTONGO hands: Pick the starting point which is one of 100 places (RECALL- DO allow wraparound.) Pick the suite, one of 7. So there are $100 \times 7$ SUPERTONGO hands. Hence the prob of getting a SUPERTONGO is

\[
\frac{700}{\binom{700}{3}}.
\]
Problem 3b

A TONGO is when all of the cards are in a row AND we DO allow wrap-around.
TONGO's: (1) \{4\Gamma, 5\Gamma, 6\Omega\}, (2) \{99\Omega, 100\Gamma, 1\Omega\}
What is the probability of getting dealt a TONGO that is NOT a SUPERTONGO?
A *TONGO* is when all of the cards are in a row AND we DO allow wrap-around.

TONGO’s: (1) \{4\Gamma, 5\Gamma, 6\Omega\}, (2) \{99\Omega, 100\Gamma, 1\Omega\}

What is the probability of getting dealt a TONGO that is NOT a SUPERTONGO?

**SOLUTION** Number of TONGO hands (including SUPERTONGO) is formed: Pick the starting point which is one of 100 places Pick 3 suite in order, so that’s $7^3$. So the number is $100 \times 7^3$. Subtract the SUPERTONGO and divide by $\binom{700}{3}$ to get

$$
\frac{100 \times 7^3 - 700}{\binom{700}{3}}
$$
Problem 4

PROBLEM Let $a_n$ be defined as follows

$a_0 = 0$

$(\forall n \geq 1)[a_n = a_{\lfloor n/3 \rfloor} + a_{\lfloor 2n/5 \rfloor} + A \cdot n]$

$a_n$ is not quite defined since don’t have $A$. YOU! will find, using Strong Const Ind, a value of $A \in \mathbb{N}$ such that

$(\forall n \in \mathbb{N})[a_n \leq 20n]$.

Try to make $A$ as large as you can.
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Try to make $A$ as large as you can.

SOLUTION

IB: $a_0 = 0 \leq 20 \times 0 = 0$ (so no constraints on $A$ yet)
Problem 4

PROBLEM Let \( a_n \) be defined as follows

\[
a_0 = 0 \\
(\forall n \geq 1)[a_n = a_{\lfloor n/3 \rfloor} + a_{\lfloor 2n/5 \rfloor} + A \cdot n]
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Try to make \( A \) as large as you can.

SOLUTION

IB: \( a_0 = 0 \leq 20 \times 0 = 0 \) (so no constraints on \( A \) yet)

IH: Assume that for all \( 0 \leq i \leq n - 1, a_i \leq A \cdot i. \)
Problem 4

**PROBLEM** Let $a_n$ be defined as follows

$a_0 = 0$

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**SOLUTION**

IB: $a_0 = 0 \leq 20 \times 0 = 0$ (so no constraints on $A$ yet)

IH: Assume that for all $0 \leq i \leq n - 1$, $a_i \leq A \cdot i$.

IS: $a_n = a_{\lfloor n/3 \rfloor} + a_{\lfloor 2n/5 \rfloor} + A \cdot n$
Problem 4

PROBLEM Let \( a_n \) be defined as follows
\[
a_0 = 0 \\
(\forall n \geq 1)[a_n = a_{\lfloor n/3 \rfloor} + a_{\lfloor 2n/5 \rfloor} + A \cdot n]
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a\(_n\) is not quite defined since don’t have \( A \). YOU! will find, using Strong Const Ind, a value of \( A \in \mathbb{N} \) such that
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IH: Assume that for all \( 0 \leq i \leq n - 1, a_i \leq A \cdot i \).
IS: \( a_n = a_{\lfloor n/3 \rfloor} + a_{\lfloor 2n/5 \rfloor} + A \cdot n \)
then by \( IH \)
Problem 4

PROBLEM Let \( a_n \) be defined as follows
\[
a_0 = 0 \\
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(\forall n \in \mathbb{N})[a_n \leq 20n].
\]
Try to make \( A \) as large as you can.

SOLUTION

IB: \( a_0 = 0 \leq 20 \times 0 = 0 \) (so no constraints on \( A \) yet)

IH: Assume that for all \( 0 \leq i \leq n - 1 \), \( a_i \leq A \cdot i \).

IS: \( a_n = a_{\lfloor n/3 \rfloor} + a_{\lfloor 2n/5 \rfloor} + A \cdot n \)
then by IH
\[
a_n \leq 20(\lfloor n/3 \rfloor) + 20(\lfloor 2n/5 \rfloor) + A \cdot n \leq 20(n/3) + 20(2n/5) + A \cdot n
\]
Problem 4

PROBLEM Let $a_n$ be defined as follows

$a_0 = 0$

$(\forall n \geq 1)[a_n = a_{\lfloor n/3 \rfloor} + a_{\lfloor 2n/5 \rfloor} + A \cdot n]$

$a_n$ is not quite defined since don’t have $A$. YOU! will find, using Strong Const Ind, a value of $A \in \mathbb{N}$ such that

$(\forall n \in \mathbb{N})[a_n \leq 20n]$.

Try to make $A$ as large as you can.

SOLUTION

IB: $a_0 = 0 \leq 20 \times 0 = 0$ (so no constraints on $A$ yet)

IH: Assume that for all $0 \leq i \leq n - 1$, $a_i \leq A \cdot i$.

IS: $a_n = a_{\lfloor n/3 \rfloor} + a_{\lfloor 2n/5 \rfloor} + A \cdot n$

then by IH

$a_n \leq 20(\lfloor n/3 \rfloor) + 20(\lfloor 2n/5 \rfloor) + A \cdot n \leq 20(n/3) + 20(2n/5) + A \cdot n$

So we want

$20(n/3) + 20(2n/5) + A \cdot n \leq 20n$

Exciting Algebra on next slide!
Problem 4 Continued

Want

\[ \frac{20n}{3} + 8n + A \cdot n \leq 20n \]

Divide by \( n \)

\[ \frac{20n}{3n} + \frac{8n}{n} + \frac{A \cdot n}{n} \leq \frac{20n}{n} \]

\[ \frac{20}{3} + 8 + A \leq 20 \]

\[ A \leq 20 - \frac{20}{3} - 8 = 12 - \frac{64}{3} = \frac{36}{3} - \frac{64}{3} = \frac{-28}{3} \]

AND \( A \in \mathbb{N} \).

SO we pick \( A = 5 \).
Problem 4 Continued

Want

$$\frac{20n}{3} + 8n + A \cdot n \leq 20n$$

Divide by \( n \)

$$\frac{20}{3} + 8 + A \leq 20$$
Problem 4 Continued

Want

\[ \frac{20n}{3} + 8n + A \cdot n \leq 20n \]

Divide by \( n \)

\[ \frac{20}{3} + 8 + A \leq 20 \]

\[ A \leq 20 - \frac{20}{3} - 8 = 12 - \frac{20}{3} = 12 - (6 + \frac{2}{3}) = 6 - \frac{2}{3} = 5.33 \ldots \]

AND \( A \in \mathbb{N} \).
Problem 4 Continued

Want

\[ \frac{20n}{3} + 8n + A \cdot n \leq 20n \]

Divide by \( n \)

\[ \frac{20}{3} + 8 + A \leq 20 \]

\[ A \leq 20 - \frac{20}{3} - 8 = 12 - \frac{20}{3} = 12 - (6 + \frac{2}{3}) = 6 - \frac{2}{3} = 5.33 \ldots \]

AND \( A \in \mathbb{N} \).

SO we pick \( A = 5 \).
Problem 4-How it Differed From the HW and Slides

On the HW and Slides you always had:

▶ A sequence that was COMPLETELY DEFINED, for example:

\[ a_n = a_{\lfloor n/3 \rfloor} + a_{\lfloor 2n/5 \rfloor} + 5 \cdot n \]

▶ A FORM for the bound (e.g., \( a_n \leq A \cdot n \)).
▶ Use CI to find A and hence THE BOUND.

In this problem we had:

▶ A FORM for the sequence

\[ a_n = a_{\lfloor n/3 \rfloor} + a_{\lfloor 2n/5 \rfloor} + A \cdot n \]

▶ A COMPLETELY DEFINED bound \( a_n \leq 20n \).
▶ Use CI to find the A and hence THE SEQUENCE.
Problem 4-A Reminder of How Induction Works

1. In the IB you **explicitly prove** the proposition correct for some natural number (often 0 or 1).
2. In the IH you **assume** the truth of the proposition for a range of natural numbers (usually 0 to \( n - 1 \) or 0 to \( n \))
3. In the IS you **assume** the IH and **prove** the proposition for the next number (usually \( n \) or \( n + 1 \)).
4. ▶ If IH is 0 to \( n - 1 \) then IS is to prove \( n \).
   ▶ If IH is 0 to \( n \) then IS is to prove \( n + 1 \).

The above hold for weak, strong, weak constructive, strong constructive, and (if modified some) structural **Induction**.
Problem 4-A Reminder of How Strong Induction Works: IH!

If want to prove \((\forall n \geq 0)[P(n)]\) then
1. IB is 0
2. IH is 0 to \(n - 1\) OR 0 to \(n\). Must BEGIN at 0.

If want to prove \((\forall n \geq 17)[P(n)]\) then
1. IB is 17
2. IH is 17 to \(n - 1\) OR 17 to \(n\). Must BEGIN at 17.
Problem 4-Which IH to use?

IH is 0 to $n - 1$ OR 0 to $n$
Which to use?
If proving a theorem about a sequence
$a_0$ given, $a_1$ given

$$(\forall n \geq 2)[a_n = \text{do the hokey pokey on } a_{n-1}, a_{n-2}, \text{etc}]$$

Then use IH 0 to $n - 1$ since in IS use information about $a_{n-1}$, $a_{n-2}$, etc to prove something about $a_n$.

If sequence defined via

$$a_{n+1} = \text{do the hokey pokey on } a_n, a_{n-1}, \text{etc}$$

Then use IH 0 to $n$ since in IS use information about $a_n$, $a_{n-1}$, etc to prove something about $a_n$.

Cause that’s! what it’s all about!
Problem 5

PROBLEM Let $A \subseteq \{1, 2, \ldots, 21\}$ of size 8. Show that $A$ has at least TWO subsets of size 3 that have the same sum. (The two subsets must be different but can overlap.)

SOLUTION $A$ has size 8, so $A$ has $\binom{8}{3} = \frac{8!}{5!3!} = \frac{8 \times 7 \times 6}{3 \times 2 \times 1} = 56$ subsets of size 3.

The largest sum possible is $21 + 20 + 19 = 60$.

The smallest sum possible is $1 + 2 + 3 = 6$.

Hence the number of sums is $60 - 5 = 55$.

Since there are 56 subsets and 55 sums there are two subsets, by Pigeonhole, that have the same sum.
**Problem 5**

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**SOLUTION**

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The smallest sum possible is $1 + 2 + 3 = 6$.
Hence the number of sums is $60 - 5 = 55$.

Since there are 56 subsets and 55 sums there are two subsets, by Pigeonhole, that have the same sum.
Problem 5- How it Differed From HW and Slides

On the HW and Slides you always looked at ALL subsets of $A$:

- The number of subsets was $2^{|A|}$
- The min sum was 0

In this problem you looked ONLY at the sets of SIZE 3

- The number of subsets of size three is $\binom{|A|}{3}$.
- The min sum was $1 + 2 + 3 = 6$. 
Problem 5 - The Perils of Memorization

- Some students wrote $2^8$ instead of $\binom{8}{3}$. **Speculation:** They memorized the other problem and hoped it would work rather than trying to truly understand the other problem.

- Some students wrote minsum=0. **Speculation:** They memorized the other problem and hoped it would work rather than trying to truly understand the other problem.

- Some students wrote maxsum=19+20+21=60 which is correct, but then minsum=0 which is incorrect. **Speculation:** They adding of the three max elements IS something that is in the problem they memorized how to do (hence this “correct” answer does not impress), but since they memorized, minsum=0 is written.
Problem 5- What $m, N$ Would Work?

Let $A \subseteq \{1, 2, \ldots, N\}$ of size $m$. Show there are two subsets of $A$ of size 3 with same sum.

There are \( \binom{m}{3} = \frac{m(m-1)(m-2)}{6} \) subsets of size 3.

The largest sum possible is $(N - 2) + (N - 1) + N = 3N - 3$.

The smallest sum possible is $1 + 2 + 3 = 6$.

Hence the number of sums is $3N - 3 - 5 = 3N - 8$.

SO we need

\[
\frac{m(m-1)(m-2)}{6} > 3N - 8
\]

\[
m(m - 1)(m - 2) > 18N - 48
\]
Problem 5- What \( m, N \) Would Work?

Let \( A \subseteq \{1, 2, \ldots, N\} \) of size \( m \). Show there are two subsets of \( A \) of size 3 with same sum.

Want to use Pigeonhole. Need:

\[
m(m-1)(m-2) > 18N - 48
\]

\[
N < \frac{m(m-1)(m-2) + 48}{18}
\]

\[
N = \left\lfloor \frac{m(m-1)(m-2) + 48}{18} \right\rfloor
\]
Problem 5- What $m, N$ Would Work?

$$N = \left\lfloor \frac{m(m - 1)(m - 2) + 48}{18} \right\rfloor$$

<table>
<thead>
<tr>
<th>$m$</th>
<th>$N$</th>
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<tbody>
<tr>
<td>3</td>
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<td>7</td>
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</tr>
<tr>
<td>8</td>
<td>21</td>
</tr>
<tr>
<td>9</td>
<td>30</td>
</tr>
<tr>
<td>10</td>
<td>42</td>
</tr>
</tbody>
</table>

NOTE: $m = 8, N = 21$ was our problem.
Problem 5- What if we take sets of size 4?

We leave this to you
Problem 5- What if we take sets of size 4?

We leave this to you
We Are Busy People!
Problem 5- What if we used different numbers?

Let $A \subseteq \{1, 2, \ldots, 30\}$ of size 16.

Using Pigeonhole: There are $X$ subsets of $A$ of size 5 with the same sum.

Work on this on your own NOW.

VOTE: $X$ is

- **BLUE:** Between 2 and 10
- **RED:** Between 11 and 100
- **GREEN:** Between 101 and 1000
- **WHITE:** Over 1000
Problem 5- What if we used different numbers?

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Work on this on your own NOW.
VOTE: $X$ is

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- RED: Between 11 and 100
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- WHITE: Over 1000

RED: Between 11 and 100 (Actually 33)
See next slide.
Problem 5- What if we used different numbers?

There are \( \binom{16}{5} = 4368 \) subsets of size 5 of \( A \).
The largest sum possible is \( 26 + 27 + 28 + 29 + 30 = 140 \).
The smallest sum possible is \( 1 + 2 + 3 + 4 + 5 = \frac{5 \times 6}{2} = 15 \).
Hence the number of sums is \( 140 - 4 = 136 \).
So the number of sets with the same sum is at least \( \lceil \frac{4368}{136} \rceil = 33 \).

TOO many calculations with large numbers to have you do on an exam!

You Are Busy People!