Find the Missing Numbers

William Gasarch-U of MD

Ground Rules

Alice recites **ALL BUT** k of the numbers in $\{1, \ldots, n\}$ in some random order.

We denote these numbers $x_1, \ldots, x_{n-k}$.

Missing numbers are $y_1, \ldots, y_k$.

Bob wants to find $y_1, \ldots, y_k$ but only has $O(k \log n)$ space

$k = 1$ Case

Bob computes $\sum_{i=1}^{n-1} x_i$ and finds

$$y_1 = \sum_{i=1}^{n} i - \sum_{i=1}^{n-1} x_i = \frac{n(n+1)}{2} - \sum_{i=1}^{n-1} x_i.$$

$k = 2$ Case

Bob computes $\sum_{i=1}^{n-2} x_i$ and $\sum_{i=1}^{n-2} x_i^2$.

$$y_1 + y_2 = \frac{n(n+1)}{2} - \sum_{i=1}^{n-2} x_i$$

$$y_1^2 + y_2^2 = \sum_{i=1}^{n} i^2 - \sum_{i=1}^{n-2} x_i^2 = \frac{n(n+1)(2n+1)}{6} - \sum_{i=1}^{n-2} x_i^2.$$

WANT $y_1 y_2$ (you'll see why soon)

$$y_1 y_2 = \frac{(y_1 + y_2)^2 - (y_1^2 + y_2^2)}{2}.$$

$k = 2$ Case Continued

KEY STEP: Form Poly

$$X^2 - (y_1 + y_2)X + y_1y_2$$

$k = 2$ Case Continued

KEY STEP: Form Poly

$$X^2 - (y_1 + y_2)X + y_1 y_2$$

$$= (X - y_1)(X - y_2)$$

Find its roots. **THEY ARE THE MISSING NUMBERS!!!!**

$k = 3$ Case- The Main Idea

NEED

$$y_1 + y_2 + y_3$$

$$y_1 y_2 + y_1 y_3 + y_2 y_3$$

$$y_1 y_2 y_3$$

Form polynomial

$$X^3 - (y_1 + y_2 + y_3)X^2 + (y_1 y_2 + y_1 y_3 + y_2 y_3)X - y_1 y_2 y_3$$

$$= (X - y_1)(X - y_2)(X - y_3)$$

Find its roots. **THEY ARE THE MISSING NUMBERS!**

Solution One

Bob computes $\sum_{i=1}^{n-3} x_i$ and $\sum_{i=1}^{n-3} x_i^2$ and $\sum_{i=1}^{n-3} x_i^3$.

$$y_1 + y_2 + y_3 = \frac{n(n+1)}{2} - \sum_{i=1}^{n-2} x_i$$

$$y_1^2 + y_2^2 + y_3^2 = \frac{n(n+1)(2n+1)}{6} - \sum_{i=1}^{n-2} x_i^2$$

$$y_1^3 + y_2^3 + y_3^3 = \sum_{i=1}^{n} i^3 - \sum_{i=1}^{n-2} x_i^3 = \frac{n^2(n+1)^2}{4} - \sum_{i=1}^{n-2} x_i^3$$

From these CAN get

$$y_1 + y_2 + y_3, \qquad y_1 y_2 + y_1 y_3 + y_2 y_3, \qquad y_1 y_2 y_3$$

Messy!- On Next Slides

Deriving Sym Functions From Sums of Powers

Have

$$y_1 + y_2 + y_3, \qquad y_1^2 + y_2^2 + y_3^2, \qquad y_1^3 + y_2^3 + y_3^3$$

Want

$$y_1 + y_2 + y_3 \text{have}, \qquad y_1 y_2 + y_1 y_3 + y_2 y_3, \qquad y_1 y_2 y_3$$

Get $y_1 y_2 + y_1 y_3 + y_2 y_3$ from:

$$(y_1 + y_2 + y_3)^2 - (y_1^2 + y_2^2 + y_3^2) = 2(y_1 y_2 + y_1 y_3 + y_2 y_3).$$

Get $y_1 y_2 y_3$ since its equal to:
$\frac{(y_1 y_2 + y_1 y_3 + y_2 y_3)(y_1 + y_2 + y_3) - (y_1 + y_2 + y_3)(y_1^2 + y_2^2 + y_3^2) + (y_1^3 + y_2^3 + y_3^3)}{3}$.

Solution Two

Bob computes (next slide shows how)

$$\sum_{1 \le i \le n-3} x_i$$

$$\sum_{1 \le i < j \le n-3} x_i x_j$$

$$\sum_{1 \le i < j < k \le n-3} x_i x_j x_k$$

From these **CAN** get (next next slide shows how)

$$y_1 + y_2 + y_3, \qquad y_1 y_2 + y_1 y_3 + y_2 y_3, \qquad y_1 y_2 y_3$$

Cleanly!

Bob Can Actually Compute Those Sums

Let

$$
\begin{aligned}
s_0^L(x_1,\ldots,x_L) &= 1 \text{ (For Notational Niceness.)} \\
s_1^L(x_1,\ldots,x_L) &= \sum_{1\le i\le L} x_i \\
s_2^L(x_1,\ldots,x_L) &= \sum_{1\le i<j\le L} x_i x_j \\
s_3^L(x_1,\ldots,x_L) &= \sum_{1\le i<j<k\le L} x_i x_j x_k
\end{aligned}
$$

Let s_i^L mean $s_i^L(x_1,\ldots,x_L)$.
We show that if Bob has

$$s_0^{L-1}, \qquad s_1^{L-1}, \qquad s_2^{L-1}, \qquad s_3^{L-1}, \qquad x_L.$$

then he can compute

$$s_0^L, \qquad s_1^L, \qquad s_2^L, \qquad s_3^L.$$

$$
\begin{aligned}
s_0^L &= 1 \\
s_1^L &= s_1^{L-1} + x_L s_0^{L-1} \\
s_2^L &= s_2^{L-1} + x_L s_1^{L-1} \\
s_3^L &= s_3^{L-1} + x_L s_2^{L-1}
\end{aligned}
$$

Getting $s_i^3(y_1, y_2, y_3)$ from s_i^{n-3}

Let s_i^3 mean $s_i^3(y_1, y_2.y_3)$. One can show:

$$
\begin{aligned}
s_1^n &= s_1^{n-3}s_0^3 + s_0^{n-3}s_1^3 \\
s_2^n &= s_2^{n-3}s_0^3 + s_1^{n-3}s_1^3 + s_0^{n-3}s_2^3 \\
s_3^n &= s_3^{n-3}s_0^3 + s_2^{n-3}s_1^3 + s_1^{n-3}s_2^3 + s_3^{n-3}s_0^3
\end{aligned}
$$

Bob knows

$$s_1^n, \quad s_1^{n-3}, \quad s_2^n, \quad s_2^{n-3}, \quad s_3^n, \quad s_3^{n-3}$$

By solving three linear equations in three variables he can find:

$$s_1^3, \quad s_2^3, \quad s_3^3$$