The Muffin Problem

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Five Muffins, Three Students

This problem appeared in the Julia Robinson Mathematics Festival. These problems were proposed by Alan Frank:

You have 5 muffins and 3 students. You want to divide the muffins evenly, but no student wants a sliver. Which division of the muffins maximized the smallest piece?
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*You have 5 muffins and 3 students. You want to divide the muffins evenly, but no student wants a sliver. Which division of the muffins maximized the smallest piece?*

1. Divide two muffins into $(\frac{1}{3}, \frac{2}{3})$.
2. Three muffins remain uncut.
3. Give two students piece of size $\frac{2}{3}$ and one muffin.
4. Give one student two pieces of size $\frac{1}{3}$ and one muffin.

The smallest piece in the above solution is $\frac{1}{3}$.

*Is there a procedure with a larger smallest piece?*

VOTE
Five Muffins, Three Students

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You have 5 muffins and 3 students. You want to divide the muffins evenly, but no student wants a sliver. Which division of the muffins maximized the smallest piece?

1. Divide two muffins into \( \left( \frac{1}{3}, \frac{2}{3} \right) \).
2. Three muffins remain uncut.
3. Give two students piece of size \( \frac{2}{3} \) and one muffin.
4. Give one student two pieces of size \( \frac{1}{3} \) and one muffin.

The smallest piece in the above solution is \( \frac{1}{3} \).

Is there a procedure with a larger smallest piece?

VOTE —YES WE CAN! (excited that we can!)
Five Muffins, Three People—Procedure

1. Divide four muffins into \( \left( \frac{5}{12}, \frac{7}{12} \right) \).
2. Divide one muffin into \( \left( \frac{1}{2}, \frac{1}{2} \right) \).
3. Give two students \( \left[ \frac{7}{12}, \frac{7}{12}, \frac{1}{2} \right] \).
4. Give one student \( \left[ \frac{5}{12}, \frac{5}{12}, \frac{5}{12}, \frac{5}{12} \right] \).

The smallest piece in the above solution is \( \frac{5}{12} \).

Is there a procedure with a larger smallest piece?  

VOTE
Five Muffins, Three People—Procedure

1. Divide four muffins into \( (\frac{5}{12}, \frac{7}{12}) \).
2. Divide one muffin into \( (\frac{1}{2}, \frac{1}{2}) \).
3. Give two students \( [\frac{7}{12}, \frac{7}{12}, \frac{1}{2}] \).
4. Give one student \( [\frac{5}{12}, \frac{5}{12}, \frac{5}{12}, \frac{5}{12}] \).

The smallest piece in the above solution is \( \frac{5}{12} \).

Is there a procedure with a larger smallest piece?

VOTE — NO WE CAN’T! (excited that we can prove we can’t)
Assumption We Can Make

There is a procedure for 5 muffins, 3 students where each student gets $\frac{5}{3}$ muffins, smallest piece $N$. We want $N \leq \frac{5}{12}$.

We **ASSUME** each muffin cut into **at least two** pieces: If not then cut that muffin ($\frac{1}{2}, \frac{1}{2}$).

**THIS TALK** ALL proofs will be about opt being $\leq 1/2$. We assume each muffin is cut into **at least two** pieces.

**PIECES VS SHARES:** They are the same. **PIECE** is muffin-view, **SHARE** is student-view.
Two Easy Principles

**Principle One:**
If a muffin is cut into $\geq u$ pieces then there is a piece $\leq \frac{1}{u}$

**Principle Two:**
If a student gets $\geq u$ shares then there is a piece $\leq \frac{m}{s} \times \frac{1}{u}$
Five Muffins, Three People—Can’t do better than $\frac{5}{12}$

There is a procedure for 5 muffins, 3 students where each student gets $\frac{5}{3}$ muffins, smallest piece $N$. We want $N \leq \frac{5}{12}$.

**Case 1:** Some muffin is cut into $\geq 3$ pieces. Then $N \leq \frac{1}{3} < \frac{5}{12}$. 
(Negation: All muffins cut into exactly 2 pieces.)

**Case 2:** Some student gets $\geq 4$ shares. Then $N \leq \frac{5}{3} \times \frac{1}{4} = \frac{5}{12}$. 
(Negation: Every student gets $\leq 3$ shares)

**Case 3:** All muffins divided 2 pieces, all students $\leq 3$ shares.
Count by muffins: $5$ (muffins) $\times$ $2$ (pieces) $= 10$ pieces.
Count by students: $3$ (students) $\times \leq 3$ (shares) $\leq 9$ shares.
Hence this case cannot occur.
General Problem

You have \( m \) muffins and \( s \) students. You want to divide the muffins evenly, but no student wants a sliver. Which division of the muffins maximized the smallest piece?

Let \( m, s \in \mathbb{N} \).

An \((m, s)\)-procedure is a way to divide and distribute \( m \) muffins to \( s \) students so that each student gets \( \frac{m}{s} \) muffins.

An \((m, s)\)-procedure is optimal if it has the largest smallest piece of any procedure.

\( f(m, s) \) be the smallest piece in an optimal \((m, s)\)-procedure. \((f(m, s) \text{ exists. Compactness argument by Douglas Ulrich.})\)

We have shown \( f(5, 3) = \frac{5}{12} \).
Floor-Ceiling Theorem

\[ f(m, s) \leq \max\left\{ \frac{1}{3}, \min\left\{ \frac{m}{s \lceil 2m/s \rceil}, 1 - \frac{m}{s \lceil 2m/s \rceil} \right\} \right\}. \]

Proof:

Case 1: Some muffin is cut into \( \geq 3 \) pieces. Some piece \( \leq \frac{1}{3} \).

Case 2: Every muffin is cut into two pieces, so \( 2m \) pieces.

Someone gets \( \geq \left\lceil \frac{2m}{s} \right\rceil \) pieces. Some piece is \( \leq \frac{(m/s)}{\lceil 2m/s \rceil} = \frac{m}{s \lceil 2m/s \rceil} \).

Someone gets \( \leq \left\lfloor \frac{2m}{s} \right\rfloor \) pieces. Some piece is \( \geq \frac{(m/s)}{\lfloor 2m/s \rfloor} = \frac{m}{s \lfloor 2m/s \rfloor} \).

The other piece from that muffin is of size \( \leq 1 - \frac{m}{s \lceil 2m/s \rceil} \).
THREE Students

CLEVERNESS, COMP PROGS for the procedure.

FLOOR-CEILING THEOREM for optimality.

\[ f(1, 3) = \frac{1}{3} \]

\[ f(3k, 3) = 1. \]

\[ f(3k + 1, 3) = \frac{3k-1}{6k}, \quad k \geq 1. \]

\[ f(3k + 2, 3) = \frac{3k+2}{6k+6}. \]
FOUR Students

CLEVERNESS, COMP PROGS for procedures.

FLOOR-CEILING THEOREM for optimality.

\[ f(4k, 4) = 1 \text{ (easy)} \]
\[ f(1, 4) = \frac{1}{4} \text{ (easy)} \]
\[ f(4k + 1, 4) = \frac{4k-1}{8k}, \ k \geq 1. \]
\[ f(4k + 2, 4) = \frac{1}{2}. \]
\[ f(4k + 3, 4) = \frac{4k+1}{8k+4}. \]

Is FIVE student case a Mod 5 pattern?

VOTE
FOUR Students

CLEVERNESS, COMP PROGS for procedures.

FLOOR-CEILING THEOREM for optimality.

\[ f(4k, 4) = 1 \text{ (easy)} \]

\[ f(1, 4) = \frac{1}{4} \text{ (easy)} \]

\[ f(4k + 1, 4) = \frac{4k-1}{8k}, \quad k \geq 1. \]

\[ f(4k + 2, 4) = \frac{1}{2}. \]

\[ f(4k + 3, 4) = \frac{4k+1}{8k+4}. \]

Is FIVE student case a Mod 5 pattern?

VOTE — NO! (excited because YES → boring)
FIVE Students, \( m = 1, 2, 3, 4, 7, 11, 10k \)

\[
\begin{align*}
  f(1, 5) &= \frac{1}{5} \text{ (easy)} \\
  f(2, 5) &= \frac{1}{5} \text{ (easy)} \\
  f(3, 5) &= \frac{1}{4} \text{ (Like } f(5, 3) = \frac{5}{12} \text{ but Muffins/Students reversed)} \\
  f(4, 5) &= \frac{3}{10} \text{ (Will come back to this later)} \\
  f(7, 5) &= \frac{1}{3} \text{ (use FLOOR-CEILING theorem)} \\
  f(11, 5) &= \text{(Will come back to this later)} \\
  f(10k, 5) &= 1 \text{ (Trivial)}
\end{align*}
\]
FIVE Students

Results on the next few slides:

CLEVERNESS, COMP PROGS for the procedure.

FLOOR-CEILING THEOREM for optimality.
FIVE Students \( m = 10k + 1, 10k + 2, 10k + 3 \)

If \( k \) not specified then \( k \geq 0 \).

\( m = 10k + 1: \)

\[ f(30k + 1, 5) = \frac{30k+1}{60k+5} \]

\[ f(30k + 11, 5) = \frac{30k+11}{60k+25} \quad (k \geq 1) \]

\[ f(30k + 21, 5) = \frac{10k+7}{20k+15} \]

\[ f(10k + 2, 5) = \frac{10k-2}{20k} \quad (k \geq 1) \]

\[ f(10k + 3, 5) = \frac{10k+3}{20k+10} \quad (k \geq 1) \]
FIVE Students $m = 10k + 4, 10k + 5, 10k + 6$

$m = 10k + 4$

$f(30k + 4, 5) = \frac{30k+1}{60k+5}$

$f(30k + 14, 5) = \frac{30k+11}{60k+25}$

$f(30k + 24, 5) = \frac{10k+7}{20k+15}$

$f(10k + 5, 5) = 1$

$m = 10k + 6$:

$f(30k + 6, 5) = \frac{30k+6}{60k+15}$

$f(30k + 16, 5) = \frac{30k+16}{60k+35}$

$f(30k + 26, 5) = \frac{30k+26}{60k+55}$
FIVE Students \( m = 10k + 7, 10k + 8, 10k + 9 \)

\[
f(10k + 7, 5) = \frac{10k+3}{20k+10}
\]

\[
f(10k + 8, 5) = \frac{5k+4}{10k+10}
\]

\( m = 10k + 9 \)

\[
f(30k + 9, 5) = \frac{10k+2}{20k+5}
\]

\[
f(30k + 19, 5) = \frac{30k+16}{60k+35}
\]

\[
f(30k + 29, 5) = \frac{30k+26}{50k+55}
\]
FIVE students, $m = 4$, Procedure

1. Divide two muffins into $(\frac{3}{10}, \frac{3}{10}, \frac{2}{5})$.
2. Divide two muffins into $(\frac{1}{2}, \frac{1}{2})$.
3. Give four students $[\frac{3}{10}, \frac{1}{2}]$.
4. Give one student $[\frac{2}{5}, \frac{2}{5}]$. 
Where Are We on $f(4, 5)$?

Procedure on last slide yields $f(4, 5) \geq \frac{3}{10}$. Theorem FLOOR-CEILING yields $f(4, 5) \leq \frac{1}{3}$. So we have

$$\frac{3}{10} \leq f(4, 5) \leq \frac{1}{3}$$
Where Are We on $f(4, 5)$?

Procedure on last slide yields $f(4, 5) \geq \frac{3}{10}$.
Theorem FLOOR-CEILING yields $f(4, 5) \leq \frac{1}{3}$.
So we have

$$\frac{3}{10} \leq f(4, 5) \leq \frac{1}{3}$$

VOTE:

1. KNOWN: $f(4, 5) = \frac{3}{10}$ - New opt technique.
2. KNOWN: $f(4, 5) = \frac{1}{4}$ - New procedure.
3. KNOWN: $\frac{3}{10} < f(4, 5) < \frac{1}{3}$ - New opt and new proc.
4. UNKNOWN TO SCIENCE!
Where Are We on \( f(4, 5) \)?

Procedure on last slide yields \( f(4, 5) \geq \frac{3}{10} \).
Theorem FLOOR-CEILING yields \( f(4, 5) \leq \frac{1}{3} \).

So we have

\[
\frac{3}{10} \leq f(4, 5) \leq \frac{1}{3}
\]

VOTE:

1. KNOWN: \( f(4, 5) = \frac{3}{10} \) - New opt technique.
2. KNOWN: \( f(4, 5) = \frac{1}{4} \) - New procedure.
3. KNOWN: \( \frac{3}{10} < f(4, 5) < \frac{1}{3} \) - New opt and new proc.
4. UNKNOWN TO SCIENCE!

\[
\text{KNOWN: } f(4, 5) = \frac{3}{10}
\]

HAPPY: New opt tech more interesting than new proc.
Three Principles: If All Students Get Two Shares Then

1) If there is a share $\geq x$ then there is a share

$$\leq \frac{m}{s} - x.$$

2) If muffin cut in $C$ pieces, one $\geq x$, then another is

$$\leq \frac{1 - x}{C - 1}.$$

3) If muffin cut in $C$ pieces, one $\leq x$, then another is

$$\geq \frac{1 - x}{C - 1}.$$

Look at the student who gets that piece to obtain a piece that is

$$\leq \frac{m}{s} - \frac{1 - x}{C - 1}.$$
\[ f(4, 5) = \frac{3}{10} : \text{Easy Case-Muffin Based} \]

Assume there is an optimal \((4, 5)\)-procedure with smallest piece \(N\).

**Case 1:** Some muffin gets cut into 4 pieces.

\[ N \leq \frac{1}{4} < \frac{3}{10}. \]

(Negation: All muffins cut into 2 or 3 pieces.)
$f(4, 5) = \frac{3}{10}$: Easy Cases-Student Based

**Case 2:** Some student gets $\geq 3$ shares.

$$N \leq \frac{4}{5} \times \frac{1}{3} = \frac{4}{15} < \frac{3}{10}.$$

**Case 3:** Some student gets 1 share. That share is size $\frac{4}{5}$. The other piece from the muffin it came from is

$$N \leq 1 - \frac{4}{5} = \frac{1}{5} < \frac{3}{10}.$$

(Negation of Case 2 and 3: Every Student gets 2 shares.)
Case 4: Every student gets 2 pieces and every muffin is cut into either two or three pieces. Let $m_2$ ($m_3$) be the number of muffins cut into 2 (3) pieces.

$$2m_2 + 3m_3 = 10$$
$$m_2 + m_3 = 4$$

$m_2 = 2$: There are 2 muffins cut into 2 pieces.
$m_3 = 2$: There are 2 muffins cut into 3 pieces.
Each ◦ is a piece.

◦ ◦ (Sums to 1)
◦ ◦ (Sums to 1)
◦ ◦ ◦ (Sums to 1)
◦ ◦ ◦ ◦ (Sums to 1)
◦ ◦ ◦ ◦ (Sums to 1)
\( f(4, 5) = \frac{3}{10} \): Fun Case

**Case 4a:** One of \( \circ \circ \circ \) (Sums to 1) is \( \geq \frac{2}{5} \)

\[ N \leq \frac{1 - (2/5)}{2} = \frac{3}{10}. \]

**Case 4b:** All six shares from

\[ \circ \circ \circ \circ \circ \circ \]

are \( < \frac{2}{5} \). No student gets two of these since sum is \( < \frac{4}{5} \). Hence every student gets one from above batch and one from below batch. But they are not equal so his case cannot occur.

\[ \circ \circ \]

\[ \circ \circ \]
FIVE Students $m = 11$: $\frac{13}{30}$ Procedure

1. Divide six muffins into ($\frac{13}{30}$, $\frac{17}{30}$).
2. Divide four muffins into ($\frac{9}{20}$, $\frac{11}{20}$).
3. Divide one muffin into ($\frac{1}{2}$, $\frac{1}{2}$).
4. Give two students [$\frac{17}{30}$, $\frac{17}{30}$, $\frac{17}{30}$, $\frac{1}{2}$].
5. Give two students [$\frac{13}{30}$, $\frac{13}{30}$, $\frac{13}{30}$, $\frac{9}{20}$, $\frac{9}{20}$].
6. Give one student [$\frac{11}{20}$, $\frac{11}{20}$, $\frac{11}{20}$, $\frac{11}{20}$].
Where Are We on $f(11, 5)$?

Procedure on last slide yields $f(11, 5) \geq \frac{13}{30}$. Theorem FLOOR-CEILING yields $f(11, 5) \leq \frac{11}{25}$. So we have

$$\frac{13}{30} \leq f(11, 5) \leq \frac{11}{25}$$
Where Are We on $f(11, 5)$?

Procedure on last slide yields $f(11, 5) \geq \frac{13}{30}$.
Theorem FLOOR-CEILING yields $f(11, 5) \leq \frac{11}{25}$.
So we have

$$\frac{13}{30} \leq f(11, 5) \leq \frac{11}{25}$$

**VOTE:**

1. **KNOWN:** $f(11, 5) = \frac{13}{30}$ - New opt technique.
2. **KNOWN:** $f(11, 5) = \frac{11}{25}$ - New procedure.
3. **KNOWN:** $\frac{13}{30} < f(11, 5) < \frac{11}{25}$. New opt and new proc.
4. **UNKNOWN TO SCIENCE!**
Where Are We on $f(11, 5)$?

Procedure on last slide yields $f(11, 5) \geq \frac{13}{30}$. Theorem FLOOR-CEILING yields $f(11, 5) \leq \frac{11}{25}$. So we have

$$\frac{13}{30} \leq f(11, 5) \leq \frac{11}{25}$$

**VOTE:**

1. KNOWN: $f(11, 5) = \frac{13}{30}$ - New opt technique.
2. KNOWN: $f(11, 5) = \frac{11}{25}$ - New procedure.
3. KNOWN: $\frac{13}{30} < f(11, 5) < \frac{11}{25}$. New opt and new proc.
4. UNKNOWN TO SCIENCE!

**KNOWN:** $f(11, 5) = \frac{13}{30}$

**HAPPY:** New opt tech more interesting than new proc.
Three Principles: If all Muffins cut in two pieces then

1) If there is a piece $\geq x$ then there is a piece

$$\leq 1 - x.$$ 

2) If student gets $C$ shares, one $\geq x$, then another is

$$\leq \frac{(m/s) - x}{C - 1}.$$ 

3) If a student gets $C$ shares, one $\leq x$, then another is

$$\geq \frac{(m/s) - x}{C - 1}.$$ 

Look at the muffin it came from to obtain a piece that is

$$\leq 1 - \frac{(m/s) - x}{C - 1}.$$
f(11, 5) = \frac{13}{30}, \text{ Easy Case-Muffin Based}

N is smallest piece.

**Case 1:** Some muffin is cut into \( \geq 3 \) pieces. \( N \leq \frac{1}{3} < \frac{13}{30} \).

(Negation: All muffins cut into 2 pieces.)
$f(11, 5) = \frac{13}{30}$, Easy Cases-Student Based

**Case 2:** Some student gets $\geq 6$ pieces.

$$N \leq \frac{11}{5} \times \frac{1}{6} = \frac{11}{30} < \frac{13}{30}.$$  

**Case 3:** Some student gets $\leq 3$ pieces. One of the shares is $x \geq \frac{11}{5} \times \frac{1}{3} = \frac{11}{15}$. Then

$$N \leq 1 - \frac{11}{15} = \frac{4}{15} < \frac{13}{30}.$$  

(Negation of Cases 2 and 3: Every student gets 4 or 5 shares.)
$f(11, 5) = \frac{13}{30}$, Fun Cases

**Case 3:** Every muffin is cut in 2 pieces and every student gets either 4 or 5 pieces. Number of pieces: 22.

- $s_4$ is number of students who get 4 shares
- $s_5$ is number of students who get 5 shares

$$4s_4 + 5s_5 = 22$$
$$s_4 + s_5 = 5$$

$s_4 = 3$: There are 3 students who have 4 shares.
$s_5 = 2$: There are 2 students who have 5 shares.
$f(11, 5) = \frac{13}{30}$, Fun Cases

Each $\circ$ is a share.

- $\circ \circ \circ \circ \circ$ Sums to $11/5$  
- $\circ \circ \circ \circ \circ$ Sums to $11/5$  
- $\circ \circ \circ \circ \circ$ Sums to $11/5$  
- $\circ \circ \circ \circ \circ$ Sums to $11/5$

**Case 3.1:** There is a 4-share $x \leq \frac{1}{2}$. Then there is a share

$$\geq \frac{(11/5) - (1/2)}{3} = \frac{17}{30}.$$  

The other piece from the muffin $x$ game from is of size

$$\leq 1 - \frac{17}{30} = \frac{13}{30}.$$
$$f(11, 5) = \frac{13}{30}, \text{ Fun Cases}$$

**Case 3.2:** All 4-students have shares \( \geq \frac{1}{2} \).

- are shares
- are shares \( > \frac{1}{2} \)

There are at \( \geq 12 \) shares \( > \frac{1}{2} \). Can’t occur.
The Techniques Generalizes!

**Good News:** The technique used to get $f(11, 5) \leq \frac{13}{30}$ and $f(4, 5) \leq \frac{3}{10}$ lead to two theorems that apply to other cases.

**Bad News:** The theorems are hard and I am time-pressed, so you don’t get to see them.

**Good News:** The theorems are hard and I am time-pressed, so you don’t have to see them.
What Have We Done?

1. Three theorems about $m > s$ case.
2. Ten theorems about the $m < s$ case.
3. Exact results for $f(s+1, s)$ and $f(s-1, s)$ cases.
4. Convinced four High School students that the most important field of Mathematics is Muffinry.
Open: Particular Values with $s \leq 15$

<table>
<thead>
<tr>
<th>Proc $\leq f(m,s) \leq$ Opt</th>
<th>Diff $\sim$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\frac{59}{126}$ $\leq f(47,9) \leq \frac{77}{162}$</td>
<td>0.00705</td>
</tr>
<tr>
<td>$\frac{29}{44}$ $\leq f(24,11) \leq \frac{91}{209}$</td>
<td>0.00359</td>
</tr>
<tr>
<td>$\frac{5}{11}$ $\leq f(52,11) \leq \frac{83}{176}$</td>
<td>0.01704</td>
</tr>
<tr>
<td>$\frac{15}{39}$ $\leq f(23,13) \leq \frac{16}{39}$</td>
<td>0.02564</td>
</tr>
<tr>
<td>$\frac{29}{65}$ $\leq f(35,13) \leq \frac{431}{962}$</td>
<td>0.00187</td>
</tr>
<tr>
<td>$\frac{11}{23}$ $\leq f(23,14) \leq \frac{23}{56}$</td>
<td>0.01786</td>
</tr>
</tbody>
</table>
Open: Depends only on $m/s$?

$$f(m, s) = f(am, as)?$$

**THOUGHTS:**

1. $f(m, s) \leq f(am, as)$.
2. All opt Theorems EXCEPT ONE give $f(am, as) \leq f(m, s)$.
3. Only ONE result with $s \leq 20$ eludes us:
   $$f(9, 10) = \frac{3}{10}$$
   $$f(18, 20) \geq \frac{3}{10}.$$
Important Open Questions

Conjectures:

1. $f(m, s)$ is always rational.
2. $f(m, s)$ is computable.
3. For all $s$, there is a mod $S$ pattern where $s$ divides $S$, that holds for almost all $m$. 