The Muffin Problem

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Five Muffins, Three Students

At

A Recreational Math Conference  
(Gathering for Gardner)

I found a pamphlet advertising

The Julia Robinson Mathematics Festival

which had this problem, proposed by Alan Frank:

How can you divide and distribute 5 muffins to 3 students so that every student gets $\frac{5}{3}$ where nobody gets a tiny sliver?
Five Muffins, Three Students, Proc by Picture

<table>
<thead>
<tr>
<th>Person</th>
<th>Color</th>
<th>What they Get</th>
</tr>
</thead>
<tbody>
<tr>
<td>Alice</td>
<td>RED</td>
<td>$1 + \frac{2}{3} = \frac{5}{3}$</td>
</tr>
<tr>
<td>Bob</td>
<td>BLUE</td>
<td>$1 + \frac{2}{3} = \frac{5}{3}$</td>
</tr>
<tr>
<td>Carol</td>
<td>GREEN</td>
<td>$1 + \frac{1}{3} + \frac{1}{3} = \frac{5}{3}$</td>
</tr>
</tbody>
</table>

Smallest Piece: $\frac{1}{3}$
Can We Do Better?

The smallest piece in the above solution is \( \frac{1}{3} \).

Is there a procedure with a larger smallest piece?

VOTE
Can We Do Better?

The smallest piece in the above solution is $\frac{1}{3}$.

Is there a procedure with a larger smallest piece?

VOTE

- YES
- NO
Can We Do Better?

The smallest piece in the above solution is $\frac{1}{3}$.

Is there a procedure with a larger smallest piece?

VOTE

▶ YES
▶ NO

YES WE CAN!

We use ! since we are excited that we can!
### Five Muffins, Three People–Proc by Picture

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<tr>
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<td>( \frac{6}{12} + \frac{7}{12} + \frac{7}{12} )</td>
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**Smallest Piece:** \( \frac{5}{12} \)
Can We Do Better?

The smallest piece in the above solution is $\frac{5}{12}$.

Is there a procedure with a larger smallest piece?

VOTE

- YES
- NO
Can We Do Better?

The smallest piece in the above solution is $\frac{5}{12}$.

Is there a procedure with a larger smallest piece?

VOTE

- YES
- NO

NO WE CAN’T!

We use ! since we are excited to prove we can’t do better!
Assumption We Can Make

There is a procedure for 5 muffins, 3 students where each student gets \( \frac{5}{3} \) muffins, smallest piece \( N \). We want \( N \leq \frac{5}{12} \).

We **ASSUME** each muffin cut into at least 2 pieces: If not then cut that muffin \( \left( \frac{1}{2}, \frac{1}{2} \right) \).

**THIS TALK** ALL proofs will be about \( \text{opt} \) being \( \leq 1/2 \). We assume each muffin is cut into at least 2 pieces.

**PIECES VS SHARES:** They are the same.
- **PIECE** is muffin-view,
- **SHARE** is student-view.
Muffin Principle

If a muffin is cut into $\geq u$ pieces then there is a piece $\leq \frac{1}{u}$.

Example: If a Muffin cut into 3 pieces:

Some piece is $\leq \frac{1}{3}$. 
Student Principle (not Principal)

If a student gets \( \geq u \) shares then there is a share \( \leq \frac{m}{5} \times \frac{1}{u} \)

Example: 5 muffins, 3 students. All student gets \( \frac{5}{3} \).
If some student gets \( \geq 4 \) shares:

Then one of these pieces is \( \leq \frac{5}{3} \times \frac{1}{4} \)
Pieces Principle

If there are $P$ pieces then:

Some student gets $\geq \lceil \frac{P}{s} \rceil$
Some student gets $\leq \lfloor \frac{P}{s} \rfloor$

Example: 5 muffins, 3 people. If there are 10 pieces:

Some student gets $\geq \left\lceil \frac{10}{3} \right\rceil = 4$
Some student gets $\leq \left\lfloor \frac{10}{3} \right\rfloor = 3$
There is a procedure for 5 muffins, 3 students where each student gets $\frac{5}{3}$ muffins, smallest piece $N$. We want $N \leq \frac{5}{12}$.

**Case 1:** Some muffin is cut into $\geq 3$ pieces. Then $N \leq \frac{1}{3} \times \frac{5}{12}$.

(Negation: All muffins are cut into 2 pieces.)

**Case 2:** All muffins are cut into 2 pieces. 10 pieces, 3 students: Someone gets $\geq 4$ pieces. He has some piece

$$\leq \frac{5}{3} \times \frac{1}{4} = \frac{5}{12}$$

Great to see $\frac{5}{12}$.
1. Procedure for 5 muffins, 3 people, smallest piece $\frac{5}{12}$.
2. NO Procedure for 5 muffins, 3 people, smallest piece $\frac{5}{12}$.

Amazing That Have Exact Result!

Prepare To Be More Amazed!

We have many results like this!:

\[
\begin{align*}
    f(47, 9) & = \frac{111}{234} \\
    f(52, 11) & = \frac{83}{176} \\
    f(35, 13) & = \frac{64}{143}
\end{align*}
\]
General Problem

How can you divide and distribute $m$ muffins to $s$ students so that each student gets $\frac{m}{s}$ AND the MIN piece is MAXIMIZED?

Let $m, s \in \mathbb{N}$.

An $(m, s)$-procedure is a way to divide and distribute $m$ muffins to $s$ students so that each student gets $\frac{m}{s}$ muffins.

An $(m, s)$-procedure is optimal if it has the largest smallest piece of any procedure.

$f(m, s)$ be the smallest piece in an optimal $(m, s)$-procedure.

We have shown $f(5, 3) = \frac{5}{12}$. 
Terminology Issue

Let $m, s \in \mathbb{N}$. 

$m$ is the number of muffins. 
$s$ is the number of students.

1. $f(m, s) \geq \alpha$ means that there is a procedure with smallest piece $\alpha$. We call this *A Procedure*.

2. $f(m, s) \leq \alpha$ means that there is NO procedure with smallest piece $> \alpha$. We call this *An Optimality Result* or *An Opt Result*.

**DO NOT** use terms *upper bound* and *lower bounds*:

1. Procedures are lower bounds, *opposite* of usual terminology.
2. Opt results are upper bounds, *opposite* of usual terminology.
Floor-Ceiling Theorem

\[ f(m, s) \leq \max \left\{ \frac{1}{3}, \min \left\{ \frac{m}{s \lceil 2m/s \rceil}, 1 - \frac{m}{s \lfloor 2m/s \rfloor} \right\} \right\}. \]

Proof:

**Case 1:** Some muffin is cut into \( \geq 3 \) pieces. Some piece \( \leq \frac{1}{3} \).

**Case 2:** Every muffin is cut into 2 pieces, so 2\( m \) pieces.

**Someone** gets \( \geq \lceil \frac{2m}{s} \rceil \) pieces. Some piece is \( \leq \frac{(m/s)}{\lceil 2m/s \rceil} = \frac{m}{s \lceil 2m/s \rceil} \).

**Someone** gets \( \leq \lfloor \frac{2m}{s} \rfloor \) pieces. Some piece is \( \geq \frac{(m/s)}{\lfloor 2m/s \rfloor} = \frac{m}{s \lfloor 2m/s \rfloor} \).

The other piece from that muffin is of size \( \leq 1 - \frac{m}{s \lfloor 2m/s \rfloor} \).
THREE Students

CLEVERNESS, COMP PROGS for the procedure.

Floor-Ceiling Theorem for optimality.

\[ f(1, 3) = \frac{1}{3} \]

\[ f(3k, 3) = 1. \]

\[ f(3k + 1, 3) = \frac{3k-1}{6k}, \quad k \geq 1. \]

\[ f(3k + 2, 3) = \frac{3k+2}{6k+6}. \]
FOUR Students

CLEVERNESS, COMP PROGS for procedures.

Floor-Ceiling Theorem for optimality.

\[ f(4k, 4) = 1 \text{ (easy)} \]

\[ f(1, 4) = \frac{1}{4} \text{ (easy)} \]

\[ f(4k + 1, 4) = \frac{4k-1}{8k}, \ k \geq 1. \]

\[ f(4k + 2, 4) = \frac{1}{2}. \]

\[ f(4k + 3, 4) = \frac{4k+1}{8k+4}. \]

Is FIVE student case a Mod 5 pattern?
VOTE YES or NO
FOUR Students

CLEVERNESS, COMP PROGS for procedures.

Floor-Ceiling Theorem for optimality.

\[ f(4k, 4) = 1 \text{ (easy)} \]
\[ f(1, 4) = \frac{1}{4} \text{ (easy)} \]
\[ f(4k + 1, 4) = \frac{4k-1}{8k}, \ k \geq 1. \]
\[ f(4k + 2, 4) = \frac{1}{2}. \]
\[ f(4k + 3, 4) = \frac{4k+1}{8k+4}. \]

Is FIVE student case a Mod 5 pattern?
VOTE YES or NO
NO! (excited because YES would be boring)
FIVE Students, \( m = 1, 2, 3, 4, 7, 11, 10k \)

\[
\begin{align*}
f(1, 5) &= \frac{1}{5} \text{ (easy)} \\
f(2, 5) &= \frac{1}{5} \text{ (easy)} \\
f(3, 5) &= \frac{1}{4} \text{ (Will discuss briefly later)} \\
f(4, 5) &= \frac{3}{10} \text{ (Will not discuss later)} \\
f(7, 5) &= \frac{1}{3} \text{ (Use Floor-Ceiling Thm)} \\
f(11, 5) &= \text{ (Will come back to this later)} \\
f(10k, 5) &= 1 \text{ (Trivial)}
\end{align*}
\]
FIVE Students

Results on the next few slides:

**CLEVERNESS, COMP PROGS** for the procedure.

**Floor-Ceiling Theorem** for optimality.
FIVE Students \( m = 10k + 1, 10k + 2, 10k + 3 \)

If \( k \) not specified then \( k \geq 0 \).

\[ m = 10k + 1: \]
\[ f(30k + 1, 5) = \frac{30k+1}{60k+5} \]
\[ f(30k + 11, 5) = \frac{30k+11}{60k+25} \quad (k \geq 1) \]
\[ f(30k + 21, 5) = \frac{10k+7}{20k+15} \]

\[ f(10k + 2, 5) = \frac{10k-2}{20k} \quad (k \geq 1) \]

\[ f(10k + 3, 5) = \frac{10k+3}{20k+10} \quad (k \geq 1) \]
FIVE Students \( m = 10k + 4, 10k + 5, 10k + 6 \)

\[
m = 10k + 4
\]

\[
f(30k + 4, 5) = \frac{30k+1}{60k+5}
\]

\[
f(30k + 14, 5) = \frac{30k+11}{60k+25}
\]

\[
f(30k + 24, 5) = \frac{10k+7}{20k+15}
\]

\[
f(10k + 5, 5) = 1
\]

\[
m = 10k + 6:
\]

\[
f(30k + 6, 5) = \frac{10k+2}{20k+5}
\]

\[
f(30k + 16, 5) = \frac{30k+16}{60k+35}
\]

\[
f(30k + 26, 5) = \frac{30k+26}{60k+55}
\]
FIVE Students $m = 10k + 7, 10k + 8, 10k + 9$

\[ f(10k + 7, 5) = \frac{10k+3}{20k+10} \]

\[ f(10k + 8, 5) = \frac{5k+4}{10k+10} \]

$m = 10k + 9$

\[ f(30k + 9, 5) = \frac{10k+2}{20k+5} \]

\[ f(30k + 19, 5) = \frac{30k+16}{60k+35} \]

\[ f(30k + 29, 5) = \frac{30k+26}{60k+55} \]
What About FIVE students, ELEVEN muffins?

Procedure:

Divide the Muffins into Pieces:

1. Divide 6 muffins into \(\left(\frac{13}{30}, \frac{17}{30}\right)\).
2. Divide 4 muffins into \(\left(\frac{9}{20}, \frac{11}{20}\right)\).
3. Divide 1 muffin into \(\left(\frac{1}{2}, \frac{1}{2}\right)\).

Distribute the Shares to Students:

1. Give 2 students \(\left\{\frac{17}{30}, \frac{17}{30}, \frac{17}{30}, \frac{1}{2}\right\}\).
2. Give 2 students \(\left\{\frac{13}{30}, \frac{13}{30}, \frac{13}{30}, \frac{9}{20}, \frac{9}{20}\right\}\).
3. Give 1 student \(\left\{\frac{11}{20}, \frac{11}{20}, \frac{11}{20}, \frac{11}{20}\right\}\).

So

\[ f(11, 5) \geq \frac{13}{30} \]
What About FIVE students, ELEVEN muffins? Opt

Recall: **Floor-Ceiling Theorem:**

\[ f(m, s) \leq \max \left\{ \frac{1}{3}, \min \left\{ \frac{m}{s \lceil 2m/s \rceil}, 1 - \frac{m}{s \lfloor 2m/s \rfloor} \right\} \right\} . \]

\[ f(11, 5) \leq \max \left\{ \frac{1}{3}, \min \left\{ \frac{11}{5 \lceil 22/5 \rceil}, 1 - \frac{11}{5 \lfloor 22/5 \rfloor} \right\} \right\} . \]

\[ f(11, 5) \leq \max \left\{ \frac{1}{3}, \min \left\{ \frac{11}{5 \times 5}, 1 - \frac{11}{5 \times 4} \right\} \right\} . \]

\[ f(11, 5) \leq \max \left\{ \frac{1}{3}, \min \left\{ \frac{11}{25}, \frac{9}{20} \right\} \right\} . \]

\[ f(11, 5) \leq \max \left\{ \frac{1}{3}, \frac{11}{25} \right\} = \frac{11}{25} . \]
Where Are We On FIVE students, ELEVEN muffins?

- By **Procedure** $\frac{13}{30} \leq f(11, 5)$.
- By **Floor-Ceiling** $f(11, 5) \leq \frac{11}{25}$.

So

\[
\frac{13}{30} \leq f(11, 5) \leq \frac{11}{25} \quad \text{Diff} = 0.0066666\ldots
\]
Where Are We On FIVE students, ELEVEN muffins?

- By Procedure $\frac{13}{30} \leq f(11, 5)$.
- By Floor-Ceiling $f(11, 5) \leq \frac{11}{25}$.

So

$$\frac{13}{30} \leq f(11, 5) \leq \frac{11}{25} \quad \text{Diff} = 0.006666\ldots$$

VOTE:

1. **KNOWN:** $f(11, 5) = \frac{13}{30}$: New opt technique.
2. **KNOWN:** $f(11, 5) = \frac{11}{25}$: New procedure.
3. **KNOWN:** $\frac{13}{30} < f(11, 5) < \frac{11}{25}$: New opt and new proc.
4. **UNKNOWN TO SCIENCE!**
5. **HARAMBE THE GORILLA!**
   (In Poll of Discrete Math Students for Presidential Election 3 wrote in Harambe.)
Where Are We On FIVE students, ELEVEN muffins?

- By Procedure $\frac{13}{30} \leq f(11, 5)$.
- By Floor-Ceiling $f(11, 5) \leq \frac{11}{25}$.

So

$$\frac{13}{30} \leq f(11, 5) \leq \frac{11}{25} \quad \text{Diff} = 0.006666\ldots$$

VOTE:

1. **KNOWN:** $f(11, 5) = \frac{13}{30}$: New opt technique.
2. **KNOWN:** $f(11, 5) = \frac{11}{25}$: New procedure.
3. **KNOWN:** $\frac{13}{30} < f(11, 5) < \frac{11}{25}$: New opt and new proc.
4. **UNKNOWN TO SCIENCE!**
5. **HARAMBE THE GORILLA!**
   (In Poll of Discrete Math Students for Presidential Election 3 wrote in Harambe.)

**KNOWN:** $f(11, 5) = \frac{13}{30}$

**HAPPY:** New opt tech more interesting than new proc.
$f(11, 5) = \frac{13}{30}$, Easy Case Based on Muffins

$N$ is smallest piece.

**Case 1:** Some muffin is cut into $\geq 3$ pieces. $N \leq \frac{1}{3} < \frac{13}{30}$.

(Negation: All muffins cut into 2 pieces.)
\( f(11, 5) = \frac{13}{30}, \) Easy Case Based on Students

**Case 2:** Some student gets \( \geq 6 \) pieces.

\[
N \leq \frac{11}{5} \times \frac{1}{6} = \frac{11}{30} < \frac{13}{30}.
\]

**Case 3:** Some student gets \( \leq 3 \) pieces.

One of the shares is

\[
\geq \frac{11}{5} \times \frac{1}{3} = \frac{11}{15}.
\]

Look at the muffin it came from to find a piece that is

\[
\leq 1 - \frac{11}{15} = \frac{4}{15} < \frac{13}{30}.
\]

*(Negation of Cases 2 and 3: Every student gets 4 or 5 shares.)*
Case 4: Every muffin is cut in 2 pieces, every student gets 4 or 5 pieces. Number of pieces: 22. Note \( \leq 11 \) pieces are \( > \frac{1}{2} \).

- \( s_4 \) is number of students who get 4 shares
- \( s_5 \) is number of students who get 5 shares

\[
4s_4 + 5s_5 = 22
\]
\[
s_4 + s_5 = 5
\]

\( s_4 = 3 \): There are 3 students who have 4 shares.
\( s_5 = 2 \): There are 2 students who have 5 shares.
\[ f(11, 5) = \frac{13}{30}, \text{ Fun Cases} \]

\[ \diamond \text{ and } \circ \text{ are shares.} \]

\[ \diamond \diamond \diamond \diamond \diamond \quad (\text{Sums to } 11/5) \]
\[ \diamond \diamond \diamond \diamond \diamond \quad (\text{Sums to } 11/5) \]

\[ \circ \circ \circ \circ \circ \quad (\text{Sums to } 11/5) \]
\[ \circ \circ \circ \quad (\text{Sums to } 11/5) \]
\[ \circ \circ \quad (\text{Sums to } 11/5) \]
\[ \circ \quad (\text{Sums to } 11/5) \]

**Case 3.1:** One of (say)

\[ \circ \quad (\text{Sums to } 11/5) \]

is \( \leq \frac{1}{2} \). Then there is a share

\[ \geq \frac{(11/5) - (1/2)}{3} = \frac{17}{30}. \]

The other piece from the muffin is

\[ \leq 1 - \frac{17}{30} = \frac{13}{30} \quad \text{Great to see } \frac{13}{30}. \]
Case 3.2: All

$$\bigcirc \bigcirc \bigcirc \bigcirc \quad (\text{Sums to } 11/5)$$
$$\bigcirc \bigcirc \bigcirc \bigcirc \bigcirc \quad (\text{Sums to } 11/5)$$
$$\bigcirc \bigcirc \bigcirc \bigcirc \bigcirc \bigcirc \quad (\text{Sums to } 11/5)$$

are $> \frac{1}{2}$.
There are $\geq 12$ shares $> \frac{1}{2}$. Can’t occur.
Good News!
The technique used to get $f(11, 5) \leq \frac{13}{30}$ lead to a theorem that apply to other cases! We call it The Interval Theorem

Bad News!
Interval Theorem is hard to state, so you don’t get to see it.

Good News!
Interval Theorem is hard to state, so you don’t have to see it.
How Research Works

**History:**

1. Obtain particular results.
2. Prove a general theorem based on those results.
3. Run into a case we cannot solve (e.g., (11,5) and (35,13)).
4. Lather, Rinse, Repeat.
What Else Have We Accomplished?-Gen Thms

1. $f(m, s) = \frac{m}{s} f(s, m)$
2. $f(m, s)$ is computable and rational.
   Uses interesting **Applied Math**: **Mixed Integer Programming**.
3. There is a nice formula for $f(s + 1, s)$. 
What Else Have We Accomplished?-Particular Thms

1. A computer program that helps us get procedures.
2. For $1 \leq s \leq 15$, for all $m$, know $f(m, s)$.
3. Convinced 4 High School students that the most important field of Mathematics is **Muffinry**.
Open Questions

1. For all $s$ there is a pattern for $f(m, s)$ that depends on $m \mod T$ where $s$ divides $T$.
2. $f(m, s) = \frac{a}{b}$ (lowest terms) where $s$ divides $b$.
3. For all $m \geq s$, $f(m, s)$ is always determined by either
   - Floor Ceiling Theorem
   - Interval Theorem
   - $f(s + 1, s)$ Theorem.