The Muffin Problem

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Cake Cutting

1. **Proportional Cake Cutting:** \( n \) people divide and distribute a cake so that everyone has \( \frac{1}{n} \) in their opinion. Exists \( O(n \log n) \) cuts discrete protocols. Optimal. **Crumbs!**

2. **Envy Free Cake Cutting:** \( n \) people divide and distribute a cake so that everyone has biggest (or tied) piece in their opinion. Exists \( O(n^n \ldots) \) (six \( n \)'s) cuts discrete protocols. No lower bounds known. **Crumbs!!!!** (Prior result had been unbounded protocol. This result was a surprise.)

3. **Cake Cutting** is a long studied problems. **Many** paper in Theory (Okay) and AI (What?).

4. **This Talk** is not about traditional cake cutting.
Our “Motivation”

1. Want to avoid crumbs.
2. All people will have uniform tastes. $\alpha$ of a cake is of value $\alpha$.
3. We use muffins rather than cakes.
4. Honesty: This is motivation after the fact.
At

**Gathering for Gardner Conference**

I found a pamphlet advertising

**The Julia Robinson Mathematics Festival**

which had this problem, proposed by Alan Frank:

*How can you divide and distribute 5 muffins to 3 students so that every student gets $\frac{5}{3}$ where nobody gets a tiny sliver?*
Five Muffins, Three Students, Proc by Picture

<table>
<thead>
<tr>
<th>Person</th>
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<th>What they Get</th>
</tr>
</thead>
<tbody>
<tr>
<td>Alice</td>
<td>RED</td>
<td>$1 + \frac{2}{3} = \frac{5}{3}$</td>
</tr>
<tr>
<td>Bob</td>
<td>BLUE</td>
<td>$1 + \frac{2}{3} = \frac{5}{3}$</td>
</tr>
<tr>
<td>Carol</td>
<td>GREEN</td>
<td>$1 + \frac{1}{3} + \frac{1}{3} = \frac{5}{3}$</td>
</tr>
</tbody>
</table>

Smallest Piece: $\frac{1}{3}$
Can We Do Better?

The smallest piece in the above solution is \( \frac{1}{3} \).

Is there a procedure with a larger smallest piece?

VOTE
Can We Do Better?

The smallest piece in the above solution is $\frac{1}{3}$.

Is there a procedure with a larger smallest piece?

VOTE

- YES
- NO
Can We Do Better?

The smallest piece in the above solution is $\frac{1}{3}$.

Is there a procedure with a larger smallest piece?

VOTE

- YES
- NO

YES WE CAN!

We use ! since we are excited that we can!
### Five Muffins, Three People—Proc by Picture

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**Smallest Piece:** $\frac{5}{12}$
Can We Do Better?

The smallest piece in the above solution is $\frac{5}{12}$. Is there a procedure with a larger smallest piece?

VOTE

- YES
- NO
Can We Do Better?

The smallest piece in the above solution is $\frac{5}{12}$.

Is there a procedure with a larger smallest piece?

VOTE

- YES
- NO

NO WE CAN’T! We use ! since we are excited to prove we can’t do better!
Assumption We Can Make

There is a procedure for 5 muffins, 3 students where each student gets \(\frac{5}{3}\) muffins, smallest piece \(N\). We want \(N \leq \frac{5}{12}\).

We **ASSUME** each muffin cut into at least 2 pieces: If not then cut that muffin \((\frac{1}{2}, \frac{1}{2})\).

**THIS TALK** ALL proofs will be about opt being \(\leq 1/2\). We assume each muffin is cut into at least 2 pieces.

**PIECES VS SHARES:** They are the same.

- **PIECE** is muffin-view,
- **SHARE** is student-view.
Muffin Principle

If a muffin is cut into $\geq u$ pieces then there is a piece $\leq \frac{1}{u}$.

Example: If a Muffin is cut into 3 pieces:

some piece is $\leq \frac{1}{3}$. 
Student Principle (not Principal)

If a student gets $\geq u$ shares then there is a share $\leq \frac{m}{s} \times \frac{1}{u}$

Example: 5 muffins, 3 students. All student gets $\frac{5}{3}$.
If some student gets $\geq 4$ shares:

Then one of these pieces is $\leq \frac{5}{3} \times \frac{1}{4}$
Pieces Principle

If there are $P$ pieces then:

Some student gets $\geq \left\lceil \frac{P}{s} \right\rceil$
Some student gets $\leq \left\lfloor \frac{P}{s} \right\rfloor$

Example: 5 muffins, 3 people. If there are 10 pieces:

Some student gets $\geq \left\lceil \frac{10}{3} \right\rceil = 4$
Some student gets $\leq \left\lfloor \frac{10}{3} \right\rfloor = 3$
There is a procedure for 5 muffins, 3 students where each student gets $\frac{5}{3}$ muffins, smallest piece $N$. We want $N \leq \frac{5}{12}$.

**Case 1:** Some muffin is cut into $\geq 3$ pieces. Then $N \leq \frac{1}{3} < \frac{5}{12}$.
(Negation: All muffins are cut into 2 pieces.)

**Case 2:** All muffins are cut into 2 pieces. 10 pieces, 3 students: **Someone** gets $\geq 4$ pieces. He has some piece

$$\leq \frac{5}{3} \times \frac{1}{4} = \frac{5}{12}$$

Great to see $\frac{5}{12}$
Be Amazed Now! And Later!

1. Procedure for 5 muffins, 3 people, smallest piece $\frac{5}{12}$.
2. NO Procedure for 5 muffins, 3 people, smallest piece $\geq \frac{5}{12}$.

Amazing That Have Exact Result!

Prepare To Be More Amazed!

We show many results like this!
General Problem

How can you divide and distribute \( m \) muffins to \( s \) students so that each student gets \( \frac{m}{s} \) AND the MIN piece is MAXIMIZED?

Let \( m, s \in \mathbb{N} \).

An \((m, s)\)-procedure is a way to divide and distribute \( m \) muffins to \( s \) students so that each student gets \( \frac{m}{s} \) muffins.

An \((m, s)\)-procedure is optimal if it has the largest smallest piece of any procedure.

\( f(m, s) \) be the smallest piece in an optimal \((m, s)\)-procedure. (\( f(m, s) \) exists. Compactness argument by Douglas Ulrich.)

We have shown \( f(5, 3) = \frac{5}{12} \).
Terminology Issue

Let $m, s \in \mathbb{N}$.
$m$ is the number of muffins.
$s$ is the number of students.

1. $f(m, s) \geq \alpha$ means that there is a procedure with smallest piece $\alpha$. We call this A Procedure.
2. $f(m, s) \leq \alpha$ means that there is NO procedure with smallest piece $> \alpha$. We call this An Optimality Result or An Opt Result.

DO NOT use terms upper bound and lower bounds:

1. Procedures are lower bounds, opposite of usual terminology.
2. Opt results are upper bounds, opposite of usual terminology.
Floor-Ceiling Theorem

\[ f(m, s) \leq \max \left\{ \frac{1}{3}, \min \left\{ \frac{m}{s \lceil 2m/s \rceil}, 1 - \frac{m}{s \lceil 2m/s \rceil} \right\} \right\}. \]

Proof:

Case 1: Some muffin is cut into \( \geq 3 \) pieces. Some piece \( \leq \frac{1}{3} \).

Case 2: Every muffin is cut into 2 pieces, so \( 2m \) pieces.

**Someone** gets \( \geq \lceil \frac{2m}{s} \rceil \) pieces. Some piece is \( \leq \frac{(m/s)}{\lceil 2m/s \rceil} = \frac{m}{s \lceil 2m/s \rceil} \).

**Someone** gets \( \leq \lfloor \frac{2m}{s} \rfloor \) pieces. Some piece is \( \geq \frac{(m/s)}{\lfloor 2m/s \rfloor} = \frac{m}{s \lfloor 2m/s \rfloor} \).

The other piece from that muffin is of size \( \leq 1 - \frac{m}{s \lfloor 2m/s \rfloor} \).
CLEVERNESS, COMP PROGS for the procedure.

Floor-Ceiling Theorem for optimality.

\[ f(1, 3) = \frac{1}{3} \]

\[ f(3k, 3) = 1. \]

\[ f(3k + 1, 3) = \frac{3k-1}{6k}, \ k \geq 1. \]

\[ f(3k + 2, 3) = \frac{3k+2}{6k+6}. \]
FOUR Students

CLEVERNESS, COMP PROGS for procedures.

Floor-Ceiling Theorem for optimality.

\[ f(4k, 4) = 1 \] (easy)

\[ f(1, 4) = \frac{1}{4} \] (easy)

\[ f(4k + 1, 4) = \frac{4k-1}{8k}, \quad k \geq 1. \]

\[ f(4k + 2, 4) = \frac{1}{2}. \]

\[ f(4k + 3, 4) = \frac{4k+1}{8k+4}. \]

Is FIVE student case a Mod 5 pattern?

VOTE YES or NO
FOUR Students

CLEVERNESS, COMP PROGS for procedures.

Floor-Ceiling Theorem for optimality.

\[ f(4k, 4) = 1 \text{ (easy)} \]

\[ f(1, 4) = \frac{1}{4} \text{ (easy)} \]

\[ f(4k + 1, 4) = \frac{4k - 1}{8k}, \quad k \geq 1. \]

\[ f(4k + 2, 4) = \frac{1}{2}. \]

\[ f(4k + 3, 4) = \frac{4k + 1}{8k + 4}. \]

Is FIVE student case a Mod 5 pattern?

VOTE YES or NO

NO! (excited because YES would be boring)
FIVE Students, \( m = 1, 2, 3, 4, 7, 11, 10k \)

\[
\begin{align*}
f(1, 5) &= \frac{1}{5} \text{ (easy)} \\
f(2, 5) &= \frac{1}{5} \text{ (easy)} \\
f(3, 5) &= \frac{1}{4} \text{ (Like } f(5, 3) = \frac{5}{12} \text{ but Muffins/Students reversed)} \\
f(4, 5) &= \frac{3}{10} \text{ (Will discuss briefly later)} \\
f(7, 5) &= \frac{1}{3} \text{ (use Floor-Ceiling Thm)} \\
f(11, 5) &= \text{ (Will come back to this later)} \\
f(10k, 5) &= 1 \text{ (Trivial)}
\end{align*}
\]
FIVE Students

Results on the next few slides:

**CLEVERNESS, COMP PROGS** for the procedure.

**Floor-Ceiling Theorem** for optimality.
FIVE Students $m = 10k + 1, 10k + 2, 10k + 3$

If $k$ not specified then $k \geq 0$.

$m = 10k + 1$:

\[
f(30k + 1, 5) = \frac{30k+1}{60k+5}
\]

\[
f(30k + 11, 5) = \frac{30k+11}{60k+25} \quad (k \geq 1)
\]

\[
f(30k + 21, 5) = \frac{10k+7}{20k+15}
\]

\[
f(10k + 2, 5) = \frac{10k-2}{20k} \quad (k \geq 1)
\]

\[
f(10k + 3, 5) = \frac{10k+3}{20k+10} \quad (k \geq 1)
\]
FIVE Students $m = 10k + 4, 10k + 5, 10k + 6$

$m = 10k + 4$

$f(30k + 4, 5) = \frac{30k+1}{60k+5}$

$f(30k + 14, 5) = \frac{30k+11}{60k+25}$

$f(30k + 24, 5) = \frac{10k+7}{20k+15}$

$f(10k + 5, 5) = 1$

$m = 10k + 6$:

$f(30k + 6, 5) = \frac{10k+2}{20k+5}$

$f(30k + 16, 5) = \frac{30k+16}{60k+35}$

$f(30k + 26, 5) = \frac{30k+26}{60k+55}$
FIVE Students $m = 10k + 7, 10k + 8, 10k + 9$

\[ f(10k + 7, 5) = \frac{10k+3}{20k+10} \]

\[ f(10k + 8, 5) = \frac{5k+4}{10k+10} \]

$m = 10k + 9$

\[ f(30k + 9, 5) = \frac{10k+2}{20k+5} \]

\[ f(30k + 19, 5) = \frac{30k+16}{60k+35} \]

\[ f(30k + 29, 5) = \frac{30k+26}{60k+55} \]
What About FIVE students, ELEVEN muffins?

Procedure:

Divide the Muffins in to Pieces:

1. Divide 6 muffins into \( \left( \frac{13}{30}, \frac{17}{30} \right) \).
2. Divide 4 muffins into \( \left( \frac{9}{20}, \frac{11}{20} \right) \).
3. Divide 1 muffin into \( \left( \frac{1}{2}, \frac{1}{2} \right) \).

Distribute the Shares to Students:

1. Give 2 students \( \left[ \frac{17}{30}, \frac{17}{30}, \frac{17}{30}, \frac{1}{2} \right] \).
2. Give 2 students \( \left[ \frac{13}{30}, \frac{13}{30}, \frac{13}{30}, \frac{9}{20}, \frac{9}{20} \right] \).
3. Give 1 student \( \left[ \frac{11}{20}, \frac{11}{20}, \frac{11}{20}, \frac{11}{20} \right] \).

So

\[ f(11, 5) \geq \frac{13}{30} \]
What About FIVE students, ELEVEN muffins? Opt

Recall: Floor-Ceiling Theorem:

\[ f(m, s) \leq \max \left\{ \frac{1}{3}, \min \left\{ \frac{m}{s \left\lceil 2m/s \right\rceil}, 1 - \frac{m}{s \left\lfloor 2m/s \right\rfloor} \right\} \right\}. \]

\[ f(11, 5) \leq \max \left\{ \frac{1}{3}, \min \left\{ \frac{11}{5 \left\lceil 22/5 \right\rceil}, 1 - \frac{11}{5 \left\lfloor 22/5 \right\rfloor} \right\} \right\}. \]

\[ f(11, 5) \leq \max \left\{ \frac{1}{3}, \min \left\{ \frac{11}{5 \times 5}, 1 - \frac{11}{5 \times 4} \right\} \right\}. \]

\[ f(11, 5) \leq \max \left\{ \frac{1}{3}, \min \left\{ \frac{11}{25}, \frac{9}{20} \right\} \right\}. \]

\[ f(11, 5) \leq \max \left\{ \frac{1}{3}, \frac{11}{25} \right\} = \frac{11}{25}. \]
Where Are We On FIVE students, ELEVEN muffins?

- By **Procedure** $\frac{13}{30} \leq f(11, 5)$.
- By **Floor-Ceiling** $f(11, 5) \leq \frac{11}{25}$.

So

$$\frac{13}{30} \leq f(11, 5) \leq \frac{11}{25} \quad \text{Diff} = 0.006666\ldots$$
Where Are We On FIVE students, ELEVEN muffins?

- By Procedure $\frac{13}{30} \leq f(11, 5)$.
- By Floor-Ceiling $f(11, 5) \leq \frac{11}{25}$.

So

$$\frac{13}{30} \leq f(11, 5) \leq \frac{11}{25} \quad \text{Diff} = 0.006666\ldots$$

VOTE:

1. **KNOWN:** $f(11, 5) = \frac{13}{30}$: New opt technique.
2. **KNOWN:** $f(11, 5) = \frac{11}{25}$: New procedure.
3. **KNOWN:** $\frac{13}{30} < f(11, 5) < \frac{11}{25}$: New opt and new proc.
4. **UNKNOWN TO SCIENCE!**
5. **HARAMBE THE GORILLA!**
   (In Poll of Discrete Math Students 3 wrote in Harambe.)
Where Are We On Five students, Eleven muffins?

- By **Procedure** \( \frac{13}{30} \leq f(11, 5) \).
- By **Floor-Ceiling** \( f(11, 5) \leq \frac{11}{25} \).

So

\[
\frac{13}{30} \leq f(11, 5) \leq \frac{11}{25}
\]

\[\text{Diff} = 0.006666\ldots\]

**VOTE:**

1. **KNOWN:** \( f(11, 5) = \frac{13}{30} \): New opt technique.
2. **KNOWN:** \( f(11, 5) = \frac{11}{25} \): New procedure.
3. **KNOWN:** \( \frac{13}{30} < f(11, 5) < \frac{11}{25} \): New opt and new proc.
4. **UNKNOWN TO SCIENCE!**
5. **HARAMBE THE GORILLA!**
   (In Poll of Discrete Math Students 3 wrote in Harambe.)

**KNOWN:** \( f(11, 5) = \frac{13}{30} \)

**HAPPY:** New opt tech more interesting than new proc.
$f(11, 5) = \frac{13}{30}$, Easy Case Based on Muffins

$N$ is smallest piece.

**Case 1:** Some muffin is cut into $\geq 3$ pieces. $N \leq \frac{1}{3} < \frac{13}{30}$.

*(Negation: All muffins cut into 2 pieces.)*
\( f(11, 5) = \frac{13}{30}, \) Easy Case Based on Students

**Case 2:** Some student gets \( \geq 6 \) pieces.

\[
N \leq \frac{11}{5} \times \frac{1}{6} = \frac{11}{30} < \frac{13}{30}.
\]

**Case 3:** Some student gets \( \leq 3 \) pieces.

One of the shares is

\[
\geq \frac{11}{5} \times \frac{1}{3} = \frac{11}{15}.
\]

Look at the muffin it came from to find a piece that is

\[
\leq 1 - \frac{11}{15} = \frac{4}{15} < \frac{13}{30}.
\]

(Negation of Cases 2 and 3: Every student gets 4 or 5 shares.)
Case 3: Every muffin is cut in 2 pieces, every student gets 4 or 5 pieces. Number of pieces: 22. Note $\leq 11$ pieces are $> \frac{1}{2}$.

- $s_4$ is number of students who get 4 shares
- $s_5$ is number of students who get 5 shares

\[
4s_4 + 5s_5 = 22 \\
\quad \quad \quad \quad s_4 + s_5 = 5
\]

$s_4 = 3$: There are 3 students who have 4 shares.
$s_5 = 2$: There are 2 students who have 5 shares.
\( f(11, 5) = \frac{13}{30}, \) Fun Cases

\( \diamond \) and \( \circ \) are shares.

\[ \begin{array}{ccccccc}
\diamond & \diamond & \diamond & \diamond & \diamond & \diamond & \quad (\text{Sums to } 11/5) \\
\diamond & \diamond & \diamond & \diamond & \diamond & \diamond & \quad (\text{Sums to } 11/5) \\
\end{array} \]

\[ \begin{array}{ccccccc}
\circ & \circ & \circ & \circ & \circ & \circ & \quad (\text{Sums to } 11/5) \\
\circ & \circ & \circ & \circ & \circ & \circ & \quad (\text{Sums to } 11/5) \\
\circ & \circ & \circ & \circ & \circ & \circ & \quad (\text{Sums to } 11/5) \\
\end{array} \]

Case 3.1: One of (say)

\[ \begin{array}{ccccccc}
\circ & \circ & \circ & \circ & \circ & \circ & \quad (\text{Sums to } 11/5) \\
\end{array} \]

is \( \leq \frac{1}{2} \). Then there is a share

\[ \geq \frac{(11/5) - (1/2)}{3} = \frac{17}{30} . \]

The other piece from the muffin is

\[ \leq 1 - \frac{17}{30} = \frac{13}{30} \quad \text{Great to see } \frac{13}{30} . \]
$f(11, 5) = \frac{13}{30}$, Fun Cases

**Case 3.2:** All

- ○ ○ ○ ○ ○ (Sums to $11/5$)
- ○ ○ ○ ○ ○ ○ (Sums to $11/5$)
- ○ ○ ○ ○ ○ ○ ○ (Sums to $11/5$)

are $> \frac{1}{2}$.

There are $\geq 12$ shares $> \frac{1}{2}$. Can’t occur.
The Techniques Generalizes!

**Good News!**
The technique used to get $f(11, 5) \leq \frac{13}{30}$ lead to a theorem that apply to other cases!

**Bad News!**
The theorem is hard to state, so you don’t *get* to see it.

**Good News!**
The theorem is hard to state, so you don’t *have* to see it.
What Else Have We Accomplished?

1. 13 theorems to help get us opt results.
   ▶ Intricate arguments!-Harder than what you’ve seen here.
   ▶ Amazing that so many optimal procedures are known.

2. A computer program that helps us get procedures.

3. For $1 \leq s \leq 15$, for almost all $m$, know $f(m, s)$.

4. Convinced 4 High School students that the most important field of Mathematics is **Muffinry**.
How Many Open Problems With $s \leq 15$?

For $1 \leq s \leq 15$, for almost all $m$, know $f(m, s)$. How many $f(m, s)$ with $1 \leq s \leq 15$ are unknown?

**VOTE:**

1. Between 1 and 10
2. Between 11 and 100
3. Between 101 and 1000
4. Over 1001
How Many Open Problems With $s \leq 15$?

For $1 \leq s \leq 15$, for almost all $m$, know $f(m,s)$. How many $f(m,s)$ with $1 \leq s \leq 15$ are unknown?

**VOTE:**

1. Between 1 and 10
2. Between 11 and 100
3. Between 101 and 1000
4. Over 1001

**Answer:** 4, so Between 1 and 10.
Open Problems With $s \leq 15$

<table>
<thead>
<tr>
<th>Proc $\leq f(m, s) \leq$ Opt</th>
<th>Diff $\sim$</th>
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<tbody>
<tr>
<td>$\frac{59}{126} \leq f(47, 9) \leq \frac{77}{162}$</td>
<td>$0.00705$</td>
</tr>
<tr>
<td>$\frac{5}{11} \leq f(52, 11) \leq \frac{83}{176}$</td>
<td>$0.01704$</td>
</tr>
<tr>
<td>$\frac{15}{39} \leq f(23, 13) \leq \frac{16}{39}$</td>
<td>$0.02564$</td>
</tr>
<tr>
<td>$\frac{33}{78} \leq f(35, 13) \leq \frac{431}{962}$</td>
<td>$0.02494$</td>
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Conjectures on those Four Problems

Consider:

\[
\begin{array}{cccc}
\text{Proc} & \leq & f(m, s) & \leq \text{Opt} \\
\frac{59}{126} & \leq & f(47, 9) & \leq \frac{77}{162} \\
\text{Diff} & \sim & 0.00705
\end{array}
\]

We can imagine a new procedure.
We can’t imagine a new opt technique.
Conjectures on those Four Problems

Consider:

\[
\begin{array}{|c|c|c|c|}
\hline
\text{Proc} & \leq & f(m,s) & \leq \text{Opt} \\
\hline
\frac{59}{126} & \leq & f(47,9) & \leq \frac{77}{162} \\
\hline
\text{Diff} & \sim & 0.00705 \\
\hline
\end{array}
\]

We can imagine a new procedure.
We can’t imagine a new opt technique.
We’ve said that 10 times before with a hard particular case

- 5 times we found a new procedure.
- 5 times we found a new opt technique (including \(f(4,5)\)).

**Our Conjecture:** The opt results are the truth.
**Our Confidence:** 4 on a scale of 1 to 10.
“Important” Open Questions

Conjectures:

1. \( f(m, s) \) is always rational.
2. \( f(m, s) \) is computable.
3. For all \( s \), there is a mod \( t \) pattern where \( s \) divides \( t \), that holds for almost all \( m \).
History and Hope

**History:**
1. Obtain particular results.
2. Prove a general theorem based on those results.
3. Run into a case we cannot solve (e.g., (4,5) and (11,5)).
4. Lather, Rinse, Repeat.

**Hope:** A *finite* set of theorems that settle all cases.

**Likely?**
1. I think *No*, but

2. was *surprised* by the $n$-person $O(n^n)$ cuts envy free cake cutting algorithm, so I could be *surprised* again!