The Muffin Problem

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Abstract

You have m muffins and s students. You want to divide the muffins into pieces and give the shares to students such that every student has $\frac{m}{s}$ muffins. Find a divide-and-distribute protocol that maximizes the minimum piece. Let $f(m, s)$ be the minimum piece in the optimal protocol. We prove that $f(m, s)$ exists, is rational, and finding it is computable (though possibly difficult).

We show that $f(m, s)$ can be derived from $f(s, m)$; hence we need only consider $m \geq s$.

We give a function $FC(m, s)$ such that, for $m \geq s + 1$, $f(m, s) \leq FC(m, s)$. It is often the case that $f(m, s) = FC(m, s)$. More formally, for all s, for all but a finite number of m, $f(m, s) = FC(m, s)$. This leads to a nice formula for $f(m, s)$, though there are exceptions to it.

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We give a formula $INT(m,s)$, which has 6 parts, such that for many of the exceptional m, $f(m,s) = INT(m,s) < FC(m,s)$. This works for most of the exceptional m where $\lceil 2m/s \rceil \geq 4$.

There are still some exceptional m with $\lceil 2m/s \rceil = 3$ (if its ≤ 2 then the problem is trivial). For these cases we have a way to *generate theorems*. For $1 \leq d \leq 7$ we have generated formulas for $f(s+d,s)$. We do not have a theorem here but we do have a methodology which leads to, for some of the m, a value $BM(m,s)$ such that often $f(m,s) = BM(m,s) < INT(m,s) < FC(m,s)$.

So far it seems like, for $m \geq s$, $f(m,s) = \min\{FC(m,s), INT(m,s), BM(m,s)\}$, though we have not prove this.

For $1 \leq s \leq 50$ and $s \leq m \leq 60$ we have obtained $f(m,s)$ for all but 20 values.

Contents

1 Introduction

There is a vast literature on the following problem (stated informally): How can n students, with different tastes, divide a cake fairly, in the best way[1, 2, 3, 4, 5, 7]. This problem has parameters: what is fair? what is best? We give some examples: (1a) There is an $O(n \log n)$-cut discrete protocol for dividing a cake among n students so that everyone thinks they have $\geq \frac{1}{n}$ [5]. (1b) Any protocol for the problem in 1a requires $\Omega(n \log n)$ cuts [4]; (2) There is a discrete protocol dividing a cake among n students so that everyone thinks they have the largest piece or are tied. This is often called an *envy free division.* The number of cuts is unbounded [3]. (3) There is a $g(n)$-cut discrete protocol dividing a cake among n students in an envy free way where $g(n)$ is a very fast growing function [1].

We raise and tackle a related but different problem. All of the students have the same valuation and its geometric- so 1/3 of a muffin is worth 1/3. There are m muffins to be divided among s students. Everyone will get exactly $\frac{m}{s}$ muffins. However, we want to *maximize the minimum piece* so that nobody gets crumbs. A real world application might be when cutting up diamonds since in that case you do not want small pieces.

We give an example:

You have 5 muffins and 3 students. You want to divide the muffins evenly, but no student wants a small piece. Find a protocol that maximizes the smallest piece?

Convention 1.1 When discussing a muffin being cut we refer to *pieces.* When discussing a student receiving we refer to *shares.* They are the same; however, it will be good to have different terminologies to focus on whats important. We treat a piece, a share, and its value as the same thing. So we may say *let $x \geq \frac{1}{3}$ be given to a student.*

The obvious protocol is to divide 2 muffins into $(\frac{1}{3}, \frac{2}{3})$, leave the other 3 muffins uncut, give 2 students $[\frac{2}{3}, 1]$ (a share of size 1 and a share of size $\frac{2}{3}$). and give 1 student $[\frac{1}{3}, \frac{1}{3}, 1]$. The smallest piece in the above solution is $\frac{1}{3}$. Can we do better? We can:

Theorem 1.2 *There is a protocol that divides five muffins evenly among three students such that the smallest piece is of size $\frac{5}{12}$. There is no protocol that yields a bigger smallest piece.*

Proof: The following protocol works:

1. Divide 4 muffins into $(\frac{5}{12}, \frac{7}{12})$.

2. Divide 1 muffin into $(\frac{1}{2}, \frac{1}{2})$.

3. Give 2 students $[\frac{7}{12}, \frac{7}{12}, \frac{1}{2}]$.

4. Give 1 student $[\frac{5}{12}, \frac{5}{12}, \frac{5}{12}, \frac{5}{12}]$.

Assume there is a protocol for dividing up 5 muffins and distributing the shares to 3 students such that every student gets $\frac{5}{3}$ muffins. We can assume that every muffin is cut since if a muffin is uncut we will cut it in $(\frac{1}{2}, \frac{1}{2})$ and give both halves to the intended recipient. Let N be the size of the smallest piece produced.

Case 1: Some muffin is split into ≥ 3 pieces. Then $N \le \frac{1}{3} < \frac{5}{12}$.

Case 2: All muffins are cut into 2 pieces. Hence there are 10 pieces. Some student gets ≥ 4 shares. One of those shares is $\le \frac{5}{3} \times \frac{1}{4} = \frac{5}{12}$. ∎

This paper will address the following problem:

You have m muffins and s students. You want to divide the muffins into pieces and give the shares to students such that every student has $\frac{m}{s}$ muffins. Find a divide-and-distribute protocol that maximizes the minimum piece.

Def 1.3 Let $m, s \in \mathbb{N}$. An (m, s)-*protocol* is a protocol to cut m muffins into pieces and then distribute them to the s students so that each student gets $\frac{m}{s}$ muffins. An (m, s)-protocol is *optimal* if it has the largest smallest piece of any protocol. $f(m, s)$ is the size of the smallest piece in an optimal (m, s)-protocol.

It is not obvious that $f(m, s)$ exists, is rational, or is computable. But it is. We will prove the following in Appendix A.

Theorem 1.4 *Let $m, s \ge 1$.*

1. *There is a mixed integer program with $O(ms)$ binary variables, $O(ms)$ real variables, $O(ms)$ constraints, and all coefficients integers of absolute value $\le \max\{m, s\}$ such that, from the solution, one can extract $f(m, s)$ and a protocol that achieves this bound. This MIP can easily be obtained given m, s.*

2. *$f(m, s)$ is always rational. This follows from part 1.*

3. *In every optimal protocol for m muffins and s students all of the pieces are of rational size. This follows from part 1.*

4. *The problem of, given m, s, determine $f(m, s)$, is decidable. This follows from part 1.*

5.

We came upon this problem in a pamphlet *Julia Robinson Mathematics Festival: A Sample of Mathematical Puzzles* compiled by Nancy Blachman. On Page 2 was *The Muffin Puzzle* which asked about the problem for

- 3 muffins and 5 students
- 5 muffins and 3 students
- 6 muffins and 10 students
- 4 muffins and 7 students

Nancy Blachman attributes the problem to Alan Frank, as described by Jeremy Copeland [6] We are the first ones to consider this problem seriously for general m, s.

We have written some programs based on theorems in this paper. They are at:

`https://github.com/jeprinz/MuffinProblem`

2 Summary of Results

In Sections 3 and 4 we give basic definitions and lemmas that we use throughout the paper.

In Section 5 we show $f(m, s) = \frac{m}{s} f(s, m)$. Hence, for the rest of the paper, we need only look at the case where $m \geq s$ Since the case of $m = s$ is trivial ($f(s, s) = 1$), for the rest of the paper we only deal with $m \geq s + 1$.

In Section 6 we give an upper bound $f(m,s) \le FC(m,s)$ (FC stands for *Floor-Ceiling*). This upper bound will permeate the rest of the paper. This is because, as we will see in Section 18, this upper bound is often the lower bound as well. The cases where $f(m,s) \ne FC(m,s)$ are, to quote a phrase, *the exceptions that test the rule.*

In Section 7 we prove $f(24,11) = \frac{19}{44} < FC(24,11)$. Hence $f(24,11) \ne FC(24,11)$. The technique used on $24, 11$ generalizes. We give an upper bound $f(m,s) \le INT(m,s)$. (INT stands for *Interval*). There are other examples like $24, 11$; however, we do not quite know what causes them or why they are distinct.

In Section 17 we, for $1 \le s \le 6$, completely solve $f(m,s)$. We obtain that, for all $1 \le s \le 6$, for almost all (that is, for all but a finite number) $f(m,s) = FC(m,s)$. We also have a set of formula for $FC(m,s)$. For each $1 \le s \le 6$ we have, for each $0 \le i \le s-1$, there are constants a_i, b_i, c_i, d_i such that, for almost all $m = ks + i$, $f(m,s) = \frac{a_ik+b_i}{c_ik+d_i}$. We abbreviate by calling a set of formulas of this type *a nice formula.* In Appendix B we give conjectures nice formulas for $1 \le s \le 50$. In the next section we prove the existence of nice formulas, though not necessarily the ones conjectured.

In Section 18 we show that, for all s, for almost all m, $f(m,s) = FC(m,s)$. We also show that, for all s, there is a nice formula that, for almost all m, computes $FC(m,s)$. The upshot is that, for all s there is a nice formula that, for almost all m, computes $f(m,s)$.

The results on FC and INT lead to a bold conjecture:

For all m, s with $m \ge s+2$ $f(m,s) = \min\{FC(m,s), INT(m,s)\}$.

The results in Section 17 prove this conjecture for $s = 1,2,3,4,5,6$. We have also shown empirically that this conjecture holds for all (m,s) with $1 \le s \le 15$ and $s+2 \le m \le 135$, though we do not present it.

In Section 19 we state our conjectures and propose a research plan.

In Appendix A we present a mixed integer program to find $f(m,s)$ and some tips on how to speed it up. This MIP establishes that $f(m,s)$ exists, is rational, and the function is computable. Note that none of this is obvious from the definition. In Appendix B we present, for $s = 1$ to 50, conjectured nice formulas for $f(m, s)$. In Appendix C we present, for $s = 1$ to 50, for $m = s + 1$ to 60, $f(m, s)$ when it is known and upper and lower bounds when it is not.

3 Basic Definitions and Lemmas About Shares and Pieces

In this section we give three sets of definitions and three lemmas that use them. We will use these lemmas throughout the paper without mentioning them.

Def 3.1 Let $m, s \in \mathbb{N}$. Assume there is an (m, s)-protocol.

1. If a muffin is cut into t pieces then let $pi_S(t)$ be the smallest piece of that muffin.

2. If a student is given u shares then let $sh_S(u)$ ($sh_L(u)$) be the smallest (largest) share that student gets. (We will later define u-student and you can note that $sh_S(u)$ ($sh_L(u)$) is the smallest (largest) share a u-student gets.)

Lemma 3.2 *Let $m, s \in \mathbb{N}$. Assume there is an (m, s)-protocol.*

1. $pi_S(t) \leq \frac{1}{t}$.

2. $sh_S(u) \leq \frac{m}{su}$.

3. $sh_L(u) \geq \frac{m}{su}$.

Proof:

1) When a muffin, which has value 1, is cut into t pieces clearly some piece has to be $\leq \frac{1}{t}$ and some piece has to be $\geq \frac{1}{t}$.

2,3) When a student, who gets $\frac{m}{s}$, is given u shares clearly some share is $\leq \frac{m}{su}$ and some share is $\geq \frac{m}{su}$. ∎

Def 3.3 Let $m, s \in \mathbb{N}$. Assume there is an (m, s)-protocol.

1. A muffin that is cut into A pieces is an A-*muffin*. A piece from an A-muffin is an A-*piece*.

2. Pieces that come from the same A-muffin are A-*buddies*. We often just say *buddies*. Note that more than 2 shares can be buddies.

3. If x is a share then $B_S(x)$ ($B_L(x)$) is the smallest (largest) buddy of x.

4. If x is a 2-piece then $B(x) = B_L(x) = B_S(x)$.

Lemma 3.4 *Let $m, s \in \mathbb{N}$. Assume there is an (m, s)-protocol. Then*

1. *If x is an A-pieces then*

   - $B_S(x) \le \frac{1-x}{A-1}$

   - $B_L(x) \ge \frac{1-x}{A-1}$

2. *If x is a 2-piece then $B(x) = 1 - x$ (This follows from Part 1.)*

Proof:

1) Take x from the A-muffin it is a piece of. Whats left is $1 - x$ in $A - 1$ pieces. Clearly the smallest share is $\le \frac{1-x}{A-1}$ and the largest share is $\ge \frac{1-x}{A-1}$. ∎

Def 3.5 Let $m, s \in \mathbb{N}$. Assume there is an (m, s)-protocol.

1. A student that gets A shares is an A-*student*. A share given to an A-student is an A-*share*.

2. Shares that are given to the same A-student are A-*matched*. We often just say *matched*. Note that more than 2 shares can be matched.

3. If x is a share then $M_S(x)$ ($M_L(x)$) is the smallest (largest) match of x.

4. If x is a 2-share then $M(x) = M_L(x) = M_S(x)$.

Lemma 3.6 *Let $m, s \in \mathbb{N}$. Assume there is an (m, s)-protocol. Then*

1. *If x is an A-shares then*
   - $M_S(x) \le \frac{(m/s)-x}{A-1}$
   - $M_L(x) \ge \frac{(m/s)-x}{A-1}$

2. *If x is a 2-share then $M(x) = \frac{m}{s} - x$ (This follows from Part 1.)*

Proof:

1) Take x from the A-student who has them. Whats left is $\frac{m}{s} - x$ in $A - 1$ shares. Clearly the smallest share is $\le \frac{(m/s)-x}{A-1}$ and the largest share is $\ge \frac{(m/s)-x}{A-1}$. ∎

Lemma 3.7 *If $m \neq s$ then no two 2-pieces can be both 2-buddies and 2-matched.*

Proof: If x, y are 2-buddies then $x + y = 1$. If x, y are 2-matched then $x + y = \frac{m}{s}$. Since $m \neq s$ these cannot both be true. ∎

4 Basic Theorems about f

Theorem 4.1 *Let $m, s \in \mathbb{N}$.*

1. *If s divides m then $f(m, s) = 1$.*

2. *If m divides s then $f(m, s) = \frac{m}{s}$.*

3. *For all A, $f(Am, As) \ge f(m, s)$.*

Proof:

1) The protocol that shows $f(m,s) \geq 1$ is: give everyone $\frac{m}{s}$ muffins. We always have $f(m,s) \leq 1$. Hence $f(m,s)=1$.

2) By Theorem 5.1 $f(m,s) = \frac{m}{s} f(s,m)$. By Part 1 of this theorem $f(s,m)=1$. Hence $f(m,s) = \frac{m}{s}$.

3) Here is an (xm, xs)-protocol whose smallest piece is $f(m,s)$: Divide the xm muffins into x piles, $\mathcal{M}_1, \ldots, \mathcal{M}_x$ of m muffins each. Divide the xs students into x piles, $\mathcal{S}_1, \ldots, \mathcal{S}_x$ of s students each. Apply the optimal (m,s)-protocol to every $(\mathcal{M}_i, \mathcal{S}_i)$ pair. ∎

Theorem 4.1.3 raises the question: does $f(m,s)$ only depend on $\frac{m}{s}$? All of our upper bound techniques depend only on $\frac{m}{s}$. One of our conjectures is that $f(m,s)$ only depends on $\frac{m}{s}$.

Theorem 4.1.1 and 4.1.2 mean that we will never have to deal with the cases where m divides s or s divides m. This leads to two conventions.

Convention 4.2 We will assume that every muffin gets cut into at least 2 pieces. Here is why we can do that. Assume you have an optimal (m,s) protocol.

Case 1: Some muffin is cut. Then $f(m,s) \leq \frac{1}{2}$. Hence you can assume that all muffins are cut. If the protocol gave a student a whole muffin then cut it in half and give the student both halves. Since $f(m,s) \leq \frac{1}{2}$ this will not affect the optimally of the protocol

Case 2: All muffins are uncut. Then every student gets a whole number of muffins. In this case s divides m and $f(m,s)=1$ by Theorem 4.1.1. We will never deal with this case.

Convention 4.3 We will assume that every student gets at least 2 shares. Here is why we can do that. Assume you have an optimal (m,s) protocol.

Case 1: Some student gets at least 2 shares. Then $f(m,s) \le \frac{m}{2s}$. If the protocol gave some other student 1 share then cut it in half and give the student both halves. Since $f(m,s) \le \frac{m}{2s}$ this will not affect the optimally of the protocol

Case 2: All students get 1 share. Hence every muffin is divided into $\frac{s}{m}$ pieces of size $\frac{m}{s}$. In this case m divides s and $f(m,s) = \frac{m}{s}$ by Theorem 4.1.2. We will never deal with this case.

5 $f(\le s, s)$

In this section we show that the case of $m \le s$ reduces to the case of $m \ge s$. Hence, in all future sections, we only look at the $m \ge s$ case.

Theorem 5.1 *Let $m, s \in \mathbb{N}$. Then $f(s,m) = \frac{s}{m} f(m,s)$.*

Proof:

Assume $f(m,s) \ge \alpha$. We show $f(s,m) \ge \frac{s}{m}\alpha$. Let $M_1, \ldots, M_m$ be the muffins. Let $S_1, \ldots, S_s$ be the students. The protocol that achieves $f(m,s) \ge \alpha$ must be of the following form:

1. For each $1 \le i \le m$ divide M_i into pieces $(a_{i1}, a_{i2}, \ldots, a_{im_i})$ where $\sum_{j=1}^{m_i} a_{ij} = 1$.

2. For each $1 \le j \le s$ give S_j the shares $[b_{1j}, b_{2j}, \ldots, b_{s_j j}]$ where $\sum_{i=1}^{s_j} b_{ij} = \frac{m}{s}$.

The following hold:

- $\bigcup_{i=1}^{m} \bigcup_{j=1}^{m_i} \{a_{ij}\} = \bigcup_{j=1}^{s} \bigcup_{i=1}^{s_j} \{b_{ij}\}$

- The min over all of the a_{ij} is α.

The following protocol shows that $f(s,m) \ge \frac{s}{m}\alpha$. Let $M_1', \ldots, M_s'$ be the muffins. Let $S_1', \ldots, S_m'$ be the students.

1. For each $1 \le j \le s$ divide M_j' into $(\frac{s}{m}b_{1j}, \frac{s}{m}b_{2j}, \ldots, \frac{s}{m}b_{s_j j})$. Note that $\sum_{i=1}^{s_j} \frac{s}{m} b_{ij} = \frac{s}{m}\sum_{i=1}^{s_j} b_{ij} = \frac{s}{m} \times \frac{m}{s} = 1$.

2. For each $1 \le i \le m$ give S'_j $[\frac{s}{m}a_{i1}, \frac{s}{m}a_{ij}, \ldots, \frac{s}{m}a_{im_i}]$. Note that $\sum_{j=1}^{m_i} \frac{s}{m}a_{ij} = \frac{s}{m}\sum_{j=1}^{m_i} a_{ij} = \frac{s}{m} \times 1 = \frac{s}{m}$.

Clearly this is a correct protocol and the minimum piece is of size $\frac{s}{m}\alpha$.

We now show that $f(s,m) = \frac{s}{m}f(m,s)$. By the above we have both (1) $f(s,m) \ge \frac{s}{m}f(m,s)$, and (2) $f(m,s) \ge \frac{m}{s}f(s,m)$. Hence

$$f(s,m) \ge \frac{s}{m}f(m,s) \ge \frac{s}{m}\frac{m}{s}f(s,m) = f(s,m).$$

Therefore $f(s,m) = \frac{s}{m}f(m,s)$. ▮

6 Upper Bounds on $f(m,s)$: The Floor-Ceiling Theorem

We will call our general theorem *the Floor-Ceiling Theorem*. The statement will tell you why.

Theorem 6.1 *Assume that $m, s \in \mathbb{N}$ and $\frac{m}{s} \notin \mathbb{N}$.*

$$f(m,s) \le \max\left\{\frac{1}{3}, \min\left\{\frac{m}{s\lceil 2m/s\rceil}, 1 - \frac{m}{s\lfloor 2m/s\rfloor}\right\}\right\}.$$

Proof:

Assume we have an optimal (m,s) protocol. Since $\frac{m}{s} \notin \mathbb{N}$ we can assume every muffin is cut into at least 2 pieces.

Case 1: Some muffin is cut into $u \ge 3$ pieces.

$$pi_S(u) \le \frac{1}{3}.$$

Case 2: All muffins are cut into 2 pieces. Since there are $2m$ shares and s students both of the following happen:

- Some student gets $t \geq \lceil 2m/s \rceil$ shares.

$$sh_S(t) \leq \frac{m}{s\lceil 2m/s \rceil}.$$

- Some student gets $t \leq \lfloor 2m/s \rfloor$ shares.

$$sh_L(t) \geq \frac{m}{s\lfloor 2m/s \rfloor}$$

$$B(sh_L(t)) \leq 1 - \frac{m}{s\lfloor 2m/s \rfloor}.$$

Putting together Cases 1 and 2 yields the theorem.

▮

The upper bound from Theorem 6.1 only depends on $\frac{m}{s}$. We formalize this thought.

Corollary 6.2 *If $f(m,s) \leq \alpha$ is shown by Theorem 6.1 then, for all A such that $Am, As \in \mathbb{N}$, $f(Am, As) \leq \alpha$.*

We will need a notation for the bound provided by Theorem 6.1

Notation 6.3 Assume that $m, s \in \mathbb{N}$ and $\frac{m}{s} \notin \mathbb{N}$.

$$FC(m,s) = \max\left\{\frac{1}{3}, \min\left\{\frac{m}{s\lceil 2m/s \rceil}, 1 - \frac{m}{s\lfloor 2m/s \rfloor}\right\}\right\}.$$

7 Upper Bounds on $f(m,s)$: The Interval Theorem

Def 7.1 Let $m, s \in \mathbb{N}$ be such that $\frac{m}{s} > 1$. Let $3 \leq V \leq m$. Then

- $g(Q) = \frac{m}{s} - Q(V-1)$. Note that $\frac{(m/s)-g}{V-1} = Q$.

- $h(Q) = \frac{m}{s} - V + 2 + Q(V-2)$. Note that $1 - \frac{(m/s)-h(Q)}{V-2} = Q$.

Henceforth we denote $g(Q)$ by g and $h(Q)$ by h.

Theorem 7.2 *Let $m, s \in \mathbb{N}$ be such that $\frac{m}{s} > 1$, $s \geq 5$, and m, s are relatively prime. Let $V = \left\lceil \frac{2m}{s} \right\rceil$. Let*

$$\begin{aligned} s_{V-1} &= Vs - 2m \\ s_V &= 2m - Vs + s \end{aligned}$$

Note that $V = \left\lceil \frac{2m}{s} \right\rceil$ is the only integer between 3 and m such that $s_V, s_{V-1} \geq 0$. Since $s \geq 3$ and m, s are relatively prime, $s_{V-1}, s_V \geq 1$. Hence we can divide by either one without worrying that they are 0.)

Let Q be such that at least one of the following STATEMENTS holds. Then

$$f(m,s) \leq \max\left\{\frac{1}{3}, \frac{m}{s(V+1)}, 1 - \frac{m}{s(V-2)}, Q\right\}.$$

STATEMENT DK-ONE (some d_k is Less than Zero, Case ONE)

1. $Q \leq 1-h \leq 1-g \leq g \leq h < 1-Q$

2. *There exists k such that:*

   (a) $(\forall i)[0 \leq i \leq k-1 \rightarrow i \times Q + (V-i) \times (1-g) \geq \frac{m}{s}]$

   (b) $(\forall i)[k+1 \leq i \leq V \rightarrow i \times (1-h) + (V-i) \times g \leq \frac{m}{s}]$

   (c) $\frac{(V-1)s_{V-1}}{k} \neq \frac{2m-2(V-1)s_{V-1}}{V-k}$ *or* $\frac{(V-1)s_{V-1}}{k} \neq s_V$.

STATEMENT DK-TWO (some d_k is Less than Zero, Case TWO)

1. $Q \leq g \leq h \leq 1-h \leq 1-g \leq 1-Q$

2. *There exists k such that:*

    (a) $(\forall i)[0 \le i \le k-1 \rightarrow i \times (1-Q) + (V-1-i) \times (1-h) \le \frac{m}{s}]$

    (b) $(\forall i)[k+1 \le i \le V-1 \rightarrow i \times (1-g) + (V-1-i) \times h \ge \frac{m}{s}]$

    (c) $\frac{Vs_V}{k} \ne \frac{2m-2Vs_V}{(V-1-k)}$ *or* $\frac{Vs_V}{k} \ne s_{V-1}$.

STATEMENT DKp-ONE (either d_k or d_{k+1} is less than zero, Case ONE)

1. $Q \le 1-h \le 1-g \le g \le h < 1-Q$

2. *There exists k such that:*

    (a) $(\forall i)[0 \le i \le k-1 \rightarrow i \times Q + (V-i) \times (1-g) \ge \frac{m}{s}]$

    (b) $(\forall i)[k+2 \le i \le V \rightarrow i \times (1-h) + (V-i) \times g \le \frac{m}{s}]$

    (c) $(1+k)s_V - (V-1)s_{V-1} < 0$ *or* $(V-1)s_{V-1} - ks_V < 0$.

STATEMENT DKp-TWO (either d_k or d_{k+1} is less than zero, Case TWO)

1. $Q \le g \le h \le 1-h \le 1-g \le 1-Q$

2. *There exists k such that:*

    (a) $(\forall i)[0 \le i \le k-1 \rightarrow i \times (1-Q) + (V-1-i) \times (1-h) \le \frac{m}{s}]$

    (b) $(\forall i)[k+2 \le i \le V-1 \rightarrow i \times (1-g) + (V-1-i) \times h \ge \frac{m}{s}]$

    (c) $(1+k)s_{V-1} - Vs_V < 0$ *or* $Vs_V - ks_{V-1} < 0$.

STATEMENTS HALF-ONE (uses shares of size $< \frac{1}{2}$ and $> \frac{1}{2}$, Case ONE)

Let

$$
\begin{aligned}
U &= \left\lceil \tfrac{(V-1)s_{V-1}}{s_V} \right\rceil \\
d_{U-1} &= Us_V - (V-1)s_{V-1} \\
d_U &= (V-1)s_{V-1} - (U-1)s_V \\
X_{U-1} &= \left\lfloor \tfrac{m-(V-1)s_{V-1}}{d_{U-1}} \right\rfloor \ \textit{(if } d_{U-1}=0 \textit{ then } X_{U-1}=\infty\textit{)} \\
X_U &= \left\lfloor \tfrac{m-(V-1)s_{V-1}}{d_U} \right\rfloor \ \textit{(if } d_U=0 \textit{ then } X_U=\infty\textit{)}
\end{aligned}
$$

Note that U is the only integer such that $d_U, d_{U-1} \ge 0$.

1. $Q < 1-h \le 1-g \le g \le h < 1-Q$ *(Note that $\frac{1}{2} \le g$.)*
2. $(U+1)(1-h) + (V-U-1)g \le \frac{m}{s}$ *OR* $U \ge s_V$.
3. $(U-2)Q + (V-U+2)(1-g) \ge \frac{m}{s}$ *OR* $U \le 1$.
4. $(V-1)s_{V-1} \le m$.
5. *At least one of the following holds:*
   (a) $X_{U-1}(1-g) + (V-U+1-X_{U-1})\frac{1}{2} + (U-1)Q \ge \frac{m}{s}$ *(If $X_{U-1}=\infty$ then this statement is still meaningful and might be true. The coefficient of X_{U-1} is $(\frac{1}{2}-g)$. If this coefficient is > 0 then $X_{U-1}(1-g)+(V-U+1-X_{U-1})\frac{1}{2}+(U-1)Q \ge \frac{m}{s}$.)*
   (b) $X_U g + (V-U-X_U)\frac{1}{2} + U(1-h) \le \frac{m}{s}$ *(If $X_{U-1}=\infty$ then this statement is still meaningful and might be true, similar to condition a above.)*

STATEMENTS HALF-TWO (uses shares of size $< \frac{1}{2}$ and $> \frac{1}{2}$, Case TWO)

Let

$$
\begin{aligned}
U &= \left\lceil \frac{Vs_V}{s_{V-1}} \right\rceil \\
d_{U-1} &= Us_{V-1} - Vs_V \\
d_U &= Vs_V - (U-1)s_{V-1} \\
X_{U-1} &= \left\lfloor \frac{m - Vs_V}{d_{U-1}} \right\rfloor \text{ (if } d_{U-1} = 0 \text{ then } X_{U-1} = \infty\text{)} \\
X_U &= \left\lfloor \frac{m - Vs_V}{d_U} \right\rfloor \text{ (if } d_U = 0 \text{ then } X_U = \infty\text{)}
\end{aligned}
$$

Note that U is the only integer such that $d_U, d_{U-1} \geq 0$.

1. $Q < g \leq h \leq 1-h \leq 1-g < 1-Q$ *(Note that $h \leq \frac{1}{2}$.)*
2. $(U+1)(1-g) + (V-U-2)h \geq \frac{m}{s}$ *OR* $U \geq s_{V-1}$.
3. $(U-2)(1-Q) + (V-U+1)(1-h) \leq \frac{m}{s}$ *OR* $U \leq 1$.
4. $Vs_V \leq m$.
5. *At least one of the following holds:*
   (a) $X_{U-1}(1-h) + (V-U-X_{U-1})\frac{1}{2} + (U-1)(1-Q) \leq \frac{m}{s}$ *(If $X_{U-1} = \infty$ then this statement is still meaningful and might be true, similar to condition 5a in STATEMENT HALF-ONE.)*
   (b) $X_U h + (V-1-U-X_U)\frac{1}{2} + U(1-g) \geq \frac{m}{s}$ *(If $X_U = \infty$ then this statement is still meaningful and might be true, similar to condition 5a in STATEMENT HALF-ONE.)*

Proof:

Case 1: Some muffin is cut into ≥ 3 pieces. $pi_S(3) \leq \frac{1}{3}$.

Case 2: Some student gets $\geq V+1$ shares. $sh_S(V+1) \leq \frac{m}{s(V+1)}$.

Case 3: Some student gets $\leq V-2$ shares. If $V = 3$ then a student is getting just one share. Since a student gets $\frac{m}{s} > 1$ this cannot occur.

If $V \geq 4$ then $B(sh_L(V-2)) \leq 1 - \frac{m}{s(V-2)}$.

Case 4: Every muffin is cut into 2 pieces. Every student gets either V or $V-1$ shares. Note that the total number of shares is $2m$. Let s_V be the number of V-students s_{V-1} be the number of $(V-1)$-students. From

$$\begin{aligned} s_V V + s_{V-1}(V-1) &= \text{Number of shares} &= 2m \\ s_V + s_{V-1} &= \text{Number of students} &= s \end{aligned}$$

we can deduce

$$\begin{aligned} s_V &= 2m - Vs + s \\ s_{V-1} &= Vs - 2m \end{aligned}$$

Case 4a: There is a V-share $x \leq Q$. This case is self-evident.

Case 4b: There is a V-share $x \geq g$.

$$M_S(x) \leq \frac{(m/s) - g}{V-1} = Q.$$

The equality follows from the definition of g.

Case 4c: There is a $(V-1)$-share $x \leq h$.

$$M_L(x) \geq \frac{(m/s) - h}{V-2}$$

$$B(M_L(x)) \leq 1 - \frac{(m/s) - h}{V-2} = Q$$

The last equality follows from the definition of h.

Case 4d: There is a $(V-1)$-share $x \geq 1 - Q$.

$$B(x) = 1 - (1 - Q) = Q$$

Case 4e: None of cases 4a, 4a, 4b, 4c, or 4d hold. Hence all V-shares are in (Q, g) and all $(V-1)$-shares are in $(h, 1-Q)$. From the premise of the theorem we have $Q \le g \le h \le 1-Q$. The following picture captures what we know so far.

(	Vs_V V-shs	)[	No shs	](	$(V-1)s_{V-1}$ $(V-1)$-shs	)
Q		g		h		$1-Q$

There are no shares $x \in [1-h, 1-g]$ since if there was such a share then $B(x) \in [g, h]$.

For the rest of this proof we assume:

$$Q \le 1-h \le 1-g \le g \le h < 1-Q.$$

This will suffice for STATEMENTS DK-ONE, DKp-ONE, and HALF-ONE. The proofs of STATEMENTS DK-TWO, DKp-TWO, and HALF-TWO are similar. The following picture captures what we know so far with some labels we describe soon.

(	SV-shs	)[	No shs	](	LV-shs	)[	No shs	](	$(V-1)$-shs	)
Q		$1-h$		$1-g$		g		h		$1-Q$

We call the two types of V-shares SV-shares (Small V-shares) and LV-shares (Large V-shares) as denoted in the picture above.

Def 7.3 $\forall i, 0 \le i \le V$, let d_i be the number of students who get i SV-shares and hence $(V-i)$ LV-shares. A d_i-*student* is a student who get d_i SV-shares and hence $(V-i)$ LV-shares. We can use other subscripts like $d_{<i}$ whose meaning is clear. We bound i by V since any V-student has at most V shares of any type.

Claim 1: B is a bijection from SV-Shares to $(V-1)$-Shares. Hence

$$\begin{aligned} |SV| &= |(V-1)\text{-shs}| = (V-1) \times s_{V-1} \\ |LV| &= 2m - |(V-1)\text{-shs}| = 2m - 2V s_{V-1} \end{aligned}$$

Proof of Claim 1: If x is a $(V-1)$-share then $x > g$ so $B(x) < 1-g$, hence $B(x)$ is an SV-share. If y is an SV-share $x < 1-g$, so $B(x) > g$, hence $B(x)$ is a $(V-1)$-share. Therefore B is a bijection of SV-shares to $(V-1)$-shares.

End of Proof of Claim 1

We use Claim 1 throughout the rest of the proof without comment.

In every subcase we have an implicit extra condition of *or the protocol produces a piece of size* $\le Q$. Hence when we show *Case 4e.DK-ONE cannot occur* we are really saying $f(m,s) \le Q$.

Case 4e.DK-ONE:

We show that, for all $0 \le i \le V$ such that $i \ne k$, $d_i = 0$. We consider the cases $0 \le i \le k-1$ and $k+1 \le i \le V$ separately.

Let $0 \le i \le k-1$. Assume Alice has exactly i SV-shares. Then she has

$$> i \times Q + (V-i) \times (1-g) \ge \frac{m}{s}$$

(the last inequality came from the premise). Since she gets $> \frac{m}{s}$ this cannot occur, so $d_i = 0$.

Let $k+1 \le i \le V$. Assume Alice has exactly i SV-shares. Then she has

$$< i \times (1-h) + (V-i) \times g \le \frac{m}{s}$$

(the last inequality came from the premise). Since she gets $< \frac{m}{s}$ this cannot occur, so $d_i = 0$.

We now show that this case cannot occur. The only nonzero d_i is d_k. Hence

$$
\begin{aligned}
kd_k &= 0d_0 + 1d_1 + \cdots + Vd_V = \#\text{ of } SV\text{-shares } = (V-1)s_{V-1} \\
(V-k)d_k &= (V-0)d_0 + (V-1)d_1 + \cdots + (V-V)d_V = \#\text{ of } LV\text{-shares } = 2m - 2(V-1)s_{V-1} \\
d_k &= d_0 + \cdots + d_V = \#\text{ of students} = s_V
\end{aligned}
$$

In order for these equations to have a solution you must have both:

1. $d_k = \frac{(V-1)s_{V-1}}{k}$ and $d_k = \frac{2m-2(V-1)s_{V-1}}{V-k}$, and

2. $d_k = s_V$.

By the premise at least one of these is false. Hence this case cannot occur.

Case 4e.DKp-ONE:

We show that for all $0 \le i \le V$ such that $i \ne k$ and $i \ne k+1$, $d_i = 0$. We consider the cases $0 \le i \le k-1$ and $k+2 \le i \le V$ separately.

Let $0 \le i \le k-1$. Assume Alice has exactly i SV-shares. Then she has

$$> i \times Q + (V-i) \times (1-g) \ge \frac{m}{s}$$

(the last inequality came from the premise). Since she gets $> \frac{m}{s}$ this cannot occur, so $d_i = 0$.

Let $k+2 \le i \le V$. Assume Alice has exactly i SV-shares. Then she has

$$< i \times (1-h) + (V-i) \times g \le \frac{m}{s}$$

(the last inequality came from the premise). Since she gets $< \frac{m}{s}$ this cannot occur, so $d_i = 0$.

The only nonzero d_i are d_k and d_{k+1}. Hence

$$\begin{aligned}
kd_k + (k+1)d_{k+1} &= 0d_0 + 1d_1 + \cdots + Vd_V = \#\text{ of } SV\text{-shares } = (V-1)s_{V-1} \\
(V-k)d_k + (V-k-1)d_{k+1} &= (V-0)d_0 + (V-1)d_1 + \cdots + (V-V)d_V = \#\text{ of } LV\text{-shares} \\
&= 2m - 2(V-1)s_{V-1} \\
d_k + d_{k+1} &= d_0 + \cdots + d_V = \#\text{ of students} = s_V
\end{aligned}$$

In order for these equations to have a solution you must have both:

1. $d_k = (1+k)s_V - (V-1)s_{V-1}$
2. $d_{k+1} = (V-1)s_{V-1} - ks_V$

By the premise at least one of these less than 0. Hence this case cannot occur.

Case 4e.HALF-ONE:

Notation 7.4 Let $0 \le \delta \le 1$. An LV-share of size $\le \delta$ is called a $(LV :\le \delta)$*-share*. If $\le$ is replaced by $\ge$ or $<$ or $>$ then we obtain other notation whose meaning is obvious.

Claim 2:

1. $\frac{1}{2}$ is in the midpoint of the LV-shares.
2. B is a bijection between $(1-f, \frac{1}{2})$ and $(\frac{1}{2}, f)$.
3. The number of $(LV :< \frac{1}{2})$-shares is $\le m - (V-1)s_{V-1}$.
4. The number of $(LV :> \frac{1}{2})$-shares is $\le m - (V-1)s_{V-1}$
5. Every V-student is either a d_U-student or a d_{U-1}-student.
6. The number of d_{U-1}-students is $d_{U-1} = Us_V - (V-1)s_{V-1}$.
7. The number of d_U-students is $d_U = (V-1)s_{V-1} - (U-1)s_V$.

8. Case 4e.HALF-ONE cannot occur.

Proof of Claim 2:

1) The midpoint of the LV-interval is $\frac{(1-f)+f}{2} = \frac{1}{2}$.

2) This is obvious.

3,4) There are $2m - 2(V-1)s_{V-1}$ LV-shares. By Part 2 $|(LV :< \frac{1}{2})| = |(LV :> \frac{1}{2})|$. Hence both are $\le m - (V-1)s_{V-1}$ (we do not have equality since there may be shares of size $\frac{1}{2}$).

5)

By premises 2 of STATEMENT HALF-ONE either a d_{U+1} student gets $< \frac{m}{s}$ (hence all d_i students with $U+1 \le i \le s_V$ get $< \frac{m}{s}$) or $U \ge s_V$ so there cannot be any d_{U+1} students.

By premises 3 of STATEMENT HALF-ONE either a d_{U-2} student gets $> \frac{m}{s}$ (hence all d_i students with $0 \le i \ge U+1$ get $> \frac{m}{s}$) or $U \le 1$ so there cannot be any d_{U-2} students.

6,7) We derive the values for d_U and d_{U-1}.

Let d_U (d_{U-1}) be the number of $(V:U)$-students (the number of $(V:U-1)$-students).

$$\begin{aligned} Ud_U + (U-1)d_{U-1} &= \text{Number of small } V\text{-shares} = (V-1)s_{V-1} \\ d_U + d_{U-1} &= \text{Number of } V\text{-students} = s_V \end{aligned}$$

These equations yield $d_{U-1} = Us_V - (V-1)s_{V-1}$ and $d_U = (V-1)s_{V-1} - (U-1)s_V$.

8) Premise 5 of STATEMENT HALF-ONE is an OR of 5a and 5b. We show that both 5a and 5b are false.

By Part 4 the number of $(LV :> \frac{1}{2})$-shares is $\le m - (V-1)s_{V-1}$. One of the d_U d_U-students gets $\le \left\lfloor \frac{m-(V-1)s_{V-1}}{d_U} \right\rfloor = X_U$ $(LV :> \frac{1}{2}$-shares). Hence she gets $\ge V - U - X_U$ $(LV :\le \frac{1}{2})$-shares. Therefore she gets less than

$$X_U g + \left(V - U - X_U\right)\frac{1}{2} + U(1-h)$$

(She gets *less than*, not *less than or equal to*: $U \geq 1$, so she gets at least one small $(V-1)$-shares of size $< 1-h$.)

If premise 5b held then the students then this quantity is $\leq \frac{m}{s}$ hence the student gets $< \frac{m}{s}$. Therefore premise 5b is false.

By Part 3 the number of $(LV :< \frac{1}{2})$-shares is $\leq m-(V-1)s_{V-1}$. By a similar argument one shows that Premise 5a is false.

End of Proof of Claim 2

∎

The proof of the upper bound from Theorem 7.2 only depends on $\frac{m}{s}$. We formalize this thought.

Corollary 7.5 *If $f(m,s) \leq \alpha$ is shown by Theorem 7.2 then, for all A such that $Am, As \in \mathbb{N}$, $f(Am, As) \leq \alpha$.*

8 A Muffin Theorem Generator for $V = 3$

We prove several general theorem that have, as corollaries, upper bounds on $f(m,s)$ when $\left\lceil \frac{2m}{s} \right\rceil = 3$. We then apply them to $f(s+d, s)$ for $1 \leq d \leq 8$. Our results are of two types:

1. Theorems you can apply immediately without any additional effort.

2. A methodology that allows you to, with a computer program, obtain results. (We have also looked at special cases of this methodology to obtain theorems you can apply immediately without any additional effort.)

3. We have written such a program. It has helped us obtain theorems. All of the theorems in this section were mostly generated by a program; though we had to simplify some of the results.

We then use our Theorems to obtain formulas for $f(3dk+a+d, 3dk+a)$ for a variety of a's and d's.

9 Notation

Notation 9.1

1. An open interval means the shares in that interval (not including endpoints). For example, the expression $(\frac{37}{108}, \frac{45}{108})$ means the set of all shares of sizes strictly between $\frac{37}{108}$ and $\frac{45}{108}$.

2. A closed interval means the shares in that interval (including endpoints). For example, the expression $[\frac{37}{108}, \frac{45}{108}]$ means the set of all shares of sizes between $\frac{37}{108}$ and $\frac{45}{108}$ including the endpoints. We only use this when there are no such shares.

3. Recall that the buddy function B is a bijection. We define it on intervals (as understood by Parts 1 and 2). Note that $B(a,b) = (1-b, 1-a)$.

4. Recall that the match function M is a bijection when restricted to 2-shares. We define it on intervals (as understood by Parts 1 and 2). For example, if (a,b) only has 2-shares then $M(a,b) = (\frac{m}{s} - b, \frac{m}{s} - a)$.

5. An interval (a,b) is *symmetric* if B, M, or some composition of B and M is a bijection from $(a, \frac{a+b}{2})$ to $(\frac{a+b}{2}, b)$. Note that

$$\left|\left(a, \frac{a+b}{2}\right)\right| = \left|\left(\frac{a+b}{2}, b\right)\right| \le \frac{|(a,b)|}{2}.$$

   This is not an equality since there could be shares of size $\frac{a+b}{2}$. Note that if (a,b) is symmetric then $B(a,b)$ and $M(a,b)$ is symmetric. Also note that if $\frac{1}{2}$ $(\frac{m}{2s})$ is the center of an interval then it is symmetric via B (via M).

Since B and M are bijections, for all i, (1) $|M_i| = |B_i| = 3s_3$, and (2) there are no shares of sizes the endpoints of M_i or B_i.

10 Example of the Buddy-Match Method

In this section we give an example of finding an upper bound using the Buddy-Match Method.

Theorem 10.1 $f(27, 25) \le \frac{17}{50}$.

Proof:

By the usual arguments the number of shares is $2 \times 27 = 54$, $s_2 = 21$, $s_3 = 4$, and:

$$\begin{array}{ccccccc} (& \text{12 3-shs} &)[& \text{No shs} &](& \text{42 2-shs} &) \\ \frac{17}{50} & & \frac{20}{50} & & \frac{21}{50} & & \frac{33}{50} \end{array}$$

For $0 \le i \le 3$ we define M_i and B_i.

$$\begin{aligned} M_0 = &\quad (\tfrac{17}{50}, \tfrac{21}{50}) \\ B_0 = &\quad B(M_0) = (\tfrac{29}{50}, \tfrac{33}{50}) \\ M_1 = &\quad M(B_0) = (\tfrac{21}{50}, \tfrac{25}{50}) \\ B_1 = &\quad B(M_1) = (\tfrac{25}{50}, \tfrac{29}{50}) \\ M_2 = &\quad M(B_1) = (\tfrac{25}{50}, \tfrac{29}{50}) \\ B_2 = &\quad B(M_2) = (\tfrac{21}{50}, \tfrac{25}{50}) \\ M_3 = &\quad M(B_2) = (\tfrac{29}{50}, \tfrac{33}{50}) \\ B_3 = &\quad B(M_3) = (\tfrac{17}{50}, \tfrac{21}{50}) \end{aligned}$$

$$M_0 \cup M_1 \cup M_2 \cup M_3 = \left(\frac{17}{50}, \frac{33}{50}\right)$$

$$|M_0 \cup M_1 \cup M_2 \cup M_3| = \left|\left(\frac{17}{50}, \frac{33}{50}\right)\right|$$

$$48 = 54$$

which is a contradiction. ∎

11 Example of the Buddy-Match Method for Upper Bounds and Linear Algebra for Lower Bounds

We prove $f(43, 39) \le \frac{53}{156}$. We then describe our general method to, given an upper bound on $f(m, s)$, find a procedure that shows the matching lower bound. We use $f(43, 39)$ as our example.

Theorem 11.1 $f(43, 39) \le \frac{53}{156}$.

Proof:

We first show $f(43, 39) \le \frac{53}{156}$. We later use the information from this proof to run a program that finds a procedure showing $f(43, 39) \ge \frac{53}{156}$.

Assume there is an $(43, 39)$-procedure. We show that there is a piece $\le \frac{53}{156}$.

Case 1: A student gets ≥ 4 shares so some share is $\le \frac{43}{39\times 4} < \frac{53}{156}$.

Case 2: A student student gets ≤ 1 shares. This cannot occur since he then has $1 < \frac{43}{39}$.

Case 3: Every muffin is cut in 2 pieces and every student gets either 2 or 3 shares. The total number of shares is 86. Let s_2 (s_3) be the number of 2-students (3-students). From (1) $2s_2 + 3s_3 = 86$ and (2) $s_2 + s_3 = 39$ we derive $s_2 = 31$ and $s_3 = 8$.

One can show that all of the 3-shares are in $(\frac{43}{156}, \frac{66}{156})$ and all of the 2-shares are in $(\frac{69}{156}, \frac{103}{156})$ (or there is a piece of size $\le \frac{53}{156}$).

The following picture captures what we know so far.

(	$---$	)[	$---$	](	$---$	)
$\frac{53}{156}$	3-shs	$\frac{66}{156}$	No shs	$\frac{69}{156}$	2-shs	$\frac{103}{156}$

The interval $(\frac{53}{156}, \frac{69}{156})$ contains all of the 3-shares and no other shares. Since there are 8 3-students, $(\frac{53}{156}, \frac{69}{156})$ has $8 \times 3 = 24$ shares, so

$24 = |(\frac{53}{156}, \frac{53}{156})| = |B(\frac{53}{156}, \frac{69}{156})| = |M(B(\frac{87}{156}, \frac{103}{156}))|$. Since $B(\frac{53}{156}, \frac{69}{156}) = (\frac{87}{156}, \frac{103}{156})$ and $M(\frac{87}{156}, \frac{103}{156}) = (\frac{69}{156}, \frac{85}{156})$ we have

$|(\frac{53}{156}, \frac{69}{156}) \cup (\frac{87}{156}, \frac{103}{156}) \cup (\frac{69}{156}, \frac{85}{156})| = 24 \times 3 = 72$, so $|(\frac{85}{156}, \frac{87}{156})| = 86 - 72 = 14$. Note that $(\frac{85}{156}, \frac{87}{156})$ is symmetric since $M(\frac{85}{156}, \frac{86}{156}) = (\frac{86}{156}, \frac{87}{156})$ (this will not be used in this proof but symmetry is used in other proofs). A sequence of B's and M's applied to $(\frac{85}{156}, \frac{87}{156})$ ends up at $(\frac{53}{156}, \frac{55}{156})$. Hence this interval has 14 shares and is symmetric. Since there are 24 3-shares, the remaining 10 3-shares are in the other interval.

Recall that $[\frac{66}{156}, \frac{69}{156}]$ is empty. A sequence of B's and M's ends up at $[\frac{55}{156}, \frac{58}{156}]$. Hence this interval is empty.

The following picture captures what we know so far about 3-shares.

$$\begin{array}{ccccccccc} (& 7 & | & 7 &)[& 0 &](& 10 &) \\ \frac{53}{156} & & \frac{54}{156} & & \frac{55}{156} & & \frac{58}{156} & & \frac{66}{156} \end{array}$$

(We assume there are no shares of size exactly $\frac{54}{156}$.)

We call the shares in $(\frac{53}{156}, \frac{55}{156})$ *small shares* and the shares in $(\frac{58}{156}, \frac{66}{156})$ *large shares*.

Notation 11.2 For $0 \le i \le 3$, d_i is the number of students who have i small shares (and hence $3 - i$ large shares). Such a student is called a d_i-*student*.

- $d_0 = 0$ since $3 \times \frac{58}{156} = \frac{174}{156} > \frac{172}{156} = \frac{43}{39}$.

- d_1: If a d_1-student has either of its large shares $\ge \frac{61}{156}$ then he will have more than $\frac{53}{156} + \frac{58}{156} + \frac{61}{156} = \frac{172}{156} = \frac{43}{39}$. Hence all the large shares of a d_1-student are in $(\frac{58}{156}, \frac{61}{156})$.

- d_2: If a d_2-student has either of its large shares $\le \frac{62}{156}$ then he will have less than $\frac{55}{156} + \frac{55}{156} + \frac{62}{156} = \frac{172}{156} = \frac{43}{39}$. Hence all the large shares of a d_2-student are in $(\frac{62}{156}, \frac{66}{156})$.

- $d_3 = 0$ since $3 \times \frac{55}{156} = \frac{165}{156} < \frac{172}{156} = \frac{43}{39}$.

Using the restrictions on d_1 and d_2 we have that $[\frac{61}{156}, \frac{62}{156}]$ is empty. A sequence of B's and M's takes $[\frac{61}{156}, \frac{62}{156}]$ to $[\frac{62}{156}, \frac{63}{156}]$, hence that interval is also empty. In addition, a sequence of B's and M's takes $(\frac{58}{156}, \frac{66}{156})$ to $(\frac{74}{156}, \frac{82}{156})$ which is symmetric. Hence $(\frac{58}{156}, \frac{66}{156})$ is symmetric.

The following picture captures what we know so far about 3-shares.

$$(\quad 7 \quad | \quad 7 \quad)[\quad 0 \quad](\quad 5 \quad)[\quad 0 \quad](\quad 5 \quad)$$

$$\frac{53}{156} \qquad \frac{54}{156} \qquad \frac{55}{156} \qquad \frac{58}{156} \qquad \frac{61}{156} \qquad \frac{63}{156} \qquad \frac{66}{156}$$

We now write down equations in the d_i's. Only the d_2-students use $(\frac{63}{156}, \frac{66}{156})$. Every d_2 student uses one share from that interval. Since there are 5 shares in it, $d_2 = 5$.

Each d_i student uses i shares from $(\frac{53}{156}, \frac{55}{156})$. Hence $1 \times d_1 + 2 \times d_2 = 14$. Since $d_2 = 5$ we obtain $d_1 = 4$. There are 8 3-students, hence we need that $d_1 + d_2 = 8$. But $d_1 + d_2 = 9$. This is a contradiction. ∎

We now describe the program that finds the procedure showing $f(43, 39) \geq \frac{53}{156}$. We *guess* that all shares are of the form $\frac{x}{156}$ where $53 \leq x \leq 103$. But we can cut down those variables *a lot* based on the proof. For example, we know there is no share of size $\frac{67}{156}$ or $\frac{68}{156}$. This is a key factor in speeding up the program.

For every way to split a muffin we have a variable for mow many muffins are split that way, as follows: $(\frac{53}{156}, \frac{103}{156})$ is associated to the variables $y_{53,103}$, $(\frac{54}{156}, \frac{102}{156})$ is associated with the variable $y_{54,102}$, etc. For every way to give muffin shares to a student we have a variable for how many students get that set of shares, as follows: $[\frac{54}{156}, \frac{59}{156}, \frac{59}{156}]$ is associated to the variable $z_{54,59,59}$, $[\frac{85}{156}, \frac{87}{156}]$ is associated to the variables $z_{85,87}$, etc.

For each share-size we have two expressions: (1) the number of pieces of that size based on the muffins, e.g., $\sum_{i=53, i \neq 60}^{103} y_{60,i}$ is the number of pieces of size 60 (if the piece is size $\frac{1}{2}$ then add a term: $\sum_{i=53, i \neq 78}^{103} y_{78,i} + 2y_{78,78}$), and (2) the number of shares of that size based on the students, e.g., the number of pieces of size 55 is

$$z_{53,55,64} + z_{53,55,64} + z_{54,55,63} + 2z_{55,55,62} + z_{55,56,61} + z_{55,57,60} + z_{55,58,59}.$$

For each share size we get an equation by equating the muffin-centered and student-centered expression. We also have equations from having 86 pieces and 39 students.

This leads to a set of linear equations whose solution leads to a procedure. The KEY is that the program runs fast, since using the proof that $f(43,39) \le \frac{53}{156}$ eliminates many share sizes.

The procedure we found that shows $f(43,39) \ge \frac{53}{156}$:

All of the numbers below have denominator 156. Here (1) the notation (a,b), when $a+156$, means that a muffin is split so that one piece is $\frac{a}{156}$ and the other is $\frac{b}{156}$, and (2) the notation $[a,b,c]$, where $a+b+c=172$, means that a student gets a shares of sizes $\frac{a}{172}$, $\frac{b}{172}$, and $\frac{c}{172}$.

1. Divide 1 muffin via (78, 78)
2. Divide 4 muffins via (81, 75)
3. Divide 4 muffins via (85, 71)
4. Divide 6 muffins via (86, 70)
5. Divide 4 muffins via (87, 69)
6. Divide 4 muffins via (91, 65)
7. Divide 2 muffins via (94, 62)
8. Divide 4 muffins via (97, 59)
9. Divide 4 muffins via (101, 55)

10. Divide 6 muffins via (102, 54)

11. Divide 4 muffins via (103, 53)

12. Give 2 students [59, 59, 54]

13. Give 2 students [62, 55, 55]

14. Give 4 students [65, 54, 53]

15. Give 3 students [86, 86]

16. Give 4 students [87, 85]

17. Give 4 students [91, 81]

18. Give 2 students [94, 78]

19. Give 4 students [97, 75]

20. Give 4 students [101, 71]

21. Give 6 students [102, 70]

22. Give 4 students [103, 69]

12 General Theorems

Theorem 12.1 *Let $d \geq 1$, $k \geq 1$, and $a \in \{1, \ldots, 2d-1\}$. Let $V = \left\lceil \frac{2(3dk+a+d)}{3dk+a} \right\rceil$. Assume a, d are relatively prime and $V = 3$. Then $f(3dk+a+d, 3dk+a) \leq \frac{dk+X}{3dk+a}$ where*

$$X = \min\left\{\frac{a}{2}, \frac{a+d}{4}\right\}.$$

Proof: Assume $f(3dk+a+d, 3dk+a) \le \frac{dk+X}{3dk+a}$. By the usual arguments $s_2 = 3dk+a-2d$, $s_3 = 2d$, and the following picture captures what we know so far

(	$3s_3$ 3-shs	)[	No Shs	](	$2s_2$ 2-shs	)
$\frac{dk+X}{3dk+a}$		$\frac{dk+a+d-2X}{3dk+a}$		$\frac{dk+d+X}{3dk+a}$		$\frac{2dk+a-X}{3dk+a}$

We define the following

$$\begin{aligned}
M_0 = \quad & (\tfrac{dk+X}{3dk+a}, \tfrac{dk+X+d}{3dk+a}) \\
B_0 = \quad & B(M_0) = (\tfrac{2dk+a-X-d}{3dk+a}, \tfrac{2dk+a-X}{3dk+a}) \\
(\forall 0 \le i \le k-1)[M_i = \quad & M(B_{i-1}) = (\tfrac{dk+X+id}{3dk+a}, \tfrac{dk+X+(i+1)d}{3dk+a})] \\
(\forall 0 \le i \le k-1)[B_i = \quad & B(M_i) = (\tfrac{2dk+a-X-(i+1)d}{3dk+a}, \tfrac{2dk+a-X+id}{3dk+a})]
\end{aligned}$$

We leave it to the reader to check that B_{k-2} (and hence $B_0, \ldots, B_{k-3}$) contain only 2-shares and hence M can be applied to it.

1) we show that if $X \ge \frac{a}{2}$ then we get a contradiction.

Using $X \ge \frac{a}{2}$ one can show that

$$B_0 \cup \cdots \cup B_{k-1} = \left(\frac{dk+X}{3dk+a}, \frac{2dk+a-X}{3dk+a}\right)$$

$$\left|B_0 \cup \cdots \cup B_{k-1}\right| = \left|\left(\frac{dk+X}{3dk+a}, \frac{2dk+a-X}{3dk+a}\right)\right|$$

$$6dk = 6dk + 2d + 2a$$

$$a + d = 0$$

which is a contradiction.

2) We show that if $X \ge \frac{a+d}{4}$ then we get a contradiction. We use a slightly different buddy-match sequence.

$$\begin{aligned}
M_0 = &\ (\tfrac{dk+X}{3dk+a}, \tfrac{dk+a-2X+d}{3dk+a}) \\
B_0 = &\ B(M_0) = (\tfrac{2dk-a-2X-d}{3dk+a}, \tfrac{2dk+a-X}{3dk+a}) \\
(\forall 0 \le i \le k-1)[M_i = &\ M(B_{i-1}) = (\tfrac{dk+X+id}{3dk+a}, \tfrac{dk+a-2X+(i+1)d}{3dk+a})] \\
(\forall 0 \le i \le k-1)[B_i = &\ B(M_i) = (\tfrac{2dk-a-2X-(i+1)d}{3dk+a}, \tfrac{2dk+a-X-id}{3dk+a})]
\end{aligned}$$

Using $X \ge \frac{a+d}{4}$ one can show that

$$B_0 \cup \cdots \cup B_{k-1} \subseteq \left(\frac{dk+d+X}{3dk+a}, \frac{2dk+a-X}{3dk+a} \right)$$

$$\left| B_0 \cup \cdots \cup B_{k-1} \right| \le 2s_2$$

$$6dk \le 6dk + 2a - 4d$$

$$2d \le a$$

which is a contradiction.

∎

Theorem 12.2 *Let $d \ge 1$, $k \ge 1$, and $a \in \{2d+1, \ldots, 3d-1\}$. Let $V = \left\lceil \frac{2(3dk+a+d)}{3dk+a} \right\rceil$. Assume a, d are relatively prime and $V = 3$. Then $f(3dk+a+d, 3dk+a) = \frac{1}{3}$.*

Proof: Assume $f(3dk+a+d, 3dk+a) \le \frac{dk+X}{3dk+a}$ where $X = \frac{a}{3}$, hence Then $f(3dk+a+d, 3dk+a) \le \frac{1}{3}$.

We use the buddy-match sequence from Theorem 12.1.1 (the $\frac{a}{2}$ part).

One can show that if $X \geq \frac{a}{3}$ then because $a \geq 2d+1$ we have a contradiction. We will need to show that the left endpoint of $B_{k-1} \leq \frac{dk+d+X}{3dk+a}$.

Observe:

$$2dk + a - X - kd \leq dk + d + X$$

$$a - X \leq d + X$$

$$X \geq \frac{a-d}{2}$$

Note that $d \geq \frac{a}{3}$, so if $X \geq \frac{a}{3}$ we have

$$X \geq \frac{a}{3} = \frac{a - \frac{a}{3}}{2} \geq \frac{a-d}{2}.$$

$$B_0 \cup \cdots \cup B_{k-1} \supseteq \left(\frac{dk+d+X}{3dk+a}, \frac{2dk+a-X}{3dk+a} \right)$$

$$\left| B_0 \cup \cdots \cup B_{k-1} \right| \geq 2s_2$$

$$a \leq 2d$$

This is a contradiction. ∎

13 A Theorem Generator

In this section we prove upper bounds of the form $f(3dk+a+d, 3dk+a) \geq \frac{dk+X}{3dk+a}$, where we try to make X as small as possible. We always have $k \geq 1$ so the procedures uses only 2-shares and 3-shares.

Def 13.1 If a system of linear equations has no solution in $\{0, \ldots, 2d\}$ then we call it $2d$-*unsolvable*.

Our contradiction is going to be that a system of equations is $2d$-unsolvable.

We revisit the Buddy-Match Methodology but use it in a more complicated way

Recall:

$$
\begin{aligned}
M_0 = & \quad (\tfrac{dk+X}{3dk+a}, \tfrac{dk+X+d}{3dk+a}) \\
B_0 = & \quad B(M_0) = (\tfrac{2dk+a-X-d}{3dk+a}, \tfrac{2dk+a-X}{3dk+a}) \\
(\forall 0 \le i \le k-1)[M_i = & \quad M(B_{i-1}) = (\tfrac{dk+X+id}{3dk+a}, \tfrac{dk+X+(i+1)d}{3dk+a})] \\
(\forall 0 \le i \le k-1)[B_i = & \quad B(M_i) = (\tfrac{2dk+a-X-(i+1)d}{3dk+a}, \tfrac{2dk+a-X+id}{3dk+a})]
\end{aligned}
$$

We will assume k is even (the case of k odd is similar). Let $L = \frac{k}{2} - 1$.

$B_0 \cup \cdots \cup B_L = (\frac{2dk+a-X-\frac{dk}{2}}{3dk+a}, \frac{2dk+a-X}{3dk+a}) = (\frac{\frac{3dk}{2}+a-X}{3dk+a}, \frac{2dk+a-X}{3dk+a})$

$M_0 \cup \cdots \cup M_L = (\frac{dk+X}{3dk+a}, \frac{dk+X+\frac{dk}{2}}{3dk+a}) = (\frac{dk+X}{3dk+a}, \frac{\frac{3dk}{2}+X}{3dk+a})$

$(\frac{dk+X}{3dk+a}, \frac{2dk+a-X}{3dk+a}) = B_0 \cup \cdots \cup B_L \cup M_0 \cup \cdots \cup M_L \cup (\frac{\frac{3dk+X}{2}+X}{3dk+a}, \frac{\frac{3dk+x}{2}+a-X}{3dk+a})$

By taking the cardinality of both sides and rearranging we obtain

$|(\frac{\frac{3dk+X}{2}+X}{3dk+a}, \frac{\frac{3dk+x}{2}+a-X}{3dk+a})| = 6dk + 2a + 2d - (2L+2)6d = 6dk + 2a + 2d - 6dk = 2a + 2d.$

Note also that the interval $(\frac{\frac{3dk+X}{2}+X}{3dk+a}, \frac{\frac{3dk+x}{2}+a-X}{3dk+a})$ is symmetric.

Recall that both Buddy and Match are bijections. By a sequence of Buddy-Match moves starting at $(\frac{\frac{3dk+X}{2}+X}{3dk+a}, \frac{\frac{3dk+x}{2}+a-X}{3dk+a})$ we obtain that the interval $(\frac{dk+X}{3dk+a}, \frac{dk+a-X}{3dk+a})$ has $2a + 2d$ shares in it and is symmetric. By another sequence of Buddy-Match moves starting at the empty closed interval $[\frac{dk+a+d-2X}{3dk+a}, \frac{dk+d+X}{3dk+a}]$ we obtain that $[\frac{dk+a-X}{3dk+a}, \frac{dk+2X}{3dk+a}]$ is an empty closed interval.

NEED TO PUT IN SOMETHING ABOUT THE REST BEING SYMM.

$$
\begin{aligned}
M_0 = {} & (\tfrac{dk+2X}{3dk+a}, \tfrac{dk-2X+a+d}{3dk+a}) \\
B_0 = {} & B(M_0) = (\tfrac{2dk+2X-d}{3dk+a}, \tfrac{2dk-2X+a}{3dk+a}) \\
M_1 = {} & M(B_0) = (\tfrac{dk+2X+d}{3dk+a}, \tfrac{dk-2X+a+2d}{3dk+a}) \\
B_1 = {} & B(M_1) = (\tfrac{2dk+2X-2d}{3dk+a}, \tfrac{2dk-2X+a-d}{3dk+a}) \\
(\forall 0 \le i \le k-1)[M_i = {} & M(B_{i-1}) = (\tfrac{dk+2X+id}{3dk+a}, \tfrac{dk-2X+a+(i+1)d}{3dk+a})] \\
(\forall 0 \le i \le k-1)[B_i = {} & B(M_i) = (\tfrac{2dk+2X-(i+1)d}{3dk+a}, \tfrac{2dk-2X-id}{3dk+a})]
\end{aligned}
$$

$$
\begin{aligned}
B_0 = {} & (\tfrac{dk+2X}{3dk+a}, \tfrac{dk-2X+a+d}{3dk+a}) \\
M_0 = {} & M(B_0) = (\tfrac{2dk+2X}{3dk+a}, \tfrac{2dk-2X+a+d}{3dk+a}) \\
B_1 = {} & B(B_0) = (\tfrac{dk+2X+d}{3dk+a}, \tfrac{dk-2X+a+2d}{3dk+a}) \\
M_1 = {} & M(M_1) = (\tfrac{2dk+2X-2d}{3dk+a}, \tfrac{2dk-2X+a-d}{3dk+a}) \\
(\forall 0 \le i \le k-1)[B_i = {} & M(B_{i-1}) = (\tfrac{dk+2X+id}{3dk+a}, \tfrac{dk-2X+a+(i+1)d}{3dk+a})] \\
(\forall 0 \le i \le k-1)[M_i = {} & B(M_i) = (\tfrac{2dk+2X-(i+1)d}{3dk+a}, \tfrac{2dk-2X-id}{3dk+a})]
\end{aligned}
$$

The following picture captures what we know so far about the 3-shares:

$$
\begin{array}{ccccccccc}
(& a+d & \mid & a+d &)[& 0 &] \\
\frac{dk+X}{3dk+a} & & \frac{dk+\frac{a}{2}}{3dk+a} & & \frac{dk+a-X}{3dk+a} & & \frac{dk+2X}{3dk+a}
\end{array}
$$

$$
\begin{array}{ccccc}
(& 2d-a & \mid & 2d-a &) \\
\frac{dk+2X}{3dk+a} & & \frac{dk+\frac{a+d}{2}}{3dk+a} & & \frac{dk+a+d-2X}{3dk+a}
\end{array}
$$

We constrain X so that the above picture is correct. Hence we impose the following conditions throughout this section:

1. $X \le a - X$ or $X \le \frac{a}{2}$.

2. $a - X \le 2X$ or $X \ge \frac{a}{3}$.

3. $2X \le a + d - 2X$ or $X \le \frac{a+d}{4}$.

4. To summarize
$$\frac{a}{3} \le X \le \min\left\{\frac{a}{2}, \frac{a+d}{4}\right\}.$$

Note 13.2

1. The constraint on X also yields $\frac{a}{3} \le \frac{a+d}{4}$ which implies $a \le 3d$. This is already a constraint and hence does not impose any additional conditions.

2. The upper bound on X is not a hindrance for applications since we already have, for $a \in \{1, \ldots, 2d\}$, $X = \min\left\{\frac{a}{2}, \frac{a+d}{4}\right\}$, by Theorem 12.1.

3. It would be great to just take $X = \frac{a}{3}$. However, in later cases we will need to take a large X to get to a contradiction.

Notation 13.3 Since all shares and students we talk about are 3-shares and 3-students we use *shares* for *3-shares* and *students* for *3-students*. The shares in the first (second) nonempty interval are *small shares* (*large shares*). For $0 \le i \le 3$, d_i is the number of students that have i small shares and $3 - i$ large shares.

The following claim allows us to put conditions on X to obtain some of the d_i's are 0.

Claim 1: If $a \in \{1, \ldots, 3d-1\}$ then the following hold.

1. If $X \ge \max\{\frac{a}{3}, \frac{a+d}{6}\}$ then $d_0 = 0$.

2. If $X \ge \max\{\frac{a}{3}, \frac{a+d}{5}\}$ then $d_1 = 0$.

3. There is no condition on X that will force d_2 to 0.

4. If $X \ge \max\{\frac{a}{3}, \frac{2a-d}{3}\}$ then $d_3 = 0$.

Proof of Claim 1:

Since in all cases we assume $X \ge \frac{a}{3}$ we can assume the picture and intervals are correct.

1) If $X \ge \frac{a+d}{6}$ then a d_0-student has more than $\frac{3(dk+2X)}{3dk+a} = \frac{3dk+6X}{3dk+a} \ge \frac{3dk+a+d}{3dk+a}$.

2) If $X \ge \frac{a+d}{5}$ then a d_1-student has more than $\frac{dk+X}{3dk+a} + \frac{2(dk+2X)}{3dk+a} = \frac{3dk+5X}{3dk+3X} \ge \frac{3dk+a+d}{3dk+a}$.

3) We leave this for the reader.

4) If $X \ge \frac{2a-d}{3}$ then a d_3-student has less than $\frac{3(dk+a-X)}{3dk+a} = \frac{3dk+3a-3X}{3dk+a} \le \frac{3dk+a+d}{3dk+a}$.

End of Proof of Claim 1

Even with just Claim 1 we can prove a result

Theorem 13.4 *If $a \in \{1, \ldots, 3d-1\}$, $a \ne d$, then $f(3dk+a+d, 3dk+a) \le \frac{dk+X}{3dk+a}$ where*

$$X = \max\left\{\frac{a}{3}, \frac{a+d}{5}, \frac{2a-d}{3}\right\}.$$

Proof:

Since $X \ge \frac{a}{3}$ we can assume the picture and intervals are correct.

Since $X \ge \frac{a+d}{6}$, by Claim 1, $d_0 = 0$.

Since $X \ge \frac{a+d}{5}$, by Claim 1, $d_1 = 0$.

Since $X \ge \frac{2a-d}{3}$, by Claim 1 $d_3 = 0$.

By the definition of the d_i's and of small shares (see Notation 13.3).

$$2d_2 = \text{ The number of small shares } = 2a + 2d.$$

Since the only 3-students are d_2-students we have

$$d_2 = \text{ the number of 3-students} = 2d.$$

Hence $a = d$ which is a contradiction. ∎

13.1 A Theorem Generator that is Useful When $a \in \{d+1, \ldots, 2d-1\}$

Recall that the following picture captures what we know:

$$
\begin{array}{ccccccccc}
(& a+d & | & a+d &)[& 0 &] \\
\frac{dk+X}{3dk+a} & & \frac{dk+\frac{a}{2}}{3dk+a} & & \frac{dk+a-X}{3dk+a} & & \frac{dk+2X}{3dk+a}
\end{array}
$$

$$
\begin{array}{ccccc}
(& 2d-a & | & 2d-a &) \\
\frac{dk+2X}{3dk+a} & & \frac{dk+\frac{a+d}{2}}{3dk+a} & & \frac{dk+a+d-2X}{3dk+a}
\end{array}
$$

Def 13.5 We need names for the intervals.

1. $I_1 = (\frac{dk+X}{3dk+a}, \frac{dk+\frac{a}{2}}{3dk+a})$
2. $I_2 = (\frac{dk+\frac{a}{2}}{3dk+a}, \frac{dk+a-X}{3dk+a})$
3. $I_3 = (\frac{dk+2X}{3dk+a}, \frac{dk+\frac{a+d}{2}}{3dk+a})$
4. $I_4 = (\frac{dk+\frac{a+d}{2}}{3dk+a}, \frac{dk+a+d-2X}{3dk+a})$

We will set up a large system of equations that we hope is $2d$-unsolvable as this will get us a contradiction.

1. (We restate the definition of d_i in terms of the intervals.) For $0 \le i \le 3$, d_i is the number of 3-students who have i 3-shares from intervals 1 or 2 and $3-i$ from intervals 3 or 4.
2. For $0 \le j_1 \le j_2 \le j_3 \le 4$ $e(j_1, j_2, j_3)$ is the number of students who have a I_{j_1}-share, a I_{j_2}-share, and an I_{j_3}-share. The j's could be equal.

Equations relating d_i, a, and d:

$$0 \times d_0 + 1 \times d_1 + 2 \times d_2 + 3 \times d_3 = |I_1 \cup I_2| = 2a + 2d$$

$$d_0 + d_1 + d_2 + d_3 = \text{the number of 3-students} = s_3 = 2d$$

Equations relating d_i to $e(j_1, j_2, j_3)$:

$$\begin{aligned}
d_0 &= \textstyle\sum_{3 \le j_1 \le j_2 \le j_3 \le 4} & e(j_1, j_2, j_3) \\
d_1 &= \textstyle\sum_{j_1=1}^{2} \sum_{3 \le j_2 \le j_3} & e(j_1, j_2, j_3) \\
d_2 &= \textstyle\sum_{1 \le j_1 \le j_2 \le 2} \sum_{j_3=3}^{4} & e(j_1, j_2, j_3) \\
d_3 &= \textstyle\sum_{1 \le j_1 \le j_2 \le j_3 \le 2} & e(j_1, j_2, j_3)
\end{aligned}$$

We use the above four equations to get rid of the variables d_0, d_1, d_2, d_3. For our program we do not need the d_i's; however, for some theorems done by hand we use them as a convenience.

Equations relating I_j to $e(j_1, j_2, j_3)$: Looking at the number of shares in I_1, I_2, I_3, I_4 we obtain the following equations

Let $ONE(j_1, j_2, j_3) = L$ be the set of all $1 \le j_1 \le j_2 \le j_3 \le 4$ where exactly one of the j's is L. Similar for $TWO(j_1, j_2, j_3) = L$ and $THREE(j_1, j_2, j_3) = L$.

$$\sum_{ONE(j_1,j_2,j_3)=1} e(j_1, j_2, j_3) + \sum_{TWO(j_1,j_2,j_3)=1} 2e(j_1, j_2, j_3) + \sum_{THREE(j_1,j_2,j_3)=1)} 3e(j_1, j_2, j_3) = |I_1| = a+d$$

$$\sum_{ONE(j_1,j_2,j_3)=2} e(j_1, j_2, j_3) + \sum_{TWO(j_1,j_2,j_3)=2} 2e(j_1, j_2, j_3) + \sum_{THREE(j_1,j_2,j_3)=2)} 3e(j_1, j_2, j_3) = |I_2| = a+d$$

$$\sum_{ONE(j_1,j_2,j_3)=3} e(j_1,j_2,j_3)+\sum_{TWO(j_1,j_2,j_3)=3} 2e(j_1,j_2,j_3)+\sum_{THREE(j_1,j_2,j_3)=3)} 3e(j_1,j_2,j_3) = |I_3| = 2d-a$$

$$\sum_{ONE(j_1,j_2,j_3)=4} e(j_1,j_2,j_3)+\sum_{TWO(j_1,j_2,j_3)=4} 2e(j_1,j_2,j_3)+\sum_{THREE(j_1,j_2,j_3)=4)} 3e(j_1,j_2,j_3) = |I_3| = 2d-a$$

We have 6 equations in 20 variables. We hope that this system of equations is $2d$-unsolvable; however, this is unlikely.

There are ways to constrain X so that some of the variables will be 0. For example, lets find the X that will make $e(1,2,2)=0$. If a student has one share in I_1, two shares in I_2 then she has $> \frac{dk+2X+\frac{a}{2}}{3dk+a}$. If $X \ge \frac{a+2d}{4}$ then $\frac{dk+2X+\frac{a}{2}}{3dk+a} \ge \frac{dk+a+d}{3dk+a}$, so $e(1,2,2)=0$. Alternatively: if a student has one share in I_1, two shares in I_2 then she has $< \frac{dk+\frac{a}{2}+2(a-X)}{3dk+a}$. If $X \ge \frac{3a-2d}{4}$ then $\frac{dk+\frac{a}{2}+2(a-X)}{3dk+a} \le \frac{dk+a+d}{3dk+a}$., so $e(1,2,2)=0$.

This leads to an algorithm: we increase X until the resulting system of equations is $2d$-unsolvable. Formally, here is the algorithm.

FIND-UPPER-BOUND

1. Input(a,d) (We seek a small X such that $f(3dk+a+d, 3dk+a) \le \frac{dk+X}{3dk+a}$.)

2. Set up the system of equations in $e(i_1,i_2,i_3)$.

3. If the system is $2d$-unsolvable then call procedure FIND-PROC below with input (a,d,X) where $X=\frac{a}{3}$.

4. For every $1 \le i_1 \le i_2 \le i_3 \le 6$ find the bound X_{i_1,i_2,i_3} such that if $X \ge X_{i_1,i_2,i_3}$ then $e(i_1,i_2,i_3)=0$ (it is possible that $X_{i_1,i_2,i_3}=\infty$).

5. Sort the X_{i_1,i_2,i_3}. Remove all X that are $\ge \min\{\frac{a}{2}, \frac{a+d}{4}\}$. Then add $\min\{\frac{a}{2}, \frac{a+d}{4}\}$ to the end of the list.

6. Starting from the smallest to the largest X_{i_1,i_2,i_3} do the following: (1) set the variable $e(i_1, i_2, i_3)$ to 0 (it will be 0 in all later stages). (2) form the new system of equations L. If L is $2d$-solvable then goto the next value of X_{i_1,i_2,i_3} if there is one (if not then goto the next step). If L is $2d$-unsolvable then call procedure FIND-PROC below with input (a, d, X) where $X = X_{i_1,i_2,i_3}$.

We can often look at the results and see what it was about a, d that gave the bound. With this technique we obtained the following theorem.

Theorem 13.6 *If $1 \le a \le 3d-1$ and $5a \ne 7d$ then $f(3dk+a+d, 3dk+a) \le \frac{dk+X}{3dk+a}$ where*

$$X = \max\left\{\frac{a}{3}, \frac{a+d}{5}, \frac{a+2d}{6}, \frac{3a-2d}{4}\right\}.$$

Proof:

Since $X \ge \frac{a}{3}$ we can assume the picture and intervals are correct.

Since $X \ge \frac{a+d}{6}$, by Claim 1, $d_0 = 0$.

Since $X \ge \frac{a+d}{5}$, by Claim 1, $d_1 = 0$.

Claim:

1. A d_2-student has no shares in I_2, or even the left endpoint of I_2.

2. A d_3-student has no shares in I_1, or even the right endpoint of I_1.

3. $5a = 7d$.

Proof of Claim:

1) If a d_2 student has an I_2-share (or the left endpoint) then he has more than

$\frac{dk+X}{3dk+a} + \frac{dk+\frac{a}{2}}{3dk+a} + \frac{dk+2X}{3dk+a} = \frac{dk+3X+\frac{a}{2}}{3dk+a} \ge \frac{dk+a+d}{3dk+a}$. This last inequality comes from $X \ge \frac{a+2d}{6}$.

2) If a d_3 student has an I_1-share (or the right endpoint) then he has less than

$\frac{dk+\frac{a}{2}}{3dk+a} + 2 \times \frac{dk+a-X}{3dk+a} = \frac{dk+\frac{5a}{2}-2X}{3dk+a} \le \frac{dk+a+d}{3dk+a}$. This last inequality comes from $X \ge \frac{3a-2d}{4}$.

3) By Parts 1 and 2 the only shares in I_1 are put there by d_2. Every d_2-student has 2 shares in I_1. Hence $a + d = |I_1| = 2d_2$.

4) By parts 1 and 2 the only shares in $I_3 \cup I_4$ are put there by d_2. Thus $4d - 2a = d_2$.

5) By parts 3 and 4 we have $8d - 4a = a + d$, so $5a = 7d$.

End of Proof of Claim

Since $5a \ne 7d$ we have a contradiction. ▮

13.2 Two Theorem Generators that are Useful when $a \in \{1, \ldots, d\}$

Recall that the following picture captures what we know:

$$\begin{array}{ccccccc} (& a+d & | & a+d &)[& 0 &] \\ \frac{dk+X}{3dk+a} & & \frac{dk+\frac{a}{2}}{3dk+a} & & \frac{dk+a-X}{3dk+a} & & \frac{dk+2X}{3dk+a} \end{array}$$

$$\begin{array}{ccccc} (& 2d-a & | & 2d-a &) \\ \frac{dk+2X}{3dk+a} & & \frac{dk+\frac{a+d}{2}}{3dk+a} & & \frac{dk+a+d-2X}{3dk+a} \end{array}$$

The following claim restricts where a d_i student can get its shares.

Claim 2:

1. A d_0-students' large shares are in $(\frac{dk+2X}{3dk+a}, \frac{dk+a+d-4X}{3dk+a})$. (Since $a+d-4X < a+d-2X$ this is always a restriction on d_0's shares.)

2. A d_1-students' large shares are in $(\frac{dk+2X}{3dk+a}, \frac{dk+a+d-3X}{3dk+a})$. (Since $a+d-3X < a+d-2X$ this is always a restriction on d_1's shares.)

3. A d_1-students' one small share is in $(\frac{dk+X}{3dk+a}, \frac{dk+a+d-4X}{3dk+a})$. (This is restriction on d_1 small shares only when $a+d-4X < a-X$ so when $X > \frac{d}{3}$.)

4. A d_2-students' one large share is in $(\frac{dk+d-a+2X}{3dk+a}, \frac{dk+a+d-2X}{3dk+a})$ (This is restriction on d_2's shares only when $2X < d-a+2X$ so when $d \geq a+1$.)

5. A d_2-students' small shares are in $(\frac{dk+X}{3dk+a}, \frac{dk+a+d-3X}{3dk+a})$ (This is restriction on d_2's shares only when $a+d-3X < a-X$ so when $d < 2X$.)

6. A d_3-students' small shares are in $(\frac{dk+d-a+2X}{3dk+a}, \frac{dk+a-X}{3dk+a})$ (This is a restriction on d_3's shares only when $X < d-a+2X$ so when $d-a+X > 0$.) (The left endpoint is the same as Part c. If this endpoint is in the size of small share then $d_2 = 0$. If it is the size of a large share then $d_3 = 0$.)

Proof of Claim 2:

1) If a d_0-student has a large share $\geq \frac{dk+a+d-4X}{3dk+a}$ then he has more than

$$2 \times \frac{dk+2X}{3dk+a} + \frac{dk+a+d-4X}{3dk+a} = \frac{3dk+a+d}{3dk+a}.$$

2) If a d_1-student has a large share $\geq \frac{dk+a+d-3X}{3dk+a}$ then he has more than

$$\frac{dk+X}{3dk+a} + \frac{dk+2X}{3dk+a} + \frac{dk+a+d-3X}{3dk+a} = \frac{3dk+a+d}{3dk+a}.$$

3) If a d_1-student has a small share $\geq \frac{dk+a+d-4X}{3dk+a}$ then he has more than

$$\frac{dk+a+d-4X}{3dk+a} + 2 \times \frac{dk+X}{3dk+a} = \frac{3dk+a+d}{3dk+a}.$$

d) If a d_2-student has a large share $\leq \frac{dk+d-a+2X}{3dk+a}$ then he has as most

$$2 \times \frac{dk+a-X}{3dk+a} + \frac{dk+d-a+2X}{3dk+a} = \frac{3dk+a+d}{3dk+a}.$$

e) If a d_2 students has a small share $\geq \frac{dk+a+d-3X}{3dk+a}$ then he has at least

$$\frac{dk+X}{3dk+a} + \frac{dk+a+d-3X}{3dk+a} + \frac{dk+2X}{3dk+a} = \frac{a+d}{3dk+a}.$$

f) If a d_3-student has a small share $\leq \frac{dk+d-a-2X}{3dk+a}$ then he has at most

$2 \times \frac{dk+a-X}{3dk+a} + \frac{dk+d-a-2X}{3dk+a} = \frac{dk+a+d}{3dk+a}$.

End of Proof of Claim 2

Def 13.7 The set of intervals within $(\frac{dk+2X}{3dk+a}, \frac{dk+a-2X}{3dk+a})$ that we know have no shares are called *No-Mans-Land*.

By Claim 2 there are no shares in

$$\left[\frac{dk+a+d-3X}{3dk+a}, \frac{dk+d-a+2X}{3dk+a}\right].$$

To make sure this interval is nonempty we have $X \geq \frac{2a}{5}$ as an assumption.

The following picture captures what we know about the 3-shares.

(	$2a+2d$	)[	0	](	y	)[	0	](	z	)
$\frac{dk+X}{3dk+a}$		$\frac{dk+a-X}{3dk+a}$		$\frac{dk+2X}{3dk+a}$		$\frac{dk+a+d-3X}{3dk+a}$		$\frac{dk+d-a+2X}{3dk+a}$		$\frac{dk+a+d-2X}{3dk+a}$

The intervals $(\frac{dk+X}{3dk+a}, \frac{dk+a-X}{3dk+a})$ and $(\frac{dk+2X}{3dk+a}, \frac{dk+a-2X}{3dk+a})$ are symmetric; however, if may be that $y \neq z$.

We will iterate B and M on $[\frac{dk+a-3X}{3dk+a}, \frac{dk+d-a+2X}{3dk+a}]$ to find another empty closed interval. In particular it will create a larger No-Mans-Land and give us a symmetry we can use.

$$\begin{aligned}
M_0 = \quad & [\tfrac{dk+a+d-3X}{3dk+a}, \tfrac{dk+d-a+2X}{3dk+a}] \\
B_0 = B(M_0) = \quad & [\tfrac{2dk+2a-d-2X}{3dk+a}, \tfrac{2dk-d+3X}{3dk+a}] \\
M_1 = M(B_0) = \quad & [\tfrac{dk+a+2d-3X}{3dk+a}, \tfrac{dk+a+2d+2X}{3dk+a}] \\
B_1 = B(M_1) = \quad & [\tfrac{2dk+2a-2d-2X}{3dk+a}, \tfrac{2dk-2d+3X}{3dk+a}] \\
M_L = M(B_{L-1}) = \quad & [\tfrac{dk+a+(L+1)d-3X}{3dk+a}, \tfrac{dk+a+(L+1)d+2X}{3dk+a}] \\
B_L = B(M_L) = \quad & [\tfrac{2dk+2a-(L+1)d-2X}{3dk+a}, \tfrac{2dk-(L+1)d+3X}{3dk+a}]
\end{aligned}$$

Note that

$$B_{k-1} = \left[\frac{2dk + 2a - dk - 2X}{3dk + a}, \frac{2dk - dk + 3X}{3dk + a}\right]$$

$$= \left[\frac{dk + 2a - 2X}{3dk + a}, \frac{dk + 3X}{3dk + a}\right]$$

We rewrite both the interval diagram and B_{k-1} on the next page.

$$\begin{array}{ccccccccccc}
(& 2a+2d &)[& 0 &](& y &)[& 0 &](& z &) \\
\frac{dk+X}{3dk+a} & & \frac{dk+a-X}{3dk+a} & & \frac{dk+2X}{3dk+a} & & \frac{dk+a+d-3X}{3dk+a} & & \frac{dk+d-a+2X}{3dk+a} & & \frac{dk+a+d-2X}{3dk+a}
\end{array}$$

$$B_{k-1} = \left[\frac{dk+2a-2X}{3dk+a}, \frac{dk+3X}{3dk+a}\right]$$

We will pick X as small as possible to cause a contradiction. There will be two cases leading to two algorithms:

- B_{k-1} is contained in the y-region.

- B_{k-1} is contained in the z-region.

Its fine if the left (right) endpoint of B_{k-1} is the left (right) end point of the y-region or the z-region.

13.2.1 B_{k-1} is contained in the y-region

In order for B_{k-1} to be in the y-region we need the following conditions:

$$\frac{2a}{5} \leq X \leq \min\left\{\frac{a}{2}, \frac{a+d}{6}\right\}$$

A consequence of these constraints on X is a constraint on a, d, namely $a \leq \frac{5d}{7}$.

The following picture captures what we know about the 3-shares. We split it into two parts:

$$
\begin{array}{ccccc}
(& 2a+2d &)[& 0 &] \\
\frac{dk+X}{3dk+a} & & \frac{dk+a-X}{3dk+a} & & \frac{dk+2X}{3dk+a}
\end{array}
$$

$$
\begin{array}{ccccccccccc}
(& y_1 &)[& 0 &](& y_2 &)[& 0 &](& z &) \\
\frac{dk+2X}{3dk+a} & & \frac{dk+2a-2X}{3dk+a} & & \frac{dk+3X}{3dk+a} & & \frac{dk+a+d-3X}{3dk+a} & & \frac{dk+d-a+2X}{3dk+a} & & \frac{dk+a+d-2X}{3dk+a}
\end{array}
$$

We know a lot about the number of shares in each interval:

- The $(\frac{dk+X}{3dk+a}, \frac{dk+a-X}{3dk+a})$ interval is symmetric. Hence we will split it into two parts of $a+d$ shares each (assuming for now that there are no shares of size exactly $\frac{dk+\frac{a}{2}}{3dk+a}$).

- The interval $(\frac{dk+2X}{3dk+a}, \frac{dk+a+d-2X}{3dk+a})$ is symmetric, hence $y_1 = z$. We will replace y_1 with z in later diagrams.

- The interval $(\frac{dk+3X}{3dk+a}, \frac{dk+a+d-3X}{3dk+a})$ is symmetric. Hence we split it into two equal parts with

$$\frac{3s_3 - (2a+2d) - 2z}{2} = \frac{6d - 2a - 2d - 2z}{2} = \frac{4d - 2a - 2z}{2} = 2d - a - z$$

shares each (assuming for now that there are no shares of size exactly $\frac{dk+\frac{a+d}{2}}{3dk+a}$).

The following picture captures what we know:

$$
\begin{array}{ccccccccc}
(& a+d & | & a+d &)[& 0 &] \\
\frac{dk+X}{3dk+a} & & \frac{dk+\frac{a}{2}}{3dk+a} & & \frac{dk+a-X}{3dk+a} & & \frac{dk+2X}{3dk+a}
\end{array}
$$

$$
\begin{array}{ccccccccccc}
(& z &)[& 0 &](& 2d-a-z & | & 2d-a-z &) \\
\frac{dk+2X}{3dk+a} & & \frac{dk+2a-2X}{3dk+a} & & \frac{dk+3X}{3dk+a} & & \frac{dk+\frac{a+d}{2}}{3dk+a} & & \frac{dk+a+d-3X}{3dk+a}
\end{array}
$$

$$
\begin{array}{ccccc}
[& 0 &](& z &) \\
\frac{dk+a+d-3X}{3dk+a} & & \frac{dk+d-a+2X}{3dk+a} & & \frac{dk+a+d-2X}{3dk+a}
\end{array}
$$

Def 13.8 We need names for the intervals.

1. $I_1 = (\frac{dk+X}{3dk+a}, \frac{dk+\frac{a}{2}}{3dk+a})$

2. $I_2 = (\frac{dk+\frac{a}{2}}{3dk+a}, \frac{dk+a-X}{3dk+a})$

3. $I_3 = (\frac{dk+2X}{3dk+a}, \frac{dk+2a-2X}{3dk+a})$

4. $I_4 = (\frac{dk+3X}{3dk+a}, \frac{dk+\frac{a+d}{2}}{3dk+a})$

5. $I_5 = (\frac{dk+\frac{a+d}{2}}{3dk+a}, \frac{dk+a+d-3X}{3dk+a})$

6. $I_6 = (\frac{dk+d-a+2X}{3dk+a}, \frac{dk+a+d-2X}{3dk+a})$

We will set up a large system of equations that we hope is $2d$-unsolvable as this will get us a contradiction.

1. (We restate the definition of d_i in terms of the intervals.) For $0 \le i \le 3$, d_i is the number of 3-students who have i 3-shares from intervals 1 or 2 and $3-i$ from intervals 3 or 4 or 5 or 6.

2. For $0 \le j_1 \le j_2 \le j_3 \le 6$ $e(j_1, j_2, j_3)$ is the number of students who have an I_{j_1}-share, an I_{j_2}-share, and an I_{j_3}-share. The j's could be equal.

We will set up a large system of equations that we hope is not s_3-solvable as this will get us a contradiction.

1. (We restate the definition of d_i in terms of the intervals.) For $0 \le i \le 3$, d_i is the number of 3-students who have i 3-shares from intervals 1 or 2 and $3-i$ from intervals 3 or 4 or 5 or 6.

2. For $0 \le j_1 \le j_2 \le j_3 \le 6$ $e(j_1, j_2, j_3)$ is the number of students who have an I_{j_1}-share, an I_{j_2}-share, and an I_{j_3}-share. The j's could be equal.

Equations relating d_i, a, and d: These are identical to those in Section 13.1

Equations relating d_i to $e(j_1, j_2, j_3)$: These are identical to those in Section 13.1

We use the above equations relating d_i to $e(j_1, j_2, j_3)$ to get rid of the variables d_0, d_1, d_2, d_3. For our program we do not need the d_i's; however, for some theorems done by hand we use them as a convenience.

Equations relating I_j to $e(j_1, j_2, j_3)$: Looking at the number of shares in $I_1, I_2, I_3, I_4, I_5, I_6$, $|I_3| = |I_6|$, and $|I_4| = |I_5|$ we obtain the following equations:

Let $ONE(j_1, j_2, j_3) = L$ be the set of all $1 \le j_1 \le j_2 \le j_3 \le 6$ where exactly one of the j's is L. Similar for $TWO(j_1, j_2, j_3) = L$ and $THREE(j_1, j_2, j_3) = L$.

$$\sum_{ONE(j_1,j_2,j_3)=1} e(j_1, j_2, j_3) + \sum_{TWO(j_1,j_2,j_3)=1} 2e(j_1, j_2, j_3) + \sum_{THREE(j_1,j_2,j_3)=1)} 3e(j_1, j_2, j_3) = |I_1| = a+d$$

$$\sum_{ONE(j_1,j_2,j_3)=2} e(j_1, j_2, j_3) + \sum_{TWO(j_1,j_2,j_3)=2} 2e(j_1, j_2, j_3) + \sum_{THREE(j_1,j_2,j_3)=2)} 3e(j_1, j_2, j_3) = |I_2| = a+d$$

$$\sum_{ONE(j_1,j_2,j_3)=3} e(j_1,j_2,j_3) + \sum_{TWO(j_1,j_2,j_3)=3} 2e(j_1,j_2,j_3) + \sum_{THREE(j_1,j_2,j_3)=3)} 3e(j_1,j_2,j_3) = \sum_{ONE(j_1,j_2,j_3)=6} e(j_1,j_2,j$$

$$\sum_{ONE(j_1,j_2,j_3)=4} e(j_1,j_2,j_3) + \sum_{TWO(j_1,j_2,j_3)=4} 2e(j_1,j_2,j_3) + \sum_{THREE(j_1,j_2,j_3)=4)} 3e(j_1,j_2,j_3) = \sum_{ONE(j_1,j_2,j_3)=5} e(j_1,j_2,j$$

We have 6 equations in 56 variables. We hope that this system of equations is $2d$-unsolvable; however, this is unlikely. We proceed as we did in Section 13.1.

After getting some results for a, d we obtained the following general theorem.

Theorem 13.9 *If $1 \le a \le \frac{5d}{7}$ and $a \ne \frac{2d}{3}$ then $f(3dk+a+d, 3dk+a) \le \frac{dk+X}{3dk+a}$ where*

$$X = \max\left\{\frac{2a}{5}, \frac{a+d}{6}\right\}.$$

Proof: When $X = \frac{a+d}{6}$ some of the intervals in our usual diagram collapse leading to this picture:

$$\begin{array}{ccccccccc}
(& & a+d & & | & & a+d & &)[& & 0 & &] \\
\frac{dk+X}{3dk+a} & & & & \frac{dk+\frac{a}{2}}{3dk+a} & & & & \frac{dk+a-X}{3dk+a} & & & & \frac{dk+2X}{3dk+a}
\end{array}$$

$$\begin{array}{ccccccccc}
(& & d_2 & &)[& & 0 & &](& & d_2 & &) \\
\frac{dk+2X}{3dk+a} & & & & \frac{dk+a+d-3X}{3dk+a} & & & & \frac{dk+d-a+2X}{3dk+a} & & & & \frac{dk+a+d-2X}{3dk+a}
\end{array}$$

Since the total number of shares is $6d$ we have $2a+2d+2d_2 = 6d$, so $d_2 = 2d-a$.

Since $\frac{a+d}{6} \ge \frac{a+d}{6}$, we have $d_0 = 0$.

Since $d \ge a$ we have $\frac{5a-d}{6} \ge a+d$, so $d_3 = 0$.

Hence we have the equations

$$\begin{aligned} d_1 + 2d_2 &= 2a + 2d \\ d_1 + d_2 &= 2d \\ d_2 = &\ 2d - a \end{aligned}$$

The first two equations imply $d_2 = 2a$. Combined with the last equation we obtain $a = \frac{2d}{3}$, contrary to hypothesis. ▮

Theorem 13.10 *If* $1 \le a \le \frac{5d}{7}$, $a \ne \frac{d}{2}$, *then* $f(3dk + a + d, 3dk + a) \le \frac{dk+X}{3dk+a}$ *where* $X \ge \max\{\frac{2a-d}{3}, \frac{7a-2d}{8}, \frac{2a}{5}, \frac{a+d}{7}, \frac{4a-d}{5}, \frac{a+2d}{10}\}$.

By the usual methods we need and have $\frac{2a}{5} \le X \le \min\{\frac{3a-d}{4}, \frac{a+d}{5}\}$

1. $I_1 = (\frac{dk+X}{3dk+a}, \frac{dk+\frac{a}{2}}{3dk+a})$
2. $I_2 = (\frac{dk+\frac{a}{2}}{3dk+a}, \frac{dk+a-X}{3dk+a})$
3. $I_3 = (\frac{dk+2X}{3dk+a}, \frac{dk+2a-2X}{3dk+a})$
4. $I_4 = (\frac{dk+3X}{3dk+a}, \frac{dk+\frac{a+d}{2}}{3dk+a})$
5. $I_5 = (\frac{dk+\frac{a+d}{2}}{3dk+a}, \frac{dk+a+d-3X}{3dk+a})$
6. $I_6 = (\frac{dk+d-a+2X}{3dk+a}, \frac{dk+a+d-2X}{3dk+a})$

Claim:

1. $d_3 = 0$: $3 \times \frac{dk+a-X}{3dk+a} \le \frac{dk+a+d}{3dk+a}$ since $X \ge \frac{2a-d}{3}$.
2. $e(1,3,3) = 0$: Student gets $< \frac{dk+\frac{a}{2}}{3dk+a} + 2 \times \frac{dk+2a-2X}{3dk+a} \le \frac{dk+a+d}{3dk+a}$ since $X \ge \frac{7a-2d}{8}$.
3. $e(1,3,6) = 0$: Student gets $> \frac{dk+X}{3dk+a} + \frac{dk+2X}{3dk+a} + \frac{dk+d-a+2X}{3dk+a} \ge \frac{dk+a+d}{3dk+a}$ since $X \ge \frac{2a}{5}$.
4. $e(1,4,4) = 0$: Student gets $> \frac{dk+X}{3dk+a} + 2 \times \frac{dk+3X}{3dk+a} \ge \frac{dk+a+d}{3dk+a}$ since $X \ge \frac{a+d}{7}$.

5. $e(2,2,5)=0$: Student gets $< 2\times \frac{dk+a-X}{3dk+a} + \times \frac{dk+a+d-3X}{3dk+a} \le \frac{dk+a+d}{3dk+a}$ since $X \ge \frac{2a}{5}$

6. $e(2,3,3)=0$: Student gets $< \frac{dk+a-X}{3dk+a} + 2\times \frac{dk+2a-2X}{3dk+a} \le \frac{dk+a+d}{3dk+a}$ since $X \ge \frac{4a-d}{5}$.

7. $e(2,3,4)=0$: Student gets $> \frac{dk+\frac{a}{2}}{3dk+a} + \frac{dk+2X}{3dk+a} + \frac{dk+3X}{3dk+a} \ge \frac{dk+a+d}{3dk+a}$ since $X \ge \frac{a+2d}{10}$.

8. $e(3,3,4)=0$: Student gets $> 2\times \frac{dk+2X}{3dk+a} + \frac{dk+3X}{3dk+a} \ge \frac{dk+a+d}{3dk+a}$ since $X \ge \frac{a+d}{7}$

9. The only $e(i,j,k)$ that might be nonzero are $e(1,1,6)$, $e(1,2,6)$, $e(1,3,4)$, $e(1,3,5)$ $e(2,2,6)$, $e(3,3,3)$, (this follows from the above parts).

10. The only d_0-students are $e(3,3,3)$-students. The only d_1-students are $e(1,3,4)$-students and $e(1,3,5)$-students. The only d_2-students are $e(1,1,6)$, $e(1,2,6)$, and $e(2,2,6)$. (this follows from the last part).

11. $|I_3| = d_1 + 3d_0$: The only students who use I_3-shares are the d_0-students, each one of which uses 3 I_3-shares, and the I_1-students, each one of which uses 1 I_3-share.

12. $|I_6| = d_2$: The only students who uses I_6-shares are the d_2-students, each one of which uses 1 I_6-share.

13. $3d_0 + d_1 - d_2 = 0$: This is obtained from $|I_3| = |I_6|$.

End of Claim

13.2.2 B_{k-1} is Contained in the z-region

In order for B_{k-1} to be in the z-region we need the following conditions:

$$\frac{2a}{5} \le X \le \min\left\{\frac{3a-d}{4}, \frac{a+d}{5}\right\}$$

A consequence of these constraints on X is a constraint on a,d, namely: $\frac{5d}{7} \le a \le d$.

By reasoning similar to that in the last section we have the following picture.

The following picture captures what we know:

$$
\begin{array}{ccccccccc}
(& a+d & | & a+d &)[& 0 &] \\
\frac{dk+X}{3dk+a} & & \frac{dk+\frac{a}{2}}{3dk+a} & & \frac{dk+a-X}{3dk+a} & & \frac{dk+2X}{3dk+a}
\end{array}
$$

$$
\begin{array}{ccccc}
(& y &)[& 0 &] \\
\frac{dk+2X}{3dk+a} & & \frac{dk+a+d-3X}{3dk+a} & & \frac{dk+d-a+2X}{3dk+a}
\end{array}
$$

$$
\begin{array}{ccccccccc}
(& 2d-a-y & | & 2d-a-y &)[& 0 &](& y &) \\
\frac{dk+d-a+2X}{3dk+a} & & \frac{dk+\frac{a+d}{2}}{3dk+a} & & \frac{dk+2a-2X}{3dk+a} & & \frac{dk+3X}{3dk+a} & & \frac{dk+a+d-2X}{3dk+a}
\end{array}
$$

Def 13.11 We need names for the intervals.

1. $I_1 = (\frac{dk+X}{3dk+a}, \frac{dk+\frac{a}{2}}{3dk+a})$
2. $I_2 = (\frac{dk+\frac{a}{2}}{3dk+a}, \frac{dk+a-X}{3dk+a})$
3. $I_3 = (\frac{dk+2X}{3dk+a}, \frac{dk+a+d-3X}{3dk+a})$
4. $I_4 = (\frac{dk+d-a+2X}{3dk+a}, \frac{dk+\frac{a+d}{2}}{3dk+a})$
5. $I_5 = (\frac{dk+\frac{a+d}{2}}{3dk+a}, \frac{dk+2a-2X}{3dk+a})$
6. $I_6 = (\frac{dk+3X}{3dk+a}, \frac{dk+a+d-2X}{3dk+a})$

We will set up a large system of equations that we hope is $2d$-unsolvable as this will get us a contradiction.

1. (We restate the definition of d_i in terms of the intervals.) For $0 \le i \le 3$, d_i is the number of 3-students who have i 3-shares from intervals 1 or 2 and $3-i$ from intervals 3 or 4 or 5 or 6.

2. For $0 \le j_1 \le j_2 \le j_3 \le 6$ $e(j_1, j_2, j_3)$ is the number of students who have an I_{j_1}-share, an I_{j_2}-share, and an I_{j_3}-share. The j's could be equal.

We then set up equations in a manner similar to what we did in Section 13.1 and have a similar algorithm.

After getting some results for a, d we obtained the following general theorem.

Theorem 13.12 *If* $\frac{5d}{7} \le a \le d-1$ *then* $f(3dk+a+d, 3dk+a) \le \frac{dk+X}{3dk+a}$ *where*

$$X = \max\left\{\frac{2a}{5}, \frac{3a-d}{4}\right\}.$$

Proof: When $X = \frac{3a-d}{4}$ some of the intervals in our usual diagram collapse leading to this picture:

$$\begin{array}{ccccccccc} (& a+d & \mid & a+d &)[& 0 &] \\ \frac{dk+X}{3dk+a} & & \frac{dk+\frac{a}{2}}{3dk+a} & & \frac{dk+a-X}{3dk+a} & & \frac{dk+2X}{3dk+a} \end{array}$$

$$\begin{array}{ccccccc} (& d_2 &)[& 0 &](& d_2 &) \\ \frac{dk+2X}{3dk+a} & & \frac{dk+a+d-3X}{3dk+a} & & \frac{dk+3X}{3dk+a} & & \frac{dk+a+d-2X}{3dk+a} \end{array}$$

Since the total number of shares is $6d$ we have $2a + 2d + 2d_2 = 6d$, so $d_2 = 2d - a$.

Since $\frac{3a-d}{4} \ge \frac{a+d}{6}$, we have $d_0 = 0$.

Since $\frac{3a-d}{4} \ge \frac{2a-d}{3}$, we have $d_3 = 0$.

Hence we have the equations

$$\begin{aligned} d_1 + 2d_2 &= 2a + 2d \\ d_1 + d_2 &= 2d \\ d_2 &= 2d - a \end{aligned}$$

The first two equations imply $d_2 = 2a$. Combined with the last equation we obtain $a = \frac{2d}{3}$. This is impossible since $a \geq \frac{5d}{7}$. ∎

Theorem 13.13 *If* $\frac{5d}{7} \leq a \leq d$, $a \neq \frac{9d}{11}$, *then* $f(3dk + a + d, 3dk + a) \leq \frac{dk+X}{3dk+a}$ *where* $X = \max\{\frac{a+d}{6}, \frac{2a-d}{3}, \frac{5a-2d}{6}, \frac{a+2d}{8}, \frac{2a}{5}\}$.

Proof:

By the usual methods we need (and have) $\frac{2a}{5} \leq X \leq \min\{\frac{a}{2}, \frac{a+d}{6}\}$, and the intervals.

1. $I_1 = (\frac{dk+X}{3dk+a}, \frac{dk+\frac{a}{2}}{3dk+a})$
2. $I_2 = (\frac{dk+\frac{a}{2}}{3dk+a}, \frac{dk+a-X}{3dk+a})$
3. $I_3 = (\frac{dk+2X}{3dk+a}, \frac{dk+a+d-3X}{3dk+a})$
4. $I_4 = (\frac{dk+d-a+2X}{3dk+a}, \frac{dk+\frac{a+d}{2}}{3dk+a})$
5. $I_5 = (\frac{dk+\frac{a+d}{2}}{3dk+a}, \frac{dk+2a-2X}{3dk+a})$
6. $I_6 = (\frac{dk+d+3X}{3dk+a}, \frac{dk+a+d-2X}{3dk+a})$

Claim:

1. $d_0 = 0$: $3 \times \frac{dk+2X}{3dk+a} \geq \frac{dk+a+d}{3dk+a}$ since $X \geq \frac{a+d}{6}$.
2. $d_3 = 0$: $3 \times \frac{dk+a-X}{3dk+a} \leq \frac{dk+a+d}{3dk+a}$ since $X \geq \frac{2a-d}{3}$.
3. $d_1 = 2d - 2a$, $d_2 = 2a$. This comes from the usual equations
4. $e(1, 2, 5) = 0$: Student gets $< \frac{dk+\frac{a}{2}}{3dk+a} + \frac{dk+a-X}{3dk+a} + \frac{dk+2a-2X}{3dk+a} \leq \frac{dk+a+d}{3dk+a}$ since $X \geq \frac{5a-2d}{6}$.
5. $e(1, 2, 6) = 0$: Student gets $> \frac{dk+X}{3dk+a} + \frac{dk+\frac{a}{2}}{3dk+a} + \frac{dk+3X}{3dk+a} \geq \frac{dk+a+d}{3dk+a}$ since $X \geq \frac{a+2d}{8}$.
6. $e(1, 3, 4) = 0$: Student gets $>$

7. $e(2,2,3) = 0$: Student gets $< 2 \times \frac{dk+a-X}{3dk+a} + \frac{dk+a+d-3X}{3dk+a} \leq \frac{dk+a+d}{3dk+a}$ since $X \geq \frac{2a}{5}$.

8. $e(2,3,3) = 0$: Student gets $> \frac{dk+\frac{a}{2}}{3dk+a} + 2 \times \frac{dk+2X}{3dk+a} \geq \frac{dk+a+d}{3dk+a}$ since $X \geq \frac{a+2d}{8}$.

9. The only $e(i,j,k)$ that might be nonzero are: $e(1,1,6)$, $e(1,3,3)$ $e(2,2,4)$, $e(2,2,5)$, (this follows from the previous parts).

10. All of the d_1-students are $e(1,3,3)$-students, hence $|I_3| = 2d_1 = 4d - 4a$ and $e(1,3,3) = \frac{|I_3|}{2} = 2d - 2a$ (this follows from the last part).

11. $|I_6| = |I_3| = 4d - 4a$.

12. The number of $e(1,1,6)$-students is $|I_6| = 4d - 4a$. This follows from (1) $|I_6| = |I_3|$, and (2) the only students who use $|I_6|$ are $e(1,1,6)$-students.

13. $|I_1| = 1 \times e(1,3,3) + 2 \times e(1,1,6) = 2d - 2a + 2(4d - 4a) = 10d - 10a$.

End of Claim

Since $|I_1| = a + d$ and $|I_1| = 10d - 10a$ we have $a = \frac{9d}{11}$, which is a contradiction. ∎

14 An Even More Sophisticated Argument!

(We later found a way to do this more morally. A program helped us. It will be in the next version of the paper we release.)

In Section 13 we described three theorem-generators. Each used a particular diagram. In this section we proof a theorem with a more sophisticated diagram that uses information about a, d to create bigger intervals.

Theorem 14.1 *If $\frac{5d}{13} \leq a \leq \frac{13d}{29}$ and $a \neq \frac{2}{5}d$ then $f(3dk + a + d, 3dk + a) \leq \frac{dk+X}{3dk+a}$ where $X = \max\{\frac{5a-d}{6}, \frac{a+d}{8}, \frac{3a}{7}\}$.*

Proof:

By the usual techniques we obtain the following picture:

$$
\begin{array}{ccccccccc}
(& a+d & | & a+d &)[& 0 &] \\
\frac{dk+X}{3dk+a} & & \frac{dk+\frac{a}{2}}{3dk+a} & & \frac{dk+a-X}{3dk+a} & & \frac{dk+2X}{3dk+a}
\end{array}
$$

$$
\begin{array}{ccccccccc}
(& d_2 &)[& 0 &](& 2d-a-d_2 & | & 2d-a-d_2 &) \\
\frac{dk+2X}{3dk+a} & & \frac{dk+2a-2X}{3dk+a} & & \frac{dk+3X}{3dk+a} & & \frac{dk+\frac{a+d}{2}}{3dk+a} & & \frac{dk+a+d-3X}{3dk+a}
\end{array}
$$

$$
\begin{array}{ccccc}
[& 0 &](& d_2 &) \\
\frac{dk+a+d-3X}{3dk+a} & & \frac{dk+d-a+2X}{3dk+a} & & \frac{dk+a+d-2X}{3dk+a}
\end{array}
$$

In order for this picture to not have some interval collapse we need the following conditions, which we have so long as $a \leq \frac{5d}{7}$ which we have.

$$\frac{2a}{5} \leq X \leq \min\left\{\frac{a}{2}, \frac{a+d}{6}\right\}$$

Def 14.2 We need names for the intervals.

1. $I_1 = (\frac{dk+X}{3dk+a}, \frac{dk+\frac{a}{2}}{3dk+a})$
2. $I_2 = (\frac{dk+\frac{a}{2}}{3dk+a}, \frac{dk+a-X}{3dk+a})$
3. $I_3 = (\frac{dk+2X}{3dk+a}, \frac{dk+2a-2X}{3dk+a})$
4. $I_4 = (\frac{dk+3X}{3dk+a}, \frac{dk+\frac{a+d}{2}}{3dk+a})$
5. $I_5 = (\frac{dk+\frac{a+d}{2}}{3dk+a}, \frac{dk+a+d-3X}{3dk+a})$
6. $I_6 = (\frac{dk+d-a+2X}{3dk+a}, \frac{dk+a+d-2X}{3dk+a})$

Claim 1:

1. $d_3 = 0$.

2. $\frac{dk+a+d-4X}{3dk+a}$ is in I_4.

3. All of the I_4-shares that a d_0 student takes are in $(\frac{dk+dk+3X}{3dk+a}, \frac{dk+a+d-4X}{3dk+a})$.

4. All of the I_4-shares that a d_1 student takes are in $(\frac{dk+dk+3X}{3dk+a}, \frac{dk+a+d-3X}{3dk+a})$.

5. d_2 students take two shares from $I_1 \cup I_2$ and one share from I_6, hence do not take I_4-shares.

6. There are no shares in $[\frac{dk+a+d-4X}{3dk+a}, \frac{dk+\frac{a+d}{2}}{3dk+a}]$ (this follows from the above points).

7. There are no shares in $[\frac{dk+\frac{a+d}{2}}{3dk+a}, \frac{dk+4X}{3dk+a}]$. (this follows from the previous point and the symmetry of $(\frac{dk+3X}{3dk+a}, \frac{dk+a+d-3X}{3dk+a})$).

Proof of Claim 1:

1) A d_3 student has less than $\frac{dk+3a-3X}{3dk+a} \le \frac{dk+a+d}{3dk+a}$ (this last inequality uses $X \ge \frac{2a-d}{3}$ which follows from the premise and $\frac{2a-d}{3} \le \frac{5a-d}{6}$).

2) We need $3X \le a+d-4X \le \frac{a+d}{2}$. The first inequality gives $X \le \frac{a+d}{7}$. To establish this we need (1) $\frac{a+d}{8} \le \frac{a+d}{7}$ which is obvious and (2) $\frac{5a-d}{6} \le \frac{a+d}{7}$ which follows from $a \le \frac{13d}{29}$. The second inequality gives $X \ge \frac{a+d}{8}$ which is true by the definition of X.

3) Let Alice be a d_0-student. We look at all of the ways she can have a share in I_4.

Alice gets I_3, I_3, I_4: The two I_3 shares gives more than $4X$, so the I_4 share is less than $a+d-4X$.

If Alice gets I_3, I_4, I_4 then she has more than $\frac{dk+8X}{3dk+a} \ge \frac{dk+a+d}{3dk+a}$, which is impossible.

Since I_3, I_4, I_4 results in more than $\frac{dk+a+d}{3dk+a}$, all other ways to include an I_4-share result in more than $\frac{dk+a+d}{3dk+a}$.

4) Let Alice be a d_1-student. We look at all of the ways she can have a share in I_4.

If Alice gets I_2, I_3, I_4 then she gets less than $\frac{dk+a+d}{3dk+a}$, using $X \geq \frac{5a-d}{6}$. Note that this also gives us that Alice can't get I_1, I_3, I_4.

If Alice gets I_1, I_4, I_4 then the I_1 and the first I_4 add up to more than $\frac{dk+X}{3dk+a} + \frac{dk+3X}{3dk+a} = \frac{dk+4X}{3dk+a}$, hence the other I_4 share is less than $\frac{dk+a+d-4X}{3dk+a}$.

If Alice gets I_2, I_4, I_4 then, by the I_1, I_4, I_4 case, the I_4's are all less than $\frac{dk+a+d-4X}{3dk+a}$.

If Alice gets I_1, I_4, I_5 then her I_4-share will be $\leq$ what it is in I_1, I_4, I_4, so the I_4-share is less than $\frac{dk+a+d-4X}{3dk+a}$.

Since I_1, I_4, I_5 results in more than $\frac{dk+a+d}{3dk+a}$, all other ways to include an I_4-share result in more than $\frac{dk+a+d}{3dk+a}$.

5) Let Alice be a d_2-student. The two $I_1 \cup I_2$ shares she gets add up to less than $\frac{dk+2a-2X}{3dk+a}$, hence the remaining share is more than $\frac{dk+a+d-2a+2X}{3dk+a} = \frac{dk+d-a+2X}{3dk+a}$ and hence is in I_6.

End of Proof of Claim 1

$$
\begin{array}{ccccccc}
(& a+d & | & a+d &)[& 0 &] \\
\frac{dk+X}{3dk+a} & & \frac{dk+\frac{a}{2}}{3dk+a} & & \frac{dk+a-X}{3dk+a} & & \frac{dk+2X}{3dk+a}
\end{array}
$$

$$
\begin{array}{ccccc}
(& d_2 &)[& 0 &] \\
\frac{dk+2X}{3dk+a} & & \frac{dk+2a-2X}{3dk+a} & & \frac{dk+3X}{3dk+a}
\end{array}
$$

$$
\begin{array}{ccccccc}
(& 2d-a-d_2 &)[& 0 &](& 2d-a-d_2 &) \\
\frac{dk+3X}{3dk+a} & & \frac{dk+a+d-4X}{3dk+a} & & \frac{dk+4X}{3dk+a} & & \frac{dk+a+d-3X}{3dk+a}
\end{array}
$$

$$
\begin{array}{ccccc}
[& 0 &](& d_2 &) \\
\frac{dk+a+d-3X}{3dk+a} & & \frac{dk+d-a+2X}{3dk+a} & & \frac{dk+a+d-2X}{3dk+a}
\end{array}
$$

Def 14.3 We use the old names but slightly change:

1. $I_1 = (\frac{dk+X}{3dk+a}, \frac{dk+\frac{a}{2}}{3dk+a})$

2. $I_2 = (\frac{dk+\frac{a}{2}}{3dk+a}, \frac{dk+a-X}{3dk+a})$

3. $I_3 = (\frac{dk+2X}{3dk+a}, \frac{dk+2a-2X}{3dk+a})$

4. $I_4 = (\frac{dk+3X}{3dk+a}, \frac{dk+a+d-4X}{3dk+a})$

5. $I_5 = (\frac{dk+4X}{3dk+a}, \frac{dk+a+d-3X}{3dk+a})$

6. $I_6 = (\frac{dk+d-a+2X}{3dk+a}, \frac{dk+a+d-2X}{3dk+a})$

We now look carefully at who can get what shares.

If a d_0-student uses an I_5 share OR two I_4 shares then he has more than $\frac{dk+8X}{3dk+a} > \frac{dk+a+d}{3dk+a}$ (this last inequality used $X \geq \frac{a+d}{8}$). Also if a student gets 3 I_3 shares then they will not have enough (follows from $\frac{5a-d}{6} \leq X$). Hence a d_0-student must have two I_3 shares and an I_4-share For clarity we call these $d_0(3,3,4)$ students.

If a d_1-student uses an I_4 and an I_5 then he has more than $\frac{dk+8X}{3dk+a} > \frac{dk+a+d}{3dk+a}$ (this last inequality used $X \geq \frac{a+d}{8}$). If a d_1-student uses an I_6 then he has more than $\frac{dk+5X+d-a}{3dk+a} > \frac{dk+a+d}{3dk+a}$ (this last inequality used $X \geq \frac{2a}{5}$). If a d_1-student uses an I_3 and an I_4 then he has less than $\frac{dk+a-X+2a-2X+a+d-4X}{3dk+a} = \frac{dk+4a+d-7X}{3dk+a} < \frac{dk+a+d}{3dk+a}$ (this last inequality uses $X \geq \frac{3a}{7}$).

Using these three fact we have that a d_1-student must either have two I_4's OR an I_3 and an I_5. For clarity we call the two types of d_1-students $d_1(4,4)$ and $d_1(3,5)$. Note that both types also have a share in $I_1 \cup I_2$.

From Claim 1 we know that a d_2-student uses one share from I_6. For clarity we call these $d_2(6)$ students. Note that these students also have two shares from $I_1 \cup I_2$.

We now have equations.

Since there are $2d$ 3-students

$d_0(3,3,4) + d_1(4,4) + d_1(3,5) + d_2(6) = 2d$

We now look at the intervals, looking at how much each type of student contributes to its shares.

$|I_1 \cup I_2| = d_1(4,4) + d_1(3,5) + 2d_2(6)$.

Hence $d_1(4,4) + d_1(3,5) + 2d_2(6) = 2a + 2d$.

$|I_3| = 2d_0(3,3,4) + d_1(3,5)$

$|I_4| = d_0(3,3,4) + 2d_1(4,4)$

$|I_5| = d_1(3,5)$

$|I_6| = d_2(6)$.

We use $|I_3| = |I_6|$ and $|I_4| = |I_5|$ and gather up all of our equations.

$d_0(3,3,4) + d_1(4,4) + d_1(3,5) + d_2(6) = 2d$

$d_1(4,4) + d_1(3,5) + 2d_2(6) = 2a + 2d$.

$2d_0(3,3,4) + d_1(3,5) = d_2(6)$.

$d_0(3,3,4) + 2d_1(4,4) = d_1(3,5)$

We replace the the variables with w, x, y, z and express the equations in matrix form:

$$\begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 2 \\ 2 & 0 & 1 & -1 \\ 1 & 2 & -1 & 0 \end{pmatrix} \begin{pmatrix} w \\ x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 2d \\ 2a+2d \\ 0 \\ 0 \end{pmatrix}$$

We show this system is $2d$-unsolvable. To column 1, add once column 3 and thrice column 4. To column 2, add twice column 3 and twice column 4.

$$\begin{pmatrix} 5 & 5 & 1 & 1 \\ 7 & 7 & 1 & 2 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & -1 & 0 \end{pmatrix} \begin{pmatrix} w' \\ x' \\ y' \\ z' \end{pmatrix} = \begin{pmatrix} 2d \\ 2a+2d \\ 0 \\ 0 \end{pmatrix}$$

Now clearly $y' = z' = 0$, so $14d = 10a + 10d$ implies $a = \frac{2}{5}d$, a contradiction.

∎

15 $f(s+d,s)$

For $1 \le d \le 8$, for $a \in \{1,\ldots,3d-1\}$ where a,d are relatively prime, we have results or conjectures (mostly results) of the following form:

If $1 \le d \le 6$, $a \in \{1,\ldots,3d-1\}$, a,d are relatively prime, $k \ge 0$, $3dk+a \ne 1$, then $f(3dk+a+d, 3dk+a) \le \frac{dk+X}{3dk+a}$ where $X = BLAH$ (BLAH will be a constant like $\frac{4}{3}$).

When $k \ge 1$ $\left\lceil \frac{2m}{s} \right\rceil = 3$ and can hence use the methods of this paper. When $k=0$ either we still have $\left\lceil \frac{2m}{s} \right\rceil = 3$, and can hence use the methods of this paper, or use Theorem 6.1 or 7.2

It is somewhat remarkable that the formula we get for the $\left\lceil \frac{2m}{s} \right\rceil = V$ case just happens to work for all cases (except when $a+d=1$). We use this observation to make conjectures for the cases of $f(a+d,a)$ that we could not solve.

In the results below we sometimes state, for completeness, results of the form $f(m,1)=1$. This is obvious so we do not comment on it.

15.1 $f(s+1,s)$

Theorem 15.1

1. $f(2,1)=1$. *For all $k \ge 1$, $f(3k+2,3k+1) \le \frac{k+X}{3k+1}$ where $X = \frac{1}{2}$.*

2. *For all $k \ge 0$, $f(3k+3,3k+2) \le \frac{k+X}{3k+2}$ where $X = 1$.*

Proof: This follow from Theorem 12.1 with $d=1$ and $a=1,2$. ∎

Theorem 15.2 *For all $k \ge 0$, $f(3k+4,3k+3) \le \frac{1}{3}$.*

Proof: These follow from Theorem 12.2 with $d=1$ and $a=3$. ∎

15.2 $f(s+2,s)$

Theorem 15.3 *$f(3,1)=1$. For all $k \ge 1$, $f(6k+3,6k+1) \le \frac{2k+X}{6k+1}$ where $X=\frac{1}{2}$.*

Proof: This follows from Theorem 12.1 with $d=2$ and $a=1$. ▮

Theorem 15.4 *For all $k \ge 0$, $f(6k+5,6k+3) \le \frac{2k+X}{6k+3}$ where $X=\frac{5}{4}$.*

Proof: When $k=1$ the result follows from Theorem 12.1 with $d=2$ and $a=3$. When $k=0$, $f(5,3) \le \frac{5}{12}$ from Theorem 6.1. ▮

Theorem 15.5 *For all $k \ge 0$, $f(6k+7,6k+5) \le \frac{1}{3}$.*

Proof: This follows from Theorem 12.2 with $d=2$ and $a=5$. ▮

15.3 $f(s+3,s)$

Theorem 15.6

1. *$f(4,1)=1$. For all $k \ge 1$ $f(9k+4,9k+1) \le \frac{3k+X}{9k+1}$ where $X=\frac{1}{2}$.*
2. *For all $k \ge 0$ $f(9k+5,9k+2) \le \frac{3k+X}{9k+2}$ where $X=1$.*

Proof: When $k=1$ these follow from Theorem 12.1 with $d=3$ and $a=1,2$. When $k=0$, $f(5,2) \le \frac{1}{2}$ by Theorem 6.1. ▮

Theorem 15.7 *For all $k \ge 0$, $f(9k+7,9k+4) \le \frac{3k+X}{9k+4}$ where $X=\frac{5}{3}$.*

Proof: When $k \ge 1$ the result follows from Theorem 13.4 with $d=3$ and $a=4$. When $k=0$, $f(7,4) \le \frac{5}{12}$ follows from Theorem 6.1. ▮

Theorem 15.8 *For all $k \ge 0$, $f(9k+8,9k+5) \le \frac{3k+X}{9k+5}$ where $X=2$.*

Proof: When $k \geq 1$ the result follows from Theorem 12.1 with $d = 3$ and $a = 5$. When $k = 0$, $f(8,5) \leq \frac{2}{5}$ by Theorem 6.1. ▮

Theorem 15.9 *For all* $k \geq 0$

1. $f(9k+10, 9k+7) \leq \frac{1}{3}$
2. $f(9k+11, 9k+8) \leq \frac{1}{3}$.

Proof: These follow from Theorem 12.2 with $d = 3$ and $a = 7, 8$. ▮

15.4 $f(s+4, s)$

Theorem 15.10 $f(5,1) = 1$. *For all* $k \geq 1$, $f(12k+5, 12k+1) \leq \frac{4k+X}{12k+1}$ *where* $X = \frac{1}{2}$.

Proof: This follows from Theorem 12.1 with $d = 4$ and $a = 1$. ▮

Theorem 15.11 *For all* $k \geq 0$, $f(12k+7, 12k+3) \leq \frac{4k+X}{12k+3}$ *where* $X = \frac{5}{4}$.

Proof: When $k \geq 1$ the result follows from Theorem 13.12 with $d = 4$ and $a = 3$. When $k = 0$, $f(7,3) \leq \frac{5}{12}$ by Theorem 6.1. ▮

Theorem 15.12 *For all* $k \geq 0$, $f(12k+9, 12k+5) \leq \frac{4k+X}{12k+5}$ *where* $X = 2$.

Proof: When $k \geq 1$ the result follows from Theorem 13.4 with $d = 4$ and $a = 5$. When $k = 0$, $f(9,5) \leq \frac{2}{5}$ by Theorem 6.1. ▮

Theorem 15.13 *For all* $k \geq 0$, $f(12k+11, 12k+7) \leq \frac{4k+X}{12k+7}$ *where* $X = \frac{11}{4}$.

Proof: When $k \geq 1$ the result follow from Theorem 12.1 with $d = 4$ and $a = 7$. When $k = 0$, $f(11,7) \leq \frac{11}{28}$ by Theorem 6.1. ▮

Theorem 15.14 *For all $k \geq 0$,*

1. $f(12k+13, 12k+9) \leq \frac{1}{3}$.

2. $(\forall k \geq 0)[f(12k+15, 12k+11) \leq \frac{1}{3}$.

Proof: These follow from Theorem 12.2 with $d = 4$ and $a = 9, 11$. ∎

15.5 $f(s+5, s)$

Theorem 15.15

1. $f(6,1) = 1$. *For all $k \geq 1$* $f(15k+6, 15k+1) \leq \frac{5k+X}{15k+1}$ *where* $X = \frac{1}{2}$.

2. *For all $k \geq 0$* $f(15k+7, 15k+2) \leq \frac{5k+X}{15k+2}$ *where* $X = 1$.

Proof: The $k \geq 1$ case follows from Theorem 12.1 with $d = 5$ and $a = 1, 2$. The $k = 0$ (second part), $f(7,2) \leq \frac{1}{2}$, by Theorem 6.1. ∎

Theorem 15.16 *For all $k \geq 0$,* $f(15k+8, 15k+3) \leq \frac{5k+X}{15k+3}$ *where* $X = \frac{4}{3}$.

Proof: When $k \geq 1$ the result follows from Theorem 13.9 with $d = 5$ and $a = 3$. When $k = 0$, $f(8,3) \leq \frac{4}{9}$ by Theorem 6.1. ∎

Theorem 15.17 *For all $k \geq 0$,* $f(15k+9, 15k+4) \leq \frac{5k+X}{15k+4}$ *where* $X = \frac{7}{4}$.

Proof: When $k \geq 1$ the result follow from Theorem 13.9 with $d = 5$ and $a = 4$. When $k = 0$, $f(9,4) \leq \frac{7}{16}$ by Theorem 6.1. ∎

Theorem 15.18 *For all $k \geq 0$,* $f(15k+11, 15k+6) \leq \frac{5k+X}{15k+6}$ *where* $X = \frac{7}{3}$.

Proof: The result follows from Theorem 13.4 with $d = 5$ and $a = 6$. ∎

Theorem 15.19 *For all $k \geq 0$, $f(15k+12, 15k+7) \leq \frac{5k+X}{15k+7}$ where $X = 3$.*

Proof: When $k \geq 1$ the result follows from Theorem 12.1 with $d = 5$ and $a = 7$. When $k = 0$, $f(12, 7) \leq \frac{3}{7}$ by Theorem 6.1. ∎

Theorem 15.20 *For all $k \geq 0$, $f(15k+13, 15k+8) \leq \frac{5k+X}{15k+8}$ where $X = \frac{13}{4}$.*

Proof: When $k = 0$ $f(13, 8) \leq \frac{13}{32}$ by Theorem 7.2. When $k \geq 1$ the result follows from Theorem 12.1 with $d = 5$ and $a = 8$. ∎

Theorem 15.21 *For all $k \geq 0$, $f(15k+14, 15k+9) \leq \frac{5k+X}{15k+9}$ where $X = \frac{7}{2}$.*

Proof: When $k \geq 1$ the result follows from Theorem 12.1 with $d = 5$ and $a = 9$. When $k = 0$, $f(14, 9) \leq \frac{7}{18}$ by Theorem 6.1. ∎

Theorem 15.22 *For all $k \geq 0$:*

1. $f(15k+16, 15k+11) \leq \frac{1}{3}$,
2. $f(15k+17, 15k+12) \leq \frac{1}{3}$,
3. $f(15k+18, 15k+13) \leq \frac{1}{3}$,
4. $f(15k+19, 15k+14) \leq \frac{1}{3}$.

Proof: These follow from Theorem 12.2 with $d = 5$ and $a = 11, 12, 13, 14$. ∎

15.6 $f(s+6,s)$

Theorem 15.23 $f(7,1)=1$. *For all* $k \geq 1$, $f(18k+7, 18k+1) \leq \frac{6k+X}{18k+1}$ *where* $X = \frac{1}{2}$.

Proof: This follow from Theorem 12.1 with $d=6$ and $a=1$. ∎

Theorem 15.24 *For all* $k \geq 0$, $f(18k+11, 18k+5) \leq \frac{6k+X}{18k+5}$ *where* $X = \frac{13}{6}$.

Proof: This follow from Theorem 13.13 with $d=6$ and $a=5$. The $k=0$, $f(11,5) \leq \frac{13}{30}$ by Theorem 7.2. ∎

Theorem 15.25 *For all* $k \geq 0$, $f(18k+13, 18k+7) \leq \frac{6k+X}{18+7}$ *where* $X = \frac{8}{3}$.

Proof: When $k=1$ the result follows from Theorem 13.4 with $d=6$ and $a=7$. When $k=0$, $f(13,7) \leq \frac{8}{21}$ by Theorem 6.1. ∎

Theorem 15.26 *For all* $k \geq 0$, $f(18k+17, 18k+11) \leq \frac{6k+X}{18k+11}$ *where* $X = \frac{17}{4}$.

Proof: When $k=1$ the result these follow from Theorem 12.1 with $d=6$ and $a=11$. When $k=0$, $f(17,11) \leq \frac{17}{44}$ by Theorem 6.1. ∎

Theorem 15.27 *For all* $k \geq 0$,

1. $f(18k+19, 18k+13) \leq \frac{1}{3}$,
2. $f(18k+23, 18k+17) \leq \frac{1}{3}$.

Proof: These follow from Theorem 12.2 with $d=6$ and $a=13,17$. ∎

15.7 $f(s+7,s)$

Theorem 15.28

1. $f(8,1)=1$. *For all* $k \geq 1$, $f(21k+8, 21k+1) \leq \frac{7k+X}{21k+1}$ *where* $X=\frac{1}{2}$.

2. *For all* $k \geq 0$, $f(21k+9, 21k+2) \leq \frac{7k+X}{21k+2}$ *where* $X=1$.

Proof: When $k \geq 1$ both follow from Theorem 12.1 with $d=7$ and $a=1,2$. When $k=0$ (second case), $f(9,2) \leq \frac{1}{2}$ by Theorem 6.1. ∎

Theorem 15.29 *For all* $k \geq 0$, $f(21k+10, 21k+3) \leq \frac{7k+X}{21k+3}$ *where* $X=\frac{4}{3}$.

Proof: When $k \geq 1$ this follows from Theorem 14.1 with $d=7$ and $a=3$. When $k=0$, $f(10,3) \leq \frac{4}{9}$ by Theorem 6.1. ∎

Theorem 15.30 *For all* $k \geq 0$, $f(21k+11, 21k+4) = \frac{7k+X}{21k+4}$ *where* $X=\frac{9}{5}$.

Proof: When $k \geq 1$ this follows from Theorem 13.10 with $d=7$ and $a=4$. When $k=0$, $f(11,4) \leq \frac{9}{20}$ by Theorem 6.1. ∎

Theorem 15.31 *For all* $k \geq 0$, $f(21k+12, 21k+5) \leq \frac{7k+X}{21k+5}$ *where* $X=2$.

Proof: When $k \geq 1$ this follows from Theorem 13.9 or 13.12 with $d=7$ and $a=5$. When $k=0$, $f(12,5) \leq \frac{2}{5}$ by Theorem 6.1. ∎

Theorem 15.32 *For all* $k \geq 0$, $f(21k+13, 21k+6) \leq frac7k+X21k+6$ *where* $X=\frac{13}{5}$.

Proof: When $k \geq 1$ this follows from Theorem 13.4 with $d=7$ and $a=6$. When $k=0$, $f(13,6) \leq \frac{13}{30}$ by Theorem 6.1. ∎

Theorem 15.33 *For all* $k \geq 0$, $f(21k+15, 21k+8) \leq \frac{7k+X}{21k+8}$ *where* $X=3$.

Proof: When $k \geq 1$ this follows from Theorem 13.4 with $d = 7$ and $a = 8$. When $k = 0$, $f(15, 8) \leq \frac{3}{8}$ by Theorem 6.1. ∎

Theorem 15.34 *For all $k \geq 0$, $f(21k + 16, 21k + 9) \leq \frac{7k+X}{21k+9}$ where $X = \frac{11}{3}$.*

Proof: When $k \geq 1$ this follows from Theorem 13.4 with $d = 7$ and $a = 9$. When $k = 0$, $f(16, 9) \leq \frac{11}{27}$ by Theorem 6.1. ∎

Theorem 15.35 *For all $k \geq 0$, $f(21k + 17, 21k + 10) \leq \frac{7k+X}{21k+10}$ where $X = 4$.*

Proof: When $k \geq 1$ this follows from Theorem 13.6 with $d = 7$ and $a = 10$. When $k = 0$, $f(17, 10) \leq \frac{2}{5}$ by Theorem 7.2. ∎

Theorem 15.36 *For all $k \geq 0$:*

1. $f(21k + 18, 21k + 11) \leq \frac{7k+X}{21k+11}$ *where* $X = \frac{9}{2}$.

2. $f(21k + 19, 21k + 12) \leq \frac{7k+X}{2ak+12}$ *where* $X = \frac{19}{4}$.

3. $f(21k + 20, 21k + 13) \leq \frac{7k+X}{21k+13}$ *where* $X = 5$.

Proof: When $k \geq 1$ these follow from Theorem 12.1 with $d = 7$ and $a = 11, 12, 13$. When $k = 0$: $f(18, 11) \leq \frac{9}{22}$, $f(19, 12) \leq \frac{19}{48}$, and $f(20, 13) \leq \frac{5}{13}$ by Theorem 6.1. ∎

Theorem 15.37 *For all $k \geq 0$:*

1. $f(21k + 22, 21k + 15) \leq \frac{1}{3}$,

2. $f(21k + 23, 21k + 16) \leq \frac{1}{3}$,

3. $f(21k + 24, 21k + 17) \leq \frac{1}{3}$,

4. $f(21k + 25, 21k + 18) \leq \frac{1}{3}$,

5. $f(21k+26, 21k+19) \le \frac{1}{3}$,

6. $f(21k+27, 21k+20) \le \frac{1}{3}$.

Proof: These follow from Theorem 12.2 with $d = 7$ and $a = 15, 16, 17, 18, 19, 20$. ▮

15.8 $f(s+8, s)$

Theorem 15.38 $f(9,1) = 1$. *For all* $k \ge 1$, $f(24k+9, 24k+1) \le \frac{8k+X}{24k+1}$ *where* $X = \frac{1}{2}$.

Proof: These follow from Theorem 12.1 with $d = 8$ and $a = 1$. ▮

Conjecture 15.39 *For all* $k \ge 0$, $f(24k+11, 24k+3) \le \frac{8k+X}{24k+3}$ *where* $X = \frac{11}{8}$.

Theorem 15.40 *For all* $k \ge 0$, $f(24k+13, 24k+5) = \frac{8k+X}{24k+5}$ *where* $X = \frac{13}{6}$.

Proof: When $k \ge 1$ this follow from Theorem 13.9 with $d = 8$ and $a = 5$. When $k = 0$, $f(13,5) \le \frac{13}{30}$. ▮

Conjecture 15.41 *For all* $k \ge 0$, $f(24k+15, 24k+7) \le \frac{8k+X}{24k+7}$ *where* $X = 3$.

Conjecture 15.42 *For all* $k \ge 0$, $f(24k+17, 24k+9) \le \frac{8k+X}{24k+9}$ *where* $X = \frac{10}{3}$.

Theorem 15.43 *For all* $k \ge 0$, $f(24k+19, 24k+11) \le \frac{8k+X}{24k+11}$ *where* $X = \frac{9}{2}$.

Proof: When $k \ge 1$ this follow from Theorem 13.6 with $d = 8$ and $a = 15$. When $k = 0$, $f(19,11) \le \frac{9}{22}$ by Theorem 7.2. ▮

Theorem 15.44 *For all* $k \ge 0$, $f(24k+21, 24k+13) \le \frac{8k+X}{24k+13}$ *where* $X = \frac{21}{4}$.

Proof: When $k \geq 1$ this follows from Theorem 12.1 with $d = 8$ and $a = 13$. When $k = 0$, $f(21, 13) \leq \frac{21}{52}$ by Theorem 6.1. ∎

Theorem 15.45 *For all $k \geq 0$, $f(24k + 23, 24k + 15) \leq \frac{8k+X}{24k+15}$ where $X = \frac{23}{4}$.*

Proof: When $k \geq 1$ this follows from Theorem 12.1 with $d = 8$ and $a = 15$. When $k = 0$, $f(23, 15) \leq \frac{23}{60}$ by Theorem 6.1. ∎

Theorem 15.46 *For all $k \geq 0$:*

1. $f(24k + 25, 24k + 17) \leq \frac{1}{3}$,
2. $f(24k + 27, 24k + 19) \leq \frac{1}{3}$.
3. $f(24k + 29, 24k + 21) \leq \frac{1}{3}$.
4. $f(24k + 31, 24k + 23) \leq \frac{1}{3}$.

Proof: These follow from Theorem 12.2 with $d = 8$ and $a =, 17, 19, 21, 23$. ∎

16 A Bold Conjecture

In Section 15 we proved many results of the form

$$(\forall k \geq 0)[f(3dk + a + d) \leq \frac{dk + X}{3dk + a} \text{ where } X = BLAH \text{ }].$$

How did we know which theorems to look at? We usually knew what happened when $k = 0$ and then used that to conjecture X, and then prove the theorem with that X. But the proof for $k \geq 1$ used the methods of this paper, whereas the proof for $k = 0$ used other techniques. This leads to a bold conjecture and a methodology

Conjecture 16.1 *For all a, d with $1 \le a < d$, a, d relatively prime, there exists a constant X such that*

$$(\forall k \ge 0)[f(3dk + a + d, 3dk + a) \le \frac{dk + X}{3dk + a} \text{ where } X = BLAH \}.$$

(Exception when $k = 0$ and $a = 1$ when $f(d + 1, 1) = 1$.)

We have so far used this in one direction: Given that we know the $k = 0$ case, we conjecture a value of X and then prove it. However, the methods in this paper are very powerful so perhaps we can turn that around. Here is a concrete example:

We do not know $f(41, 19)$. Note that $f(41, 19) = f(66k+22+19, 66k+19)$. For $k \ge 1$ we may be able to solve this problem with the technique of this paper. So use those techniques to find $f(107, 85)$. You will then have a conjecture for X (likely you will have gotten the upper bound for all $k \ge 1$) and then, you have a good conjecture for $f(41, 19)$. Use other techniques for this problem; however, this will be much easier since we know an answer as opposed to trying to find the answer.

17 $f(m, s)$ for $s = 1, 2, 3, 4, 5, 6$

In this section we determine $f(m, s)$ for $1 \le s \le 6$ and any m. Since $f(s, m) = \frac{m}{s} f(m, s)$ (Theorem 5.1) and $f(m, m) = 1$ (Theorem 4.1.1) we need only consider the case where $m > s$.

We freely use that $f(Am, As) = f(m, s)$ since this holds for all of our techniques, as noted in Theorem 4.1 and Corollaries 6.2,7.5,7.5.

Throughout this section we state a theorem in such a way that the proof of the theorem is embedded in the statement of it.

We use the following compact notation for protocols:

Notation 17.1

1. $a - (\alpha, \beta)$ means that a muffins are split into two pieces, one of size α, one of size β. For example:

$$(24k + 18) - \left(\frac{10k+7}{20k+15}, \frac{10k+8}{20k+15}\right)$$

means that $24k + 18$ muffins are split in the ratio indicated. There is one time will need to split a muffin into 3 pieces. In this one case we use the obvious extension of the notation. If $a = 1$ we may omit it.

2. $a - [a_1 : \alpha_1, \ldots, a_i : \alpha_L]$ means that a students gets, for $1 \le i \le L$, a_i shares of size α_i. These are all of the shares those student will get. For example:

$$3 - \left[(8k+6) : \frac{10k+8}{20k+15}, (2k+1) : \frac{1}{2}\right].$$

means that 3 students get $8k+6$ shares of size $\frac{10k+8}{20k+15}$ and $2k+1$ shares of size $\frac{1}{2}$. Note that $a_1\alpha_1 + \cdots + a_L\alpha_L = \frac{m}{s}$. We will only use using this for $L = 1, 2, 3$. If $a = 1$ ($a_i = 1$) we may omit it.

Convention 17.2 We label cases based on what congruence class they are dealing with. We give an example:

Case 1: $m = 10k + 1$

Case 1.1: $m = 30k + 1$

Case 1.11: $m = 30k + 11$

Case 1.21: $m = 30k + 21$

17.1 $f(m, 1)$

Theorem 17.3 *$f(m, 1) = 1$ by Theorem 4.1.1.*

17.2 $f(m, 2)$

Theorem 17.4

If $m = 2k + i$ where $0 \le i \le 1$ then $f(m, 2)$ depends only on i via a formula. Hence there are 2 cases.

Case 0: *$s = 2k + 0$ with $k \ge 1$. Then $f(2k, 2) = f(k, 1) = 1$ by Theorem 17.3.*

Case 1: *$s = 2k + 1$ with $k \ge 1$. Then $f(2k + 1, 2) = \frac{1}{2}$. The following protocol show $f(2k + 1, 2) \ge \frac{1}{2}$: divide every muffin into 2 pieces of size $\frac{1}{2}$ and give every student $2k + 1$ half-muffins. By Convention 4.2 (which holds when s does not divide m which is the case here) every muffin is cut in 2 pieces, so $f(2k + 1, s) \le \frac{1}{2}$.*

17.3 $f(m, 3)$

Theorem 17.5

If $m = 3k + i$ where $0 \le i \le 2$ then $f(m, 3)$ depends only on k, i via a formula. Hence there are 3 cases.

Case 0: *$m = 3k + 0$ with $k \ge 1$. Then $f(3k, 3) = 1$ by Theorem 4.1.1.*

For the rest of the cases the upper bound is obtained by Theorem 6.1. Hence we just give the protocol.

Case 1: *$m = 3k + 1$ with $k \ge 1$. Then $f(3k + 1, 3) = \frac{3k-1}{6k}$.*

1. $2k$-$(\frac{3k-1}{6k}, \frac{3k+1}{6k})$, $(k + 1)$-$(\frac{1}{2}, \frac{1}{2})$.
2. $[2k : \frac{3k+1}{6k}]$, 2-$[k : \frac{3k-1}{6k}, k + 1 : \frac{1}{2}]$

Case 2: *$m = 3k + 2$ with $k \ge 1$. Then $f(3k + 2, 3) = \frac{3k+2}{6k+6}$.*

1. $2k + 2$-$(\frac{3k+2}{6k+6}, \frac{3k+4}{6k+6})$, k-$(\frac{1}{2}, \frac{1}{2})$.
2. $[2k + 2 : \frac{3k+2}{6k+6}]$, 2-$[k + 1 : \frac{3k+4}{6k+6}, k : \frac{1}{2}]$.

17.4 $f(m,4)$

Theorem 17.6

If $m=4k+i$ where $0 \le i \le 3$ then $f(m,4)$ depends only on k,i via a formula. Hence there are 4 cases.

Cases 1 and 3 both use the upper bound from Theorem 6.1; hence, in those cases, we only present the lower bound (the protocol).

Case 0: $m=4k$ *where* $k \ge 1$.. *Then* $f(4k,4)=1$ *by Theorem 4.1.1.*

Case 1: $m=4k+1$ *where* $k \ge 1$. *Then* $f(4k+1,4)=\frac{4k-1}{8k}$.

1. $4k$-$(\frac{4k-1}{8k}, \frac{4k+1}{8k})$, 1-$(\frac{1}{2}, \frac{1}{2})$.

2. 2-$[2k : \frac{4k+1}{8k}]$, 2-$[2k : \frac{4k-1}{8k}, 1 : \frac{1}{2}]$

Case 2: $m=4k+2$. $f(4k+2,4)=f(2k+1,2)=\frac{1}{2}$ *(by Theorem 17.4).*

Case 3: $m=4k+3$ *where* $k \ge 1$. *Then* $f(4k+3,4)=\frac{4k+1}{8k+4}$

1. $4k+2$-$(\frac{4k+1}{8k+4}, \frac{4k+3}{8k+4})$.

2. 1-$(\frac{1}{2}, \frac{1}{2})$.

3. 2-$[2k+1 : \frac{4k+3}{8k+4}]$, 2-$[2k+1 : \frac{4k+1}{8k+4}, 1 : \frac{1}{2}]$

17.5 $f(m,5)$

We first state and prove a theorem that tells us what $f(m,5)$ is for all but two values of $m \ge 5$. We then state and prove the theorem that covers the missing two cases. We also explain why they were missing.

Theorem 17.7 *If $m=5k+i$ where $0 \le i \le 4$ then $f(m,5)$ depends only on k,i via a formula, given below, with 2 exceptions (we will note the exceptions).*

Case 0: $m = 5k + 0$ *with* $k \geq 1$*. Then* $f(5k, 5) = 1$*. (This is by Theorem 4.1.1 so we won't prove it.)*

Case 1: $m = 5k + 1$ *with* $k \geq 1$, $k \neq 2$*. Then* $f(5k + 1, 5) = \frac{5k+1}{10k+5}$*. (Exception:* $f(11, 5) = \frac{13}{30}$*.)*

Case 2: $m = 5k + 2$ *with* $k \geq 2$*. Then* $f(5k + 2, 5) = \frac{5k-2}{10k}$*. (Exception:* $f(7, 5) = \frac{1}{3}$*.)*

Case 3: $m = 5k + 3$ *with* $k \geq 1$*. Then* $f(5k + 3, 5) = \frac{5k+3}{10k+10}$*.*

Case 4: $m = 5k + 4$ *with* $k \geq 1$*. Then* $f(5k + 4, 5) = \frac{5k+1}{10k+5}$*.*

Proof:

Case 1: $m = 5k + 1$ with $k \geq 3$. Then $f(5k + 1, 5) = \frac{5k+1}{10k+5}$. There are cases mod 30.

$m = 30k + 1$ with $k \geq 1$, $f(30k + 1) = \frac{30k+1}{60k+5}$

1. $(24k + 2)$-$(\frac{30k+1}{60k+5}, \frac{30k+4}{60k+5})$; 2-$(\frac{30k+2}{60k+5}, \frac{30k+3}{60k+5})$; $(6k - 3)$-(1).

2. 2-$[12k + 1 : \frac{30k+1}{60k+5}]$; 2-$[8k + 1 : \frac{30k+4}{60k+5}, \frac{30k+2}{60k+5}, 2k - 1 : 1]$; $[8k : \frac{30k+4}{60k+5}, 2 : \frac{30k+3}{60k+5}, 2k - 1 : 1]$.

$m = 30k + 6$ with $k \geq 0$, $f(30k + 6) = \frac{30k+6}{60k+15}$

1. $(24k + 6)$-$(\frac{30k+6}{60k+15}, \frac{30k+9}{60k+15})$; $(6k)$-(1).

2. 2-$[12k + 3 : \frac{30k+6}{60k+15}]$; 3-$[8k + 2 : \frac{30k+9}{60k+15}, , 2k : 1]$;

$m = 30k + 11$ with $k \geq 1$, $f(30k + 11) = \frac{30k+11}{60k+25}$

1. $(24k + 10)$-$(\frac{30k+11}{60k+25}, \frac{30k+14}{60k+25})$, 2-$(\frac{30k+12}{60k+25}, \frac{30k+13}{60k+25})$, $(6k - 1)$-(1).

2. 2-$[12k+5 : \frac{30k+11}{60k+25}]$, 2-$[8k+3 : \frac{30k+14}{60k+25}, \frac{30k+13}{60k+25}, 2k : 1]$, $[8k+4 : \frac{30k+14}{60k+25}, 2 : \frac{30k+12}{60k+25}, 2k-1 : 1]$.

$m = 30k + 16$ with $k \geq 0$, $f(30k + 16) = \frac{30k+16}{60k+35}$

1. $(24k + 14)$-$(\frac{30k+16}{60k+35}, \frac{30k+19}{60k+35})$, 2-$(\frac{30k+17}{60k+35}, \frac{30k+18}{60k+35})$, $6k$-(1).

2. 2-$[12k+7 : \frac{30k+16}{60k+35}]$, 2-$[8k+5 : \frac{30k+19}{60k+35}, \frac{30k+17}{60k+35}, 2k : 1]$, $[8k+4 : \frac{30k+19}{60k+35}, 2 : \frac{30k+18}{60k+35}, 2k : 1]$.

$m = 30k+21$ with $k \geq 0$, $f(30k+21) = \frac{10k+7}{20k+15}$

1. $(24k+18)$-$(\frac{10k+7}{20k+15}, \frac{10k+8}{20k+15})$, $(6k+3)$-(1),
2. 2-$[12k+9 : \frac{10k+7}{20k+15}]$, 3-$[8k+6 : \frac{10k+8}{20k+15}, (2k+1) : 1]$,

$m = 30k+26$ with $k \geq 0$, $f(30k+26) = \frac{30k+26}{60k+55}$

1. $(24k+22)$-$(\frac{30k+26}{60k+55}, \frac{30k+29}{60k+55})$, 2-$(\frac{30k+27}{60k+55}, \frac{30k+28}{60k+55})$, $(6k+2)$-(1).
2. 2-$[12k+11 : \frac{30k+26}{60k+55}]$, 2-$[8k+7 : \frac{30k+29}{60k+55}, \frac{30k+28}{60k+55}, 2k+1 : 1]$, 1-$[8k+8 : \frac{30k+29}{60k+55}, 2 : \frac{30k+27}{60k+55}, 2k : 1]$.

Case 2: $m = 5k+2$ with $k \geq 1$. Then $f(5k+2, 5) = \frac{5k-2}{10k}$. There are cases mod 10.

$m = 10k+2$ with $k \geq 1$, $f(10k+2, 5) = \frac{10k-2}{20k}$.

1. $4k$-$(\frac{10k-2}{20k}, \frac{10k+2}{20k})$, $(6k+2)$-$(\frac{1}{2}, \frac{1}{2})$.
2. $[4k : \frac{10k+2}{20k}]$, 4-$[k : \frac{10k-2}{20k}, (3k+1) : \frac{1}{2}]$.

$m = 10k+7$ with $k \geq 1$, $f(10k+7) = \frac{10k+3}{20k+10}$.

1. $4k+2$-$(\frac{10k+3}{20k+10}, \frac{10k+7}{20k+10})$, $4k+2$-$(\frac{10k+4}{20k+10}, \frac{10k+6}{20k+10})$, $(2k+3)$-$(\frac{1}{2}, \frac{1}{2})$.
2. 1-$[4k+2 : \frac{10k+7}{20k+10}]$, 2-$[(2k+1) : \frac{10k+3}{20k+10}, 2k+1 : \frac{10k+6}{20k+10}, 1 : \frac{1}{2}]$ 2-$[(2k+1) : \frac{10k+4}{20k+10}, 2k+2 : \frac{1}{2}]$

Case 3: $m = 5k+3$ with $k \geq 1$. Then $f(5k+3, 5) = \frac{5k+3}{10k+10}$. There are cases mod 10.

$m = 10k+3$ with $k \geq 1$, $f(10k+3) = \frac{10k+3}{20k+10}$.

1. $4k+2$-$(\frac{10k+3}{20k+10}, \frac{10k+7}{20k+10})$, 3-$(\frac{10k+4}{20k+10}, \frac{10k+6}{20k+10})$, $(6k-2)$-$(\frac{1}{2}, \frac{1}{2})$.

2. $[4k+2 : \frac{10k+3}{20k+10}]$, 3-$[\frac{10k+4}{20k+10}, (k+1) : \frac{10k+7}{20k+10}, 3k-1 : \frac{1}{2}]$ $[3 : \frac{10k+6}{20k+10}, (k-1)\frac{10k+7}{20k+10}, 3k-1 : \frac{1}{2}]$

$m = 10k+7$ with $k \geq 1$, $f(10k+7) = \frac{10k+3}{20k+10}$.

1. $4k+2$-$(\frac{10k+3}{20k+10}, \frac{10k+7}{20k+10})$, $4k+2$-$(\frac{10k+4}{20k+10}, \frac{10k+6}{20k+10})$, $(2k+3)$-$(\frac{1}{2}, \frac{1}{2})$.

2. 1-$[4k+2 : \frac{10k+7}{20k+10}]$, 2-$[(2k+1) : \frac{10k+3}{20k+10}, 2k+1 : \frac{10k+6}{20k+10}, 1 : \frac{1}{2}]$ 2-$[(2k+1) : \frac{10k+4}{20k+10}, 2k+2 : \frac{1}{2}]$

Case 4: $m = 5k+4$ with $k \geq 1$. Then $f(5k+4, 5) = \frac{5k+1}{10k+5}$.

$m = 30k+4$ with $k \geq 1$, $f(30k+4) = \frac{30k+1}{60k+5}$

1. $(24k+2)$-$(\frac{30k+1}{60k+5}, \frac{30k+4}{60k+5})$; 2-$(\frac{30k+2}{60k+5}, \frac{30k+3}{60k+5})$; $(6k)$-(1).

2. 2-$[12k+1 : \frac{30k+4}{60k+5}]$; 2-$[8k+1 : \frac{30k+1}{60k+5}, \frac{30k+3}{60k+5}, 2k : 1]$; $[8k : \frac{30k+1}{60k+5}, 2 : \frac{30k+2}{60k+5}, 2k : 1]$.

$m = 30k+9$ with $k \geq 0$, $f(30k+9) = \frac{10k+2}{20k+5}$

1. $(24k+6)$-$(\frac{10k+2}{20k+5}, \frac{10k+3}{20k+5})$; $(6k+3)$-(1).

2. 2-$[12k+3 : \frac{10k+3}{20k+5}]$; 3-$[8k+2 : \frac{10k+2}{20k+5}, 2k+1 : 1]$;

$m = 30k+14$ with $k \geq 0$, $f(30k+14) = \frac{30k+11}{60k+25}$

1. $(24k+10)$-$(\frac{30k+11}{60k+25}, \frac{30k+14}{60k+25})$, 2-$(\frac{30k+12}{60k+25}, \frac{30k+13}{60k+25})$, $(6k+2)$-(1).

2. 2-$[12k+5 : \frac{30k+14}{60k+25}]$, 2-$[8k+3 : \frac{30k+11}{60k+25}, \frac{30k+12}{60k+25}, 2k+1 : 1]$, $[8k+4 : \frac{30k+11}{60k+25}, 2 : \frac{30k+13}{60k+25}, 2k : 1]$.

$m = 30k+19$ with $k \geq 0$, $f(30k+19) = \frac{30k+16}{60k+35}$

1. $(24k+14)$-$(\frac{30k+16}{60k+35}, \frac{30k+19}{60k+35})$, 2-$(\frac{30k+17}{60k+35}, \frac{30k+18}{60k+35})$, $(6k+3)$-(1).

2. 2-$[12k+7 : \frac{30k+19}{60k+35}]$, 2-$[8k+5 : \frac{30k+16}{60k+35}, \frac{30k+18}{60k+35}, 2k+1 : 1]$, $[8k+4 : \frac{30k+16}{60k+35}, 2 : \frac{30k+17}{60k+35}, 2k+1 : 1]$.

$m = 30k+24$ with $k \geq 0$, $f(30k+24) = \frac{10k+7}{20k+15}$

1. $(24k+18)$-$(\frac{10k+7}{20k+15}, \frac{10k+8}{20k+15})$, $(6k+6)$-(1),

2. 2-$[12k+9 : \frac{10k+8}{20k+15}]$, 3-$[8k+6 : \frac{10k+7}{20k+15}, 2k+2 : 1]$,

$m = 30k+29$ with $k \geq 0$, $f(30k+29) = \frac{30k+26}{60k+55}$

1. $(24k+22)$-$(\frac{30k+26}{60k+55}, \frac{30k+29}{60k+55})$, 2-$(\frac{30k+27}{60k+55}, \frac{30k+28}{60k+55})$, $(6k+5)$-(1),

2. 2-$[12k+11 : \frac{30k+29}{60k+55}]$, 2-$[8k+7 : \frac{30k+26}{60k+55}, \frac{30k+27}{60k+55}, 2k+2 : 1]$, 1-$[8k+8 : \frac{30k+26}{60k+55}, 2 : \frac{30k+28}{60k+55}, 2k+1 : 1]$.

$\blacksquare$

There were a few cases that Theorem 17.7 did not cover. we cover them now and discuss why they were not covered.

Theorem 17.8

1. $f(7,5) = \frac{1}{3}$.

2. $f(11,5) = \frac{13}{30}$.

Proof:

1) $f(7,5) = \frac{1}{3}$.

By Theorem 6.1 $f(7,5) \leq \frac{1}{3}$. By the following protocol $f(7,5) \geq \frac{1}{3}$.

1. $(\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$; 6-$(\frac{7}{15}, \frac{8}{15})$.

2. 3-$[\frac{1}{3}, \frac{8}{15}, \frac{8}{15}]$; 2-$[\frac{7}{15}, \frac{7}{15}, \frac{7}{15}]$.

How come this result did not fall out of Theorem 17.7 Case 2? The protocol given in that case would imply $f(7,5) \geq \frac{3}{10}$ which is true but not optimal. We will see later in this paper that, for fixed s, the bound of Theorem 6.1 gives a nice form similar to the one in Theorem 17.7; however, that form does not work when $f(m,s) = \frac{1}{3}$.

2) $f(11,5) = \frac{13}{30}$.

By Theorem 7.2.STATEMENT3 with $V = 5$, $f(11,5) \leq \frac{13}{30}$. By the following protocol $f(11,5) \geq \frac{13}{30}$.

1. 6-$(\frac{13}{30}, \frac{17}{30})$; 4-$(\frac{9}{20}, \frac{11}{20})$; $(\frac{1}{2}, \frac{1}{2})$.

2. 2-$[\frac{17}{30}, \frac{17}{30}, \frac{17}{30}, \frac{1}{2}]$; 2-$[\frac{13}{30}, \frac{13}{30}, \frac{13}{30}, \frac{9}{20}, \frac{9}{20}]$; $[\frac{11}{20}, \frac{11}{20}, \frac{11}{20}, \frac{11}{20}]$.

How come this result did not fall out of Theorem 17.7? This is one of the rare cases where the upper bound from Theorem 7.2, $\frac{11}{25}$, is smaller than the upper bound from Theorem 6.1, $\frac{13}{30}$. Hence there could not be a protocol showing $f(11,5) \geq \frac{11}{25}$. Nevertheless, what goes wrong with the protocol given in the $m = 30k + 11$ case for $k = 0$? The protocol given require having -1 muffins split $(\frac{1}{2}, \frac{1}{2})$. This is, of course, impossible. ∎

17.6 $f(m,6)$

We first state and prove a theorem that tells us what $f(m,6)$ is for all but two values of $m \geq 5$. We then state and prove the theorem that covers the missing two cases. We also explain why they were missing.

Theorem 17.9 *If $m = 6k + i$ where $0 \leq i \leq 5$ then $f(m,6)$ depends only on k, i via a formula, given below, with 1 exceptions (we will note the exception).*

Case 0: $m = 6k + 0$ *with* $k \geq 1$. *Then* $f(6k,6) = 1$. *(This is by Theorem 4.1.1 so we won't prove it.)*

Case 1: $m = 6k+1$ *with* $k \geq 2$. *Then* $f(6k+1,6) = \frac{6k+1}{12k+6}$. *(Exception:* $f(7,6)$.*)*

Case 2: $m = 6k+2$ *with* $k \geq 1$. *Then* $f(6k+2,6) = f(3k+1,3) = \frac{3k-1}{6k}$.

Case 3: $m = 6k+3$ *with* $k \geq 1$. *Then* $f(6k+3,6) = f(2k+1,2) = \frac{1}{2}$.

Case 4: $m = 6k+4$ *with* $k \geq 1$. *Then* $f(6k+4,6) = f(3k+2,3) = \frac{3k+2}{6k+6}$.

Case 5: $m = 6k+5$ *with* $k \geq 1$. *Then* $f(6k+5,6) = \frac{6k+1}{12k+6}$.

Proof:

Case 1: $m = 6k+1$ with $k \geq 2$. Then $f(6k+1,6) = \frac{6k+1}{12k+6}$.

1. $4k+2$-$(\frac{6k+1}{12k+6}, \frac{6k+5}{12k+6})$, 2-$(\frac{6k+2}{12k+6}, \frac{6k+4}{12k+6})$, $2k-3$-$(\frac{1}{2}, \frac{1}{2})$.
2. 2-$[2k+1 : \frac{6k+1}{12k+6}]$, 2-$[k : \frac{6k+5}{12k+6} 1 : \frac{6k+4}{12k+6}, k-1 : \frac{1}{2}]$, 2-$[k+1 : \frac{6k+5}{12k+6} 1 : \frac{6k+2}{12k+6}, k-2 : \frac{1}{2}]$.

Case 2: By Theorem 4.1.3 $f(6k+2,6) \geq f(3k+1,3)$.

Let $FC(m,s)$ be the upper bound on $f(m,s)$ from Theorem 6.1 (we will use FC for something else later so this notation is only for this proof). Note that, for all $x \in \mathbb{N}$, $FC(m,s) = FC(mx,sx)$.

By Theorem 6.1

$$f(6k+2,6) \leq FC(6k+2,6) = FC(3k+1,3)$$

By Theorem 17.5 $FC(3k+1,3) = f(3k+1,3)$. Hence we have $f(6k+2) \leq f(3k+1,3)$.

Case 3,4: Similar to Case 2.

Case 5: $m = 6k+5$ with $k \geq 1$. Then $f(6k+5,6) = \frac{6k+1}{12k+6}$. There are cases mod 18.
$m = 18k+5$ with $k \geq 1$. $f(18k+5,6) = \frac{18k+1}{30k+6}$.

1. $12k+2$-$(\frac{18k+1}{36k+6}, \frac{18k+5}{36k+6})$, 2-$(\frac{18k+2}{36k+6}, \frac{18k+4}{36k+6})$, $6k+1$-$(\frac{1}{2}, \frac{1}{2})$.

2. 2-$[6k+1 : \frac{18k+5}{36k+6}]$, 2-$[3k : \frac{18k+1}{36k+6}, \frac{18k+2}{36k+6}, 3k+1 : \frac{1}{2}]$, 2-$[3k+1 : \frac{18k+1}{36k+6}, \frac{18k+4}{36k+6}, 3k : \frac{1}{2}]$.

$m = 18k+11$ with $k \geq 0$. $f(18k+11, 6) = \frac{18k+7}{36k+18}$.

1. $12k+6$-$(\frac{18k+7}{36k+18}, \frac{18k+11}{36k+18})$, 2-$(\frac{18k+8}{36k+18}, \frac{18k+10}{36k+18})$, $6k+3$-$(\frac{1}{2}, \frac{1}{2})$.

2. 2-$[6k+3 : \frac{18k+11}{36k+18}]$, 2-$[3k+1 : \frac{18k+7}{36k+18}, \frac{18k+8}{36k+18}, 3k+2 : \frac{1}{2}]$, 2-$[3k+2 : \frac{18k+7}{36k+18}, \frac{18k+10}{36k+18}, 3k+1 : \frac{1}{2}]$.

$m = 18k+17$ with $k \geq 0$. $f(18k+17, 6) = \frac{18+13}{36k+30}$.

1. $12k+10$-$(\frac{18k+13}{36k+30}, \frac{18k+17}{36k+30})$, 2-$(\frac{18k+14}{36k+30}, \frac{18k+16}{36k+30})$, $6k+5$-$(\frac{1}{2}, \frac{1}{2})$.

2. 2-$[6k+5 : \frac{18k+17}{36k+30}]$, 2-$[3k+2 : \frac{18k+13}{36k+30}, \frac{18k+14}{36k+30}, 3k+3 : \frac{1}{2}]$, 2-$[3k+3 : \frac{18k+13}{36k+30}, \frac{18k+16}{36k+30}, 3k+2 : \frac{1}{2}]$.

▮

Theorem 17.10 $f(7,6) = \frac{13}{36}$

Proof: This is obtained by the Buddy-Match Method. ▮

18 For fixed s, For Almost All m, $f(m,s) = FC(m,s)$

Recall Theorem 6.1:

Theorem 18.1 *Assume that $m, s \in \mathbb{N}$, $s < m$, and $\frac{m}{s} \notin \mathbb{N}$. Then*

$$f(m,s) \leq \max\left\{\frac{1}{3}, \min\left\{\frac{m}{s\lceil 2m/s\rceil}, 1 - \frac{m}{s\lfloor 2m/s\rfloor}\right\}\right\}.$$

In this section we will show that, for fixed s, for large enough m, the bound in Theorem 18.1 is $f(m,s)$.

Lemma 18.2 *If $m > 2s$ then*

$$\frac{1}{3} < \min\left\{\frac{m}{s\lceil 2m/s\rceil}, 1 - \frac{m}{s\lfloor 2m/s\rfloor}\right\}$$

Proof:

1) We show

$$\frac{1}{3} < \frac{m}{s\lceil 2m/s \rceil}.$$

This is equivalent to

$$\frac{m}{s\lceil 2m/s \rceil} < \frac{2}{3}.$$

Note that:

$$\lceil 2m/s \rceil < 2m/s + 1 = \frac{2m+s}{s}$$

$$\frac{m}{s\lceil 2m/s \rceil} > \frac{m}{2m+s}$$

Hence we need

$$\frac{m}{2m+s} > \frac{1}{3}$$

$$3m > 2m + s$$

$$m > s$$

2) We show $\frac{1}{3} < 1 - \frac{m}{s\lfloor 2m/s \rfloor}$

$$\lfloor 2m/s \rfloor > 2m/s - 1 = ((2m - s)/s)$$

$$\frac{m}{s\lfloor 2m/s \rfloor} < \frac{m}{2m-s}$$

So need

$$\frac{m}{2m-s} < \frac{2}{3}$$

$$3m < 4m - 2s$$

$$2s < m$$

▮

Using Lemma 18.2 and some notation that will come in handy later we restate Theorem 18.1

Theorem 18.3 *Let m, s be relatively prime such that $m > 2s$. Let $V = \left\lceil \frac{2m}{s} \right\rceil$. Note that $V \notin \mathbb{N}$ and hence $\left\lceil \frac{2m}{s} \right\rceil = V + 1$. Then*

$$f(m,s) \le \min\left\{ \frac{m}{s(V+1)}, 1 - \frac{m}{sV} \right\}$$

Notation 18.4 Let $V = \left\lfloor \frac{2m}{s} \right\rfloor$.

$$FC(m,s) = \min\left\{ \frac{m}{s(V+1)}, 1 - \frac{m}{sV} \right\}.$$

Note 18.5 Since our goal is to show $f(m,s) \ge FC(m,s)$, and with $m > 2s$ $FC(m,s) > \frac{1}{3}$, our protocols will never cut a muffin into ≥ 3 pieces. By our convention, our protocols never leave a muffin uncut. Hence every muffin will be cut into two pieces.

For the rest of this section:

- $s \geq 3$.
- $m > 2s$ and m, s are relatively prime.
- $V = \left\lfloor \frac{2m}{s} \right\rfloor$. (Each student will get either V or $V+1$ pieces.)

Let s_V (s_{V+1}) be how many students get V ($V+1$) pieces. Since every muffin is cut into two pieces there will be $2m$ total pieces. Hence

$$\begin{aligned} s_V + s_{V+1} &= s \\ Vs_V + (V+1)s_{V+1} &= 2m \end{aligned}$$

Algebra shows that:

- $s_V = s + Vs - 2m$
- $s_{V+1} = 2m - Vs$

18.1 Case I: $s_V > s_{V+1}$

We show that if $s_V > s_{V+1}$ and m is large enough then $f(m, s) = FC(m, s)$.

Let q, r be such that $(V+1)s_{V+1} = qs_V + r$ with $0 \leq r \leq s_V - 1$.

Lemma 18.6 *If* $m > \dfrac{s^2+s}{4}$ *and* $s_V > s_{V+1}$, *then* $\dfrac{m}{s(V+1)} \leq 1 - \dfrac{m}{sV}$.

Proof: By definition, $s_V > s_{V+1} \implies s + Vs - 2m > 2m - Vs$, which can be simplified to $\frac{2m}{s} < V + \frac{1}{2}$. Letting $\left\{\frac{2m}{s}\right\} = \frac{2m}{s} - \left\lfloor \frac{2m}{s} \right\rfloor$, $\left\{\frac{2m}{s}\right\} < \frac{1}{2}$. Since $\left\{\frac{2m}{s}\right\}$ is a fraction with integer numerator and denominator s, it can be at most $\frac{s-1}{2s}$. We have

$$\begin{aligned} m > \frac{s^2+s}{4} &\implies \frac{2m}{s} - 1 > \frac{s-1}{2} \\ &\implies V = \left\lfloor \frac{2m}{s} \right\rfloor > \frac{s-1}{2} \\ &\implies \frac{s-1}{2s} < \frac{V}{2V+1} \\ &\implies \left\{ \frac{2m}{s} \right\} < \frac{V}{2V+1} \\ &\implies \frac{m}{s} < \frac{\frac{2m}{s} - \left\{ \frac{2m}{s} \right\}}{2} + \frac{V}{4V+2} \\ &\implies \frac{m}{s} < \frac{V}{2} + \frac{V}{4V+2} \\ &\implies \frac{m}{s}\left(\frac{1}{V+1} + \frac{1}{V} \right) \le 1 \\ &\implies \boxed{\frac{m}{s(V+1)} \le 1 - \frac{m}{sV}} \end{aligned}$$

∎

We present a protocol that will, if m, s satisfy conditions to be named later (though they will include the premise of Lemma 18.6) yield $f(m,s) = FC(m,s)$.

1. Divide $s_{V+1}(V+1)$ muffins $\left[\frac{m}{s(V+1)}, 1 - \frac{m}{s(V+1)}\right]$. (Need $\frac{m}{s(V+1)} \le 1 - \frac{m}{sV}$.)
2. Divide $(s_V - r)r$ muffins $\left[\frac{1}{2} - \frac{1}{s_V}\left(\frac{1}{2} - \frac{m}{s(V+1)}\right), \frac{1}{2} + \frac{1}{s_V}\left(\frac{1}{2} - \frac{m}{s(V+1)}\right)\right]$
3. Divide $\frac{1}{2}(s_V(V-q-2r) + 2r^2 - r) = m - s_{V+1}(V+1) - (s_V - r)r$ muffins $(\frac{1}{2}, \frac{1}{2})$. (We will later see that this equality holds and does not need a condition on m, s.)
4. Give s_{V+1} students $(V+1 : \frac{m}{s(V+1)})$. (These students have $\frac{m}{s}$ muffins.)
5. Give $s_V - r$ students $\left(q : 1 - \frac{m}{s(V+1)}, r : \frac{1}{2} + \frac{1}{s_V}\left(\frac{1}{2} - \frac{m}{s(V+1)}\right), V - q - r : \frac{1}{2}\right)$ (Need $V - q - r \ge 0$.)

6. Give r students $\left(q+1 : 1-\frac{m}{s(V+1)}, s_V - r : \frac{1}{2} - \frac{1}{s_V}\left(\frac{1}{2} - \frac{m}{s(V+1)}\right), V-q-1-s_V+r : \frac{1}{2}\right)$. (Need $V - q - 1 - s_V + r \geq 0$.)

Claim 1: $\frac{1}{2}(s_V(V-q-2r)+2r^2-r) = m - s_{V+1}(V+1) - (s_V - r)r$.

Proof:

We give two proofs.

Proof 1: A Conceptual Approach

Consider steps 1,2,3 with step 3 dividing $m - s_{V+1}(V+1) - (s_V - r)r$ muffins into $(\frac{1}{2}, \frac{1}{2})$. Step three creates $2(m - s_{V+1}(V+1) - (s_V - r)r)$ pieces of size $\frac{1}{2}$.

Distribute all of the pieces as in steps 4, 5, and 6, except do not distribute the $\frac{1}{2}$ pieces yet. We can compute that the students in the s_V group still need $\frac{1}{2}(s_V(V-q-2r)+2r^2-r) = m - s_{V+1}(V+1) - (s_V - r)$ pieces of muffin, and nobody else needs any more pieces. After step 3, we have cut every muffin into two pieces. Thus, we have exactly enough pieces to give s_V students V pieces and s_{V+1} students $V+1$ pieces. We have computed already that we have $2(m - s_{V+1}(V+1) - (s_V - r)r)$ pieces left to give out, and that the students still need to receive $\frac{1}{2}(s_V(V-q-2r)+2r^2-r) = m - s_{V+1}(V+1) - (s_V - r)$, so those values must be equal.

Proof 2: An Algebraic Approach

It is clear by algebra that

$$V(s + Vs - 2m) - 2m + (V+1)(2m - Vs) = 0$$

By definition of s_V and s_{V+1},

$$\implies Vs_V - 2m + (V+1)s_{V+1} = 0$$

Since $qs_V + r = (V+1)s_{V+1}$,

$$\implies Vs_V - qs_V - r = 2m - 2s_{V+1}(V+1)$$

$$\implies Vs_V - qs_V - r - 2rs_V + 2r^2 = 2m - 2s_{V+1}(V+1) - 2rs_V + 2r^2$$

$$\implies \frac{1}{2}(s_V(V-q-2r) + 2r^2 - r) = m - s_{V+1}(V+1) - (s_V - r)r$$

End of Proof of Claim 1

Claim 2: Every student gets $\frac{m}{s}$.

Proof:

Clearly the s_{V+1} students will receive $\frac{m}{s}$ muffins. Thus if we distribute the remaining muffin evenly among the s_V students, they will each receive $\frac{m}{s}$ muffin also. We may compute

$$q\left(1 - \frac{m}{s(V+1)}\right) + r\left(\frac{1}{2} + \frac{1}{s_V}\left(\frac{1}{2} - \frac{m}{s(V+1)}\right)\right) + \frac{1}{2}(V-q-r)$$
$$- \left((q+1)\left(1 - \frac{m}{s(V+1)}\right) + (s_V - r)\left(\frac{1}{2} - \frac{1}{s_V}\left(\frac{1}{2} - \frac{m}{s(V+1)}\right)\right) + \frac{1}{2}(V-q-1-s_V+r)\right)$$
$$= \frac{m}{s(V+1)} - 1 + \left(\frac{1}{2} - \frac{m}{s(V+1)}\right) + \frac{1}{2}$$
$$= 0$$

So each student receives $\frac{m}{s}$.

End of Proof of Claim 2

Lemma 18.7 *If* $m \geq \dfrac{s^3 + 2s^2 + s}{2}$ *and* $s_V > s_{V+1}$*, then* $V-q-r \geq 0$ *and* $V-q-1-s+r \geq 0$ *are satisfied.*

Proof:

$$
\begin{aligned}
& s_V - 1 \geq s_{V+1} \text{ and } (V+1)s_{V+1} = qs_V + r \\
& \implies (V+1)(s_V - 1) \geq qs_V + r \\
& \implies V - q \geq \frac{r+V+1}{s_V} - 1
\end{aligned}
$$

Also,

$$
\begin{aligned}
m \geq \frac{s^3+2s^2+s}{2} \geq \frac{s^3+s^2+s}{2} & \implies \frac{2m}{s} - 1 \geq s^2 + s \\
& \implies V = \left\lfloor \frac{2m}{s} \right\rfloor \geq s^2 + s \\
& \implies V \geq s_V r + s_V \\
& \implies V \geq s_V r + s_V - r - 1 \\
& \implies \frac{r+V+1}{s_V} - 1 \geq r
\end{aligned}
$$

The two inequalities give us $\boxed{V - q - r \geq 0}$

Also,

$$
\begin{aligned}
m \geq \frac{s^3+2s^2+s}{2} & \implies \frac{2m}{s} - 1 \geq s^2 + 2s \\
& \implies V = \left\lfloor \frac{2m}{s} \right\rfloor \geq s^2 + 2s \\
& \implies V \geq ss_V + 2s_V \\
& \implies V \geq ss_V + 2s_V - rs_V - r - 1 \\
& \implies \frac{r+V+1}{s_V} - 2 - s + r \geq 0
\end{aligned}
$$

Thus, $\boxed{V - q - 1 - s + r \geq 0}$ so we are done. ∎

Putting this all together we have the following theorem.

Theorem 18.8 *If $s_V > s_{V+1}$ and $m \ge \frac{s^3+2s^2+s}{2}$ then $f(m,s) = FC(m,s)$.*

18.2 Case II: $s_V < s_{V+1}$

We show that if $s_V < s_{V+1}$ and m is large enough then $f(m,s) = FC(m,s)$.

Let q, r be such that $Vs_V = qs_{V+1} + r$ with $0 \le r \le s_{V+1} - 1$.

Lemma 18.9 *If $s_V < s_{V+1}$ then $\frac{m}{s(V+1)} \ge 1 - \frac{m}{sV}$.*

Proof: In fact, we will prove that $1 - \frac{m}{sV} \le \frac{m}{s(V+1)}$ if and only if $Vs_V \le (V+1)s_{V+1}$. Since $V < V+1$, the lemma follows.
Note that $(Vs_V)\left(\frac{m}{sV}\right) + ((V+1)s_{V+1})\left(\frac{m}{s(V+1)}\right) = m = \frac{1}{2}(Vs_V) + \frac{1}{2}(Vs_{V+1})$. Let $x = \frac{m}{sV} - \frac{1}{2}$ and let $y = \frac{1}{2} - \frac{m}{s(V+1)}$.
Then we have $(Vs_V)\left(\frac{1}{2}+x\right) + ((V+1)s_{V+1})\left(\frac{1}{2}-y\right) = \frac{1}{2}(Vs_V) + \frac{1}{2}(Vs_{V+1})$, so $(x)(Vs_V) = (y)((V+1)s_{V+1})$, so $\frac{x}{y} = \frac{(V+1)s_{V+1}}{Vs_V}$. The lemma follows. ∎

We present a protocol that will, if m, s satisfy conditions to be named later (though they will include the premise of Lemma 18.9) yield $f(m,s) = FC(m,s)$.

1. Divide $s_V(V)$ muffins $\left[\frac{m}{sV}, 1 - \frac{m}{sV}\right]$.
2. Divide $(s_{V+1} - r)r$ muffins $\left[\frac{1}{2} - \frac{1}{s_{V+1}}\left(\frac{1}{2} - \frac{m}{sV}\right), \frac{1}{2} + \frac{1}{s_{V+1}}\left(\frac{1}{2} - \frac{m}{sV}\right)\right]$.
3. Divide $\frac{1}{2}(s_{V+1}(V - q - 2r) + 2r^2 - s_{V+1} + r) = m - s_V(V) - (s_{V+1} - r)r$ muffins $(\frac{1}{2}, \frac{1}{2})$.
4. Give s_V students $(V : \frac{m}{sV})$. (These students have $\frac{m}{s}$ muffins.)
5. Give $s_{V+1} - r$ students $\left(q : 1 - \frac{m}{sV}, r : \frac{1}{2} + \frac{1}{s_{V+1}}\left(\frac{1}{2} - \frac{m}{sV}\right), V + 1 - q - r : \frac{1}{2}\right)$ (Need $V + 1 - q - r \ge 0$.)

6. Give r students $\left(q+1 : 1-\frac{m}{sV}, s_{V+1}-r : \frac{1}{2}-\frac{1}{s_{V+1}}\left(\frac{1}{2}-\frac{m}{sV}\right), V-q-s_{V+1}+r : \frac{1}{2}\right)$. (Need $V-q-s_{V+1}+r \geq 0$.)

Claims 1 and 2 below have proofs very similar to Claims 1 and 2 in Section 18.1.

Claim 1: $\frac{1}{2}(s_{V+1}(V-q-2r)+2r^2-s_{V+1}+r) = m - s_V(V) - (s_{V+1}-r)r$ is identical.

Claim 2: Every student gets $\frac{m}{s}$.

Theorem 18.10 *If* $m \geq \dfrac{s^3+s}{2}$ *and* $s_V < s_{V+1}$*, then* $V+1-q-r \geq 0$ *and* $V-q-s_{V+1}+r \geq 0$ *are satisfied.*

Proof: From Lemma 18.9, we know that $s_V < s_{V+1}$ implies Case 2. $s_V < s_{V+1}$ and $Vs_V = qs_{V+1}+r$

$$\begin{aligned} &\implies V(s_{V+1}-1) \geq qs_{V+1}+r \\ &\implies V-q \geq \frac{V+r}{s_{V+1}} \end{aligned}$$

Also,

$$\begin{aligned} m \geq \frac{s^3+s}{2} &\implies \frac{2m}{s}-1 \geq s^2 \\ &\implies V = \left\lfloor \frac{2m}{s} \right\rfloor \geq s^2 \\ &\implies V \geq rs_{V+1} \\ &\implies V \geq rs_{V+1}-s_{V+1}-r \\ &\implies \frac{V+r}{s_{V+1}} \geq r-1 \end{aligned}$$

Thus, $\boxed{V+1-q-r \geq 0}$

Also,

$$\begin{aligned} m \geq \frac{s^3+s}{2} &\implies \frac{2m}{s} - 1 \geq s^2 \\ &\implies V = \left\lfloor \frac{2m}{s} \right\rfloor \geq s^2 \\ &\implies V \geq (s_{V+1})^2 \\ &\implies V \geq (s_{V+1})^2 - rs_{V+1} - r \\ &\implies \frac{V+r}{s_{V+1}} \geq s_{V+1} - r \end{aligned}$$

The two inequalities give us $\boxed{V - q - s_{V+1} + r \geq 0}$ so we are done. ▮

Theorem 18.11 *If* $s_V < s_{V+1}$ *and* $m \geq \dfrac{s^3+s}{2}$ *then* $f(m,s) = FC(m,s)$.

18.3 $s_V = s_{V+1}$

Lemma 18.12 $s_V = s_{V+1} \implies s = 4$.

Proof: Assume $s_V = s_{V+1}$. Then $s + Vs - 2m = 2m - Vs$ so:

$$s + 2Vs = 4m \implies \frac{2m}{s} = V + \frac{1}{2}$$

So $\{\frac{2m}{s}\} = \frac{1}{2}$. But since $2m$ is even, s must be a multiple of 4. Letting $s = 4k$, $2m = 4k\left(V + \frac{1}{2}\right) = 2k(2V+1)$ so $m = k(2V+1)$. Therefore, (m,s) is of the form $(k[2V+1], 4k)$, and m, s relatively prime implies that $k = 1$ and $s = 4$, which we have proven in section 17.4. ▮

18.4 For almost all m, $f(m,s) = FC(m,s)$ and Has a Nice Form

Combining Theorem's 18.8 and 18.11, and stating the premises at the beginning of Section 18, we obtain

Theorem 18.13 *If $s \geq 3$, m, s are relatively prime, and $m \geq \dfrac{s^3+2s^2+s}{2}$ then $f(m,s) = FC(m,s)$.*

For large m, $FC(m,s)$ has a very nice form.

Theorem 18.14 *Let $s \geq 3$. There exists $\{a_i\}_{i=0}^{s-1}$, $\{b_i\}_{i=0}^{s-1}$, $\{c_i\}_{i=0}^{s-1}$, $\{d_i\}_{i=0}^{s-1}$ such that, for all $m \geq \dfrac{s^2+s}{4}$ if $m = ks+i$ with $0 \leq i \leq s-1$ then*

$$FC(m,s) = \frac{a_i k + b_i}{c_i k + d_i}$$

For all $m \geq \dfrac{s^3+2s^2+s}{2}$ $f(m,s)$ follows the formula in Part 1. (this follows from Part 1 and Theorem 18.13.

Fix s. Then $f(m,s)$ can be computed in $O(s^3 M(L)$ time where L is the length of $\lfloor m/s \rfloor$ and $M(L)$ is the time to multiply two L-bit numbers. Hence $f(m,s)$ is fixed parameter tractable. (By Part 1 $f(m,s)$ can be computed with a mod, 2 multiplications by constants, 2 additions, 1 division, with all number of magnitude $O(m/s)$. The Newton-Raphson division algorithm takes $O(M(L)$ time.

Proof: Given $m \geq \dfrac{s^2+s}{4}$ Lemma 18.6 and Lemma 18.9 show which of $\{\frac{m}{s(V+1)}, 1 - \frac{m}{sV}\}$ is smaller. It is easy to see whether $\{\frac{2m}{s}\} < \frac{1}{2}$, or whether equivalently $s_V > s_{V+1}$ (see proof of Lemma 1.6), for each i, and substituting $m = ks+i$ gives the following result:

Case 1: $0 \leq i \leq \lceil \frac{s}{4} \rceil - 1$. $FC(m,s) = \frac{sk+i}{2sk+s}$.

Case 2: $\lceil \frac{s}{4} \rceil \leq i \leq \lceil \frac{s}{2} \rceil - 1$. $FC(m,s) = \frac{sk-i}{2sk}$.

Case 3: $\lceil \frac{s}{2} \rceil \leq i \leq \lceil \frac{3s}{4} \rceil - 1$. $FC(m,s) = \frac{sk+i}{2sk+2s}$.

Case 4: $\lceil \frac{3s}{4} \rceil \leq i \leq s-1$. $FC(m,s) = \frac{sk+s-i}{2sk+s}$. ∎

19 Open Questions

The following conjectures are supported by the evidence in this paper.

All Techniques Conjecture:: For all $m \geq s+1$ $f(m,s) \in \{FC(m,s), INT(m,s), BM(m,s)\}$. If true then Theorems 6.1,7.2, and the buddy-match method cover all cases.

Ratio Conjecture: $f(m,s)$ is a function of $\frac{m}{s}$. This would follow from the **All Techniques Conjecture**

Complexity Conjecture: The problem of computing $f(m,s)$ is in P This would follow from the **All Techniques Conjecture**

For $1 \leq s \leq 50$, $s+1 \leq m \leq 60$ we list the open problems, the known upper and lower bounds, and the method used to get the upper bound. FC stands for Floor-Ceiling (Theorem 6.1), $DK{-}ONE$ ($DK{-}TWO$, $HALF{-}TWO$) is the subcase of the Interval Method (Theorem 7.2 with that name, BM is the Buddy-Match Methodology from Section 13, $ERIK$ is a very sophisticated argument by Erik Metz which we have not yet gotten into a general theorem.

M	S	LB	UB	UB-METHOD
41	19	980/2280	983/2280	ERIK
59	22	58/132	59/132	FC
41	23	27/69	28/69	FC
54	25	53/125	54/125	FC
59	26	44/104	45/104	FC
47	29	46/116	47/116	FC
49	30	48/120	49/120	FC
52	31	50/124	51/124	DK-TWO
55	31	37/93	38/93	FC
59	33	39/99	40/99	FC
55	34	54/136	55/136	FC
57	35	56/140	57/140	FC
47	36	24/72	25/72	FC
48	37	102/296	103/296	BM
50	41	58/164	59/164	DK-ONE
55	42	28/84	29/84	FC
53	43	45/129	46/129	DK-ONE
55	43	90/258	91/258	HALF-TWO
56	43	14/43	15/43	FC
59	45	30/90	31/90	FC

20 Acknowledgments

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A Appendix A: $f(m,s)$ Exists, is Rational, and Computable

Theorem A.1 *Let $m, s \geq 1$.*

1. *There is a mixed integer program with $O(ms)$ binary variables, $O(ms)$ real variables, $O(ms)$ constraints, and all coefficients integers of absolute value $\leq \max\{m,s\}$ such that, from the solution, one can extract $f(m,s)$ and a protocol that achieves this bound. This MIP can easily be obtained given m, s.*

2. *$f(m,s)$ is always rational. This follows from part 1.*

3. *In every optimal protocol for m muffins and s students all of the pieces are of rational size. This follows from part 1.*

4. *The problem of, given m, s, determine $f(m,s)$, is decidable. This follows from part 1.*

5.

Proof:

Consider the following (failed) attempt to solve the problem using linear programming.

1. The variables are x_{ij} where $1 \leq i \leq m$ and $1 \leq j \leq s$. The intent is that x_{ij} is the fraction of muffin i that student j gets.

2. For all $1 \leq i \leq m$, $1 \leq j \leq s$, $0 \leq x_{ij} \leq 1$.

3. For each $1 \le i \le m$, $\sum_{j=1}^{s} x_{ij} = 1$.

   This says that the amount of muffin i that student 1 gets, students 2 gets, ..., student s gets all adds up to 1.

4. For each $1 \le j \le s$, $\sum_{i=1}^{m} x_{ij} = \frac{m}{s}$.

   This says that the amount that student j gets from muffin 1, muffin 2, ..., muffin m all adds up to $\frac{m}{s}$.

5. For all $1 \le i \le m$, $1 \le j \le s$, $x_{ij} \ge z$.

6. Maximize z.

This does not work. The problem is that (say) x_{13} could be 0. In fact it is likely that some x_{ij} is 0. This makes $z = 0$. What we really want is

$$x_{ij} \ne 0 \implies x_{ij} \ge z$$

It is easy to show that $f(m,s) \ge \frac{1}{s}$. Hence every nonzero x_{ij} is $\ge \frac{1}{s}$. We will use this in our proof.

For $1 \le i \le m$, $1 \le j \le s$ modify the linear program above as follows.

1. Add variable y_{ij} which is in $\{0,1\}$.

2. Add the constraint $x_{ij} + y_{ij} \le 1$. Note that

   - $x_{ij} = 0 \implies x_{ij} + y_{ij} \le 1$, so the constraint imposes no condition on y_{ij}.
   - $x_{ij} > 0 \implies y_{ij} < 1 \implies y_{ij} = 0 \implies x_{ij} + y_{ij} = x_{ij}$.

3. Add the constraint $x_{ij} + y_{ij} \ge \frac{1}{s}$. Note that

   - $x_{ij} = 0 \implies y_{ij} \ge \frac{1}{s} \implies y_{ij} = 1 \implies x_{ij} + y_{ij} = 1$

- $x_{ij} > 0 \implies x_{ij} \geq \frac{1}{s}$ (since we know all non-zero pieces are $\geq \frac{1}{s}$) $\implies x_{ij}+y_{ij} \geq \frac{1}{s}$, so the constraint imposes no condition on y_{ij}.

4. Replace the constraint $z \leq x_{ij}$ with $z \leq x_{ij} + y_{ij}$.

If $x_{ij} = 0$ then the constraint

$$z \leq x_{ij} + y_{ij} = 1$$

is always met and hence is (as it should be) irrelevant. If $x_{ij} > 0$ then the constraint

$$z \leq x_{ij} + y_{ij} = x_{ij}$$

is the constraint we want.

Solve the resulting mixed integer program. Since all of the coefficients are rational the answer will be rational.

▮

B Appendix B: Conjectures for $s = 1$ to 50 and $m = s + 1$ to 100

For $1 \le s \le 50$ we make a conjecture that $f(m,s)$ is a nice mod-s formula (which is the same as the FC bound with one caveat) for $f(m,s)$ with some exceptions.

When there is not an exception, there are only 4 cases for $m \le 50$ where there is no matching lower bound. For higher m, our programs cannot reliably find a solution or determine lack of solution. These cases are: $f(41,23)$, $f(47,29)$, $f(47,36)$

For $1 \le s \le 50$ the conjecture is a nice mod-s formula (which is the same as the FC bound with one caveat) for $f(m,s)$ with some exceptions. We label the exceptions as follows

1. FC-exception is when the $f(m,s) = \frac{1}{3}$ which is what the FC formula gives. Note that the FC-formula has two cases- one is the normal case which results in a nice formula and one is when the answer is $\frac{1}{3}$. This occurred 130 times. Since $f(m,s) \ge \frac{1}{3}$ the lower bound always matched.

2. INT-exception is when (1) $INT(m,s) < FC(m,s)$ so the answer cannot be the formula for $FC(m,s)$, and (2) $INT(m,s) \le BM(m,s)$ so it could not be a BM-exception. This occurred 332 times. In all but one case ($f(50,41)$) when we had an INT-exception we had a matching lower bound. We do not know the value of $f(50,41)$.

3. BM-exception is when (1) $BM(m,s) < \min\{INT(m,s), FC(m,s)\}$ so the answer cannot be the formula for $FC(m,s)$ and this is not an INT-exception. This occurred 97 times. In all but one case ($f(48,37)$) when we had an BM-exception we had a matching lower bound. We do not know the value of $f(48,37)$.

4. ERIK-exception is when Erik Metz obtained an upper bound that was less than $\min\{BM(m,s), INT(m,s), FC(m,s)\}$. This happens four times. In all but one case ($f(41,19)$) when we had an ERIK-exception we had a matching lower bound. We do not know the value of $f(41,19)$.

Conjecture B.1 *For all m, $f(m,1) = 1$*

Conjecture B.2 *If $m = 2k$ then $f(m,2) = 1$. If $m = 2k+1$ then $f(m,2) = \frac{1}{2}$ For all m, $f(m,1) = 1$*

Conjecture B.3 *If $m = 3k+i$ where $0 \le i \le 2$ then $f(m,3)$ depends only on k,i via a formula, given below, with 1 exceptions (we will note the exceptions).*

Case 0: *$m = 3k+0$ with $k \ge 1$. Then $f(3k,3) = 1$.*

Case 1: *$m = 3k+1$ with $k \ge 2$. Then $f(3k+1,3) = \frac{3k-1}{6k}$.(Exception: $f(4,3) = 1/3$ FC-exception.)*

Case 2: *$m = 3k+2$ with $k \ge 1$. Then $f(3k+2,3) = \frac{3k+2}{6k+6}$.*

Conjecture B.4 *If $m = 4k+i$ where $0 \le i \le 3$ then $f(m,4)$ depends only on k,i via a formula, given below.*

Case 0: *$m = 4k+0$ with $k \ge 1$. Then $f(4k,4) = 1$.*

Case 1: *$m = 4k+1$ with $k \ge 1$. Then $f(4k+1,4) = \frac{4k-1}{8k}$.*

Case 3: *$m = 4k+3$ with $k \ge 1$. Then $f(4k+3,4) = \frac{4k+1}{8k+4}$.*

Conjecture B.5 *If $m = 5k+i$ where $0 \le i \le 4$ then $f(m,5)$ depends only on k,i via a formula, given below, with 2 exceptions (we will note the exceptions).*

Case 0: *$m = 5k+0$ with $k \ge 1$. Then $f(5k,5) = 1$.*

Case 1: *$m = 5k+1$ with $k \ge 1$. Then $f(5k+1,5) = \frac{5k+1}{10k+5}$.(Exception: $f(11,5) = 13/30$ INT-exception.)*

Case 2: *$m = 5k+2$ with $k \ge 2$. Then $f(5k+2,5) = \frac{5k-2}{10k}$.(Exception: $f(7,5) = 1/3$ FC-exception.)*

Case 3: $m = 5k + 3$ *with* $k \geq 1$. *Then* $f(5k + 3, 5) = \frac{5k+3}{10k+10}$.

Case 4: $m = 5k + 4$ *with* $k \geq 1$. *Then* $f(5k + 4, 5) = \frac{5k+1}{10k+5}$.

Conjecture B.6 *If* $m = 6k + i$ *where* $0 \leq i \leq 5$ *then* $f(m, 6)$ *depends only on* k, i *via a formula, given below, with 2 exceptions (we will note the exceptions).*

Case 0: $m = 6k + 0$ *with* $k \geq 1$. *Then* $f(6k, 6) = 1$.

Case 1: $m = 6k + 1$ *with* $k \geq 2$. *Then* $f(6k + 1, 6) = \frac{6k+1}{12k+6}$.*(Exception:* $f(7, 6) = 1/3$ *INT-exception.)*

Case 5: $m = 6k + 5$ *with* $k \geq 1$. *Then* $f(6k + 5, 6) = \frac{6k+1}{12k+6}$.

Conjecture B.7 *If* $m = 7k + i$ *where* $0 \leq i \leq 6$ *then* $f(m, 7)$ *depends only on* k, i *via a formula, given below, with 3 exceptions (we will note the exceptions).*

Case 0: $m = 7k + 0$ *with* $k \geq 1$. *Then* $f(7k, 7) = 1$.

Case 1: $m = 7k + 1$ *with* $k \geq 2$. *Then* $f(7k + 1, 7) = \frac{7k+1}{14k+7}$.*(Exception:* $f(8, 7) = 5/14$ *INT-exception.)*

Case 2: $m = 7k + 2$ *with* $k \geq 1$. *Then* $f(7k + 2, 7) = \frac{7k-2}{14k}$.

Case 3: $m = 7k + 3$ *with* $k \geq 2$. *Then* $f(7k + 3, 7) = \frac{7k-3}{14k}$.*(Exception:* $f(10, 7) = 1/3$ *FC-exception.)*

Case 4: $m = 7k + 4$ *with* $k \geq 1$. *Then* $f(7k + 4, 7) = \frac{7k+4}{14k+14}$.

Case 5: $m = 7k + 5$ *with* $k \geq 1$. *Then* $f(7k + 5, 7) = \frac{7k+5}{14k+14}$.*(Exception:* $f(19, 7) = 25/56$ *INT-exception.)*

Case 6: $m = 7k + 6$ *with* $k \geq 1$. *Then* $f(7k + 6, 7) = \frac{7k+1}{14k+7}$.

Conjecture B.8 *If $m = 8k + i$ where $0 \le i \le 7$ then $f(m,8)$ depends only on k, i via a formula, given below, with 1 exceptions (we will note the exceptions).*

Case 0: $m = 8k + 0$ *with* $k \ge 1$. *Then* $f(8k, 8) = 1$.

Case 1: $m = 8k + 1$ *with* $k \ge 1$. *Then* $f(8k + 1, 8) = \frac{8k+1}{16k+8}$.

Case 3: $m = 8k + 3$ *with* $k \ge 2$. *Then* $f(8k + 3, 8) = \frac{8k-3}{16k}$.*(Exception: $f(11, 8) = 1/3$ FC-exception.)*

Case 5: $m = 8k + 5$ *with* $k \ge 1$. *Then* $f(8k + 5, 8) = \frac{8k+5}{16k+16}$.

Case 7: $m = 8k + 7$ *with* $k \ge 1$. *Then* $f(8k + 7, 8) = \frac{8k+1}{16k+8}$.

Conjecture B.9 *If $m = 9k + i$ where $0 \le i \le 8$ then $f(m,9)$ depends only on k, i via a formula, given below, with 7 exceptions (we will note the exceptions).*

Case 0: $m = 9k + 0$ *with* $k \ge 1$. *Then* $f(9k, 9) = 1$.

Case 1: $m = 9k + 1$ *with* $k \ge 2$. *Then* $f(9k + 1, 9) = \frac{9k+1}{18k+9}$.*(Exception: $f(10, 9) = 1/3$ INT-exception.)*

Case 2: $m = 9k+2$ *with* $k \ge 2$. *Then* $f(9k+2, 9) = \frac{9k+2}{18k+9}$.*(Exception: $f(11, 9) = 13/36$ INT-exception, $f(29, 9) = 41/90$ INT-exception, $f(38, 9) = 59/126$ INT-exception, $f(47, 9) = 37/78$ INT-exception.)*

Case 4: $m = 9k + 4$ *with* $k \ge 2$. *Then* $f(9k + 4, 9) = \frac{9k-4}{18k}$.*(Exception: $f(13, 9) = 1/3$ FC-exception.)*

Case 5: $m = 9k + 5$ *with* $k \ge 1$. *Then* $f(9k + 5, 9) = \frac{9k+5}{18k+18}$.

Case 7: $m = 9k + 7$ *with* $k \ge 1$. *Then* $f(9k + 7, 9) = \frac{9k+2}{18k+9}$.

Case 8: $m = 9k + 8$ *with* $k \ge 1$. *Then* $f(9k + 8, 9) = \frac{9k+1}{18k+9}$.

Conjecture B.10 *If $m = 10k + i$ where $0 \le i \le 9$ then $f(m,10)$ depends only on k, i via a formula, given below, with 4 exceptions (we will note the exceptions).*

Case 0: $m = 10k + 0$ *with* $k \geq 1$. *Then* $f(10k, 10) = 1$.

Case 1: $m = 10k+1$ *with* $k \geq 2$. *Then* $f(10k+1, 10) = \frac{10k+1}{20k+10}$.*(Exception:* $f(11, 10) = 7/20$ *INT-exception.)*

Case 3: $m = 10k + 3$ *with* $k \geq 1$. *Then* $f(10k + 3, 10) = \frac{10k-3}{20k}$.

Case 7: $m = 10k + 7$ *with* $k \geq 2$. *Then* $f(10k + 7, 10) = \frac{10k+7}{20k+20}$.*(Exception:* $f(17, 10) = 2/5$ *INT-exception.)*

Case 9: $m = 10k + 9$ *with* $k \geq 1$. *Then* $f(10k + 9, 10) = \frac{10k+1}{20k+10}$.

Conjecture B.11 *If* $m = 11k + i$ *where* $0 \leq i \leq 10$ *then* $f(m, 11)$ *depends only on* k, i *via a formula, given below, with 7 exceptions (we will note the exceptions).*

Case 0: $m = 11k + 0$ *with* $k \geq 1$. *Then* $f(11k, 11) = 1$.

Case 1: $m = 11k + 1$ *with* $k \geq 1$. *Then* $f(11k + 1, 11) = \frac{11k+1}{22k+11}$.

Case 2: $m = 11k + 2$ *with* $k \geq 3$. *Then* $f(11k + 2, 11) = \frac{11k+2}{22k+11}$.*(Exception:* $f(13, 11) = 1/3$ *INT-exception,* $f(24, 11) = 19/44$ *INT-exception.)*

Case 3: $m = 11k + 3$ *with* $k \geq 1$. *Then* $f(11k + 3, 11) = \frac{11k-3}{22k}$.

Case 4: $m = 11k + 4$ *with* $k \geq 2$. *Then* $f(11k + 4, 11) = \frac{11k-4}{22k}$.*(Exception:* $f(15, 11) = 1/3$ *FC-exception.)*

Case 5: $m = 11k + 5$ *with* $k \geq 2$. *Then* $f(11k + 5, 11) = \frac{11k-5}{22k}$.*(Exception:* $f(16, 11) = 1/3$ *FC-exception.)*

Case 6: $m = 11k + 6$ *with* $k \geq 1$. *Then* $f(11k + 6, 11) = \frac{11k+6}{22k+22}$.

Case 7: $m = 11k + 7$ *with* $k \geq 1$. *Then* $f(11k + 7, 11) = \frac{11k+7}{22k+22}$.

Case 8: $m = 11k+8$ *with* $k \geq 2$. *Then* $f(11k+8, 11) = \frac{11k+8}{22k+22}$.*(Exception:* $f(19, 11) = 9/22$ *INT-exception,* $f(41, 11) = 61/132$ *INT-exception,* $f(52, 11) = 83/176$ *INT-exception.)*

Case 9: $m = 11k + 9$ *with* $k \geq 1$. *Then* $f(11k + 9, 11) = \frac{11k+2}{22k+11}$.

Case 10: $m = 11k + 10$ *with* $k \geq 1$. *Then* $f(11k + 10, 11) = \frac{11k+1}{22k+11}$.

Conjecture B.12 *If $m = 12k + i$ where $0 \le i \le 11$ then $f(m, 12)$ depends only on k, i via a formula, given below, with 4 exceptions (we will note the exceptions).*

Case 0: $m = 12k + 0$ *with* $k \ge 1$. *Then* $f(12k, 12) = 1$.

Case 1: $m = 12k + 1$ *with* $k \ge 2$. *Then* $f(12k + 1, 12) = \frac{12k+1}{24k+12}$.*(Exception:* $f(13, 12) = 1/3$ *INT-exception.)*

Case 5: $m = 12k + 5$ *with* $k \ge 2$. *Then* $f(12k + 5, 12) = \frac{12k-5}{24k}$.*(Exception:* $f(17, 12) = 1/3$ *FC-exception.)*

Case 7: $m = 12k + 7$ *with* $k \ge 1$. *Then* $f(12k + 7, 12) = \frac{12k+7}{24k+24}$.

Case 11: $m = 12k + 11$ *with* $k \ge 1$. *Then* $f(12k + 11, 12) = \frac{12k+1}{24k+12}$.

Conjecture B.13 *If $m = 13k + i$ where $0 \le i \le 12$ then $f(m, 13)$ depends only on k, i via a formula, given below, with 13 exceptions (we will note the exceptions).*

Case 0: $m = 13k + 0$ *with* $k \ge 1$. *Then* $f(13k, 13) = 1$.

Case 1: $m = 13k+1$ *with* $k \ge 2$. *Then* $f(13k+1, 13) = \frac{13k+1}{26k+13}$.*(Exception:* $f(14, 13) = 9/26$ *INT-exception.)*

Case 2: $m = 13k+2$ *with* $k \ge 2$. *Then* $f(13k+2, 13) = \frac{13k+2}{26k+13}$.*(Exception:* $f(15, 13) = 9/26$ *INT-exception.)*

Case 3: $m = 13k + 3$ *with* $k \ge 3$. *Then* $f(13k + 3, 13) = \frac{13k+3}{26k+13}$.*(Exception:* $f(16, 13) = 14/39$ *INT-exception,* $f(29, 13) = 45/104$ *INT-exception,* $f(55, 13) = 85/182$ *INT-exception,* $f(68, 13) = 37/78$ *INT-exception,* $f(81, 13) = 137/286$ *INT-exception,* $f(94, 13) = 119/247$ *INT-exception.)*

Case 4: $m = 13k + 4$ *with* $k \ge 1$. *Then* $f(13k + 4, 13) = \frac{13k-4}{26k}$.

Case 5: $m = 13k + 5$ *with* $k \ge 2$. *Then* $f(13k + 5, 13) = \frac{13k-5}{26k}$.*(Exception:* $f(18, 13) = 1/3$ *FC-exception.)*

Case 6: $m = 13k+6$ *with* $k \geq 2$. *Then* $f(13k+6, 13) = \frac{13k-6}{26k}$.*(Exception:* $f(19, 13) = 1/3$ *FC-exception.)*

Case 7: $m = 13k+7$ *with* $k \geq 1$. *Then* $f(13k+7, 13) = \frac{13k+7}{26k+26}$.

Case 8: $m = 13k+8$ *with* $k \geq 1$. *Then* $f(13k+8, 13) = \frac{13k+8}{26k+26}$.

Case 9: $m = 13k+9$ *with* $k \geq 3$. *Then* $f(13k+9, 13) = \frac{13k+9}{26k+26}$.*(Exception:* $f(22, 13) = 21/52$ *INT-exception,* $f(35, 13) = 64/143$ *INT-exception.)*

Case 10: $m = 13k+10$ *with* $k \geq 2$. *Then* $f(13k+10, 13) = \frac{13k+3}{26k+13}$.*(Exception:* $f(23, 13) = 53/130$ *INT-exception.)*

Case 11: $m = 13k+11$ *with* $k \geq 1$. *Then* $f(13k+11, 13) = \frac{13k+2}{26k+13}$.

Case 12: $m = 13k+12$ *with* $k \geq 1$. *Then* $f(13k+12, 13) = \frac{13k+1}{26k+13}$.

Conjecture B.14 *If* $m = 14k+i$ *where* $0 \leq i \leq 13$ *then* $f(m, 14)$ *depends only on* k, i *via a formula, given below, with 9 exceptions (we will note the exceptions).*

Case 0: $m = 14k+0$ *with* $k \geq 1$. *Then* $f(14k, 14) = 1$.

Case 1: $m = 14k+1$ *with* $k \geq 1$. *Then* $f(14k+1, 14) = \frac{14k+1}{28k+14}$.

Case 3: $m = 14k+3$ *with* $k \geq 5$. *Then* $f(14k+3, 14) = \frac{14k+3}{28k+14}$.*(Exception:* $f(17, 14) = 5/14$ *INT-exception,* $f(31, 14) = 3/7$ *INT-exception,* $f(45, 14) = 16/35$ *INT-exception,* $f(59, 14) = 131/280$ *INT-exception.)*

Case 5: $m = 14k+5$ *with* $k \geq 2$. *Then* $f(14k+5, 14) = \frac{14k-5}{28k}$.*(Exception:* $f(19, 14) = 1/3$ *FC-exception.)*

Case 9: $m = 14k+9$ *with* $k \geq 2$. *Then* $f(14k+9, 14) = \frac{14k+9}{28k+28}$.*(Exception:* $f(23, 14) = 17/42$ *INT-exception.)*

Case 11: $m = 14k+11$ *with* $k \geq 1$. *Then* $f(14k+11, 14) = \frac{14k+3}{28k+14}$.

Case 13: $m = 14k+13$ *with* $k \geq 1$. *Then* $f(14k+13, 14) = \frac{14k+1}{28k+14}$.

Conjecture B.15 *If $m = 15k + i$ where $0 \le i \le 14$ then $f(m, 15)$ depends only on k, i via a formula, given below, with 11 exceptions (we will note the exceptions).*

Case 0: $m = 15k + 0$ *with* $k \ge 1$. *Then* $f(15k, 15) = 1$.

Case 1: $m = 15k+1$ *with* $k \ge 2$. *Then* $f(15k+1, 15) = \frac{15k+1}{30k+15}$.*(Exception:* $f(16, 15) = 12/35$ *BM-exception.)*

Case 2: $m = 15k+2$ *with* $k \ge 2$. *Then* $f(15k+2, 15) = \frac{15k+2}{30k+15}$.*(Exception:* $f(17, 15) = 7/20$ *INT-exception.)*

Case 4: $m = 15k+4$ *with* $k \ge 2$. *Then* $f(15k+4, 15) = \frac{15k-4}{30k}$.*(Exception:* $f(19, 15) = 7/20$ *INT-exception.)*

Case 7: $m = 15k+7$ *with* $k \ge 2$. *Then* $f(15k+7, 15) = \frac{15k-7}{30k}$.*(Exception:* $f(22, 15) = 1/3$ *FC-exception.)*

Case 8: $m = 15k+8$ *with* $k \ge 1$. *Then* $f(15k+8, 15) = \frac{15k+8}{30k+30}$.

Case 11: $m = 15k+11$ *with* $k \ge 3$. *Then* $f(15k+11, 15) = \frac{15k+11}{30k+30}$.*(Exception:* $f(26, 15) = 37/90$ *INT-exception,* $f(41, 15) = 67/150$ *INT-exception,* $f(71, 15) = 113/240$ *INT-exception,* $f(86, 15) = 143/300$ *INT-exception.)*

Case 13: $m = 15k+13$ *with* $k \ge 1$. *Then* $f(15k+13, 15) = \frac{15k+2}{30k+15}$.

Case 14: $m = 15k+14$ *with* $k \ge 1$. *Then* $f(15k+14, 15) = \frac{15k+1}{30k+15}$.

Conjecture B.16 *If $m = 16k + i$ where $0 \le i \le 15$ then $f(m, 16)$ depends only on k, i via a formula, given below, with 6 exceptions (we will note the exceptions).*

Case 0: $m = 16k + 0$ *with* $k \ge 1$. *Then* $f(16k, 16) = 1$.

Case 1: $m = 16k+1$ *with* $k \ge 2$. *Then* $f(16k+1, 16) = \frac{16k+1}{32k+16}$.*(Exception:* $f(17, 16) = 11/32$ *BM-exception.)*

Case 3: $m = 16k+3$ *with* $k \ge 2$. *Then* $f(16k+3, 16) = \frac{16k+3}{32k+16}$.*(Exception:* $f(19, 16) = 1/3$ *INT-exception.)*

Case 5: $m = 16k+5$ *with* $k \geq 1$. *Then* $f(16k+5, 16) = \frac{16k-5}{32k}$.

Case 7: $m = 16k+7$ *with* $k \geq 2$. *Then* $f(16k+7, 16) = \frac{16k-7}{32k}$.*(Exception:* $f(23, 16) = 1/3$ *FC-exception.)*

Case 9: $m = 16k+9$ *with* $k \geq 1$. *Then* $f(16k+9, 16) = \frac{16k+9}{32k+32}$.

Case 11: $m = 16k+11$ *with* $k \geq 3$. *Then* $f(16k+11, 16) = \frac{16k+11}{32k+32}$.*(Exception:* $f(27, 16) = 13/32$ *INT-exception,* $f(43, 16) = 25/56$ *INT-exception.)*

Case 13: $m = 16k+13$ *with* $k \geq 1$. *Then* $f(16k+13, 16) = \frac{16k+3}{32k+16}$.

Case 15: $m = 16k+15$ *with* $k \geq 1$. *Then* $f(16k+15, 16) = \frac{16k+1}{32k+16}$.

Conjecture B.17 *If* $m = 17k+i$ *where* $0 \leq i \leq 16$ *then* $f(m, 17)$ *depends only on* k, i *via a formula, given below, with 13 exceptions (we will note the exceptions).*

Case 0: $m = 17k+0$ *with* $k \geq 1$. *Then* $f(17k, 17) = 1$.

Case 1: $m = 17k+1$ *with* $k \geq 1$. *Then* $f(17k+1, 17) = \frac{17k+1}{34k+17}$.

Case 2: $m = 17k+2$ *with* $k \geq 2$. *Then* $f(17k+2, 17) = \frac{17k+2}{34k+17}$.*(Exception:* $f(19, 17) = 1/3$ *INT-exception.)*

Case 3: $m = 17k+3$ *with* $k \geq 3$. *Then* $f(17k+3, 17) = \frac{17k+3}{34k+17}$.*(Exception:* $f(20, 17) = 1/3$ *INT-exception,* $f(37, 17) = 59/136$ *INT-exception.)*

Case 4: $m = 17k+4$ *with* $k \geq 4$. *Then* $f(17k+4, 17) = \frac{17k+4}{34k+17}$.*(Exception:* $f(21, 17) = 6/17$ *INT-exception,* $f(38, 17) = 59/136$ *INT-exception,* $f(55, 17) = 31/68$ *INT-exception,* $f(89, 17) = 145/306$ *INT-exception.)*

Case 5: $m = 17k+5$ *with* $k \geq 1$. *Then* $f(17k+5, 17) = \frac{17k-5}{34k}$.

Case 6: $m = 17k+6$ *with* $k \geq 2$. *Then* $f(17k+6, 17) = \frac{17k-6}{34k}$.*(Exception:* $f(23, 17) = 1/3$ *FC-exception.)*

Case 7: $m = 17k+7$ *with* $k \geq 2$. *Then* $f(17k+7, 17) = \frac{17k-7}{34k}$.*(Exception:* $f(24, 17) = 1/3$ *FC-exception.)*

Case 8: $m = 17k+8$ *with* $k \geq 2$. *Then* $f(17k+8, 17) = \frac{17k-8}{34k}$.*(Exception:* $f(25,17) = 1/3$ *FC-exception.)*

Case 9: $m = 17k+9$ *with* $k \geq 1$. *Then* $f(17k+9, 17) = \frac{17k+9}{34k+34}$.

Case 10: $m = 17k+10$ *with* $k \geq 1$. *Then* $f(17k+10, 17) = \frac{17k+10}{34k+34}$.

Case 11: $m = 17k+11$ *with* $k \geq 1$. *Then* $f(17k+11, 17) = \frac{17k+11}{34k+34}$.

Case 12: $m = 17k+12$ *with* $k \geq 4$. *Then* $f(17k+12, 17) = \frac{17k+12}{34k+34}$.*(Exception:* $f(29,17) = 27/68$ *BM-exception,* $f(46,17) = 61/136$ *INT-exception,* $f(63,17) = 173/374$ *INT-exception.)*

Case 13: $m = 17k+13$ *with* $k \geq 1$. *Then* $f(17k+13, 17) = \frac{17k+4}{34k+17}$.*(Exception:* $f(47,17) = 91/204$ *ERIK-exception.)*

Case 14: $m = 17k+14$ *with* $k \geq 1$. *Then* $f(17k+14, 17) = \frac{17k+3}{34k+17}$.

Case 15: $m = 17k+15$ *with* $k \geq 1$. *Then* $f(17k+15, 17) = \frac{17k+2}{34k+17}$.

Case 16: $m = 17k+16$ *with* $k \geq 1$. *Then* $f(17k+16, 17) = \frac{17k+1}{34k+17}$.

Conjecture B.18 *If* $m = 18k+i$ *where* $0 \leq i \leq 17$ *then* $f(m, 18)$ *depends only on* k, i *via a formula, given below, with 14 exceptions (we will note the exceptions).*

Case 0: $m = 18k+0$ *with* $k \geq 1$. *Then* $f(18k, 18) = 1$.

Case 1: $m = 18k+1$ *with* $k \geq 2$. *Then* $f(18k+1, 18) = \frac{18k+1}{36k+18}$.*(Exception:* $f(19,18) = 43/126$ *BM-exception.)*

Case 5: $m = 18k+5$ *with* $k \geq 2$. *Then* $f(18k+5, 18) = \frac{18k-5}{36k}$.*(Exception:* $f(23,18) = 19/54$ *INT-exception.)*

Case 7: $m = 18k+7$ *with* $k \geq 2$. *Then* $f(18k+7, 18) = \frac{18k-7}{36k}$.*(Exception:* $f(25,18) = 1/3$ *FC-exception.)*

Case 11: $m = 18k+11$ *with* $k \geq 1$. *Then* $f(18k+11, 18) = \frac{18k+11}{36k+36}$.

Case 13: $m = 18k+13$ *with* $k \geq 4$. *Then* $f(18k+13, 18) = \frac{18k+13}{36k+36}$.*(Exception:* $f(31,18) = 11/27$ *INT-exception,* $f(49,18) = 4/9$ *INT-exception,* $f(67,18) = 25/54$ *INT-exception.)*

Case 17: $m = 18k + 17$ *with* $k \geq 1$. *Then* $f(18k + 17, 18) = \frac{18k+1}{36k+18}$.

Conjecture B.19 *If* $m = 19k + i$ *where* $0 \leq i \leq 18$ *then* $f(m, 19)$ *depends only on* k, i *via a formula, given below, with 16 exceptions (we will note the exceptions).*

Case 0: $m = 19k + 0$ *with* $k \geq 1$. *Then* $f(19k, 19) = 1$.

Case 1: $m = 19k+1$ *with* $k \geq 2$. *Then* $f(19k+1, 19) = \frac{19k+1}{38k+19}$.*(Exception:* $f(20, 19) = 13/38$ *BM-exception.)*

Case 2: $m = 19k+2$ *with* $k \geq 2$. *Then* $f(19k+2, 19) = \frac{19k+2}{38k+19}$.*(Exception:* $f(21, 19) = 13/38$ *INT-exception.)*

Case 3: $m = 19k+3$ *with* $k \geq 2$. *Then* $f(19k+3, 19) = \frac{19k+3}{38k+19}$.*(Exception:* $f(22, 19) = 13/38$ *INT-exception,* $81/90 \leq f(41, 19) < 983/2280$ *ERIK-exception[OPEN].)*

Case 4: $m = 19k+4$ *with* $k \geq 5$. *Then* $f(19k+4, 19) = \frac{19k+4}{38k+19}$.*(Exception:* $f(23, 19) = 27/76$ *INT-exception,* $f(42, 19) = 49/114$ *INT-exception,* $f(61, 19) = 313/684$ *INT-exception,* $f(80, 19) = 231/494$ *INT-exception.)*

Case 5: $m = 19k+5$ *with* $k \geq 3$. *Then* $f(19k+5, 19) = \frac{19k-5}{38k}$.*(Exception:* $f(24, 19) = 27/76$ *INT-exception,* $f(43, 19) = 115/266$ *INT-exception.)*

Case 6: $m = 19k + 6$ *with* $k \geq 1$. *Then* $f(19k + 6, 19) = \frac{19k-6}{38k}$.

Case 7: $m = 19k + 7$ *with* $k \geq 2$. *Then* $f(19k + 7, 19) = \frac{19k-7}{38k}$.*(Exception:* $f(26, 19) = 1/3$ *FC-exception.)*

Case 8: $m = 19k + 8$ *with* $k \geq 2$. *Then* $f(19k + 8, 19) = \frac{19k-8}{38k}$.*(Exception:* $f(27, 19) = 1/3$ *FC-exception.)*

Case 9: $m = 19k + 9$ *with* $k \geq 2$. *Then* $f(19k + 9, 19) = \frac{19k-9}{38k}$.*(Exception:* $f(28, 19) = 1/3$ *FC-exception.)*

Case 10: $m = 19k + 10$ *with* $k \geq 1$. *Then* $f(19k + 10, 19) = \frac{19k+10}{38k+38}$.

Case 11: $m = 19k + 11$ *with* $k \geq 1$. *Then* $f(19k + 11, 19) = \frac{19k+11}{38k+38}$.

Case 12: $m = 19k + 12$ *with* $k \geq 1$. *Then* $f(19k + 12, 19) = \frac{19k+12}{38k+38}$.*(Exception:* $f(31, 19) = 54/133$ *ERIK-exception.)*

Case 13: $m = 19k + 13$ *with* $k \geq 2$. *Then* $f(19k + 13, 19) = \frac{19k+13}{38k+38}$.*(Exception:* $f(32, 19) = 31/76$ *INT-exception.)*

Case 14: $m = 19k + 14$ *with* $k \geq 4$. *Then* $f(19k + 14, 19) = \frac{19k+14}{38k+38}$.*(Exception:* $f(33, 19) = 47/114$ *INT-exception,* $f(52, 19) = 17/38$ *INT-exception,* $f(71, 19) = 123/266$ *INT-exception.)*

Case 15: $m = 19k + 15$ *with* $k \geq 1$. *Then* $f(19k + 15, 19) = \frac{19k+4}{38k+19}$.

Case 16: $m = 19k + 16$ *with* $k \geq 1$. *Then* $f(19k + 16, 19) = \frac{19k+3}{38k+19}$.

Case 17: $m = 19k + 17$ *with* $k \geq 1$. *Then* $f(19k + 17, 19) = \frac{19k+2}{38k+19}$.

Case 18: $m = 19k + 18$ *with* $k \geq 1$. *Then* $f(19k + 18, 19) = \frac{19k+1}{38k+19}$.

Conjecture B.20 *If* $m = 20k + i$ *where* $0 \leq i \leq 19$ *then* $f(m, 20)$ *depends only on* k, i *via a formula, given below, with 8 exceptions (we will note the exceptions).*

Case 0: $m = 20k + 0$ *with* $k \geq 1$. *Then* $f(20k, 20) = 1$.

Case 1: $m = 20k + 1$ *with* $k \geq 1$. *Then* $f(20k + 1, 20) = \frac{20k+1}{40k+20}$.

Case 3: $m = 20k+3$ *with* $k \geq 2$. *Then* $f(20k+3, 20) = \frac{20k+3}{40k+20}$.*(Exception:* $f(23, 20) = 7/20$ *INT-exception.)*

Case 7: $m = 20k + 7$ *with* $k \geq 2$. *Then* $f(20k + 7, 20) = \frac{20k-7}{40k}$.*(Exception:* $f(27, 20) = 1/3$ *FC-exception.)*

Case 9: $m = 20k + 9$ *with* $k \geq 2$. *Then* $f(20k + 9, 20) = \frac{20k-9}{40k}$.*(Exception:* $f(29, 20) = 1/3$ *FC-exception.)*

Case 11: $m = 20k + 11$ *with* $k \geq 1$. *Then* $f(20k + 11, 20) = \frac{20k+11}{40k+40}$.

Case 13: $m = 20k + 13$ *with* $k \geq 2$. *Then* $f(20k + 13, 20) = \frac{20k+13}{40k+40}$.*(Exception:* $f(33, 20) = 49/120$ *INT-exception.)*

Case 17: $m = 20k + 17$ *with* $k \geq 1$. *Then* $f(20k + 17, 20) = \frac{20k+3}{40k+20}$.

Case 19: $m = 20k + 19$ *with* $k \geq 1$. *Then* $f(20k + 19, 20) = \frac{20k+1}{40k+20}$.

Conjecture B.21 *If* $m = 21k + i$ *where* $0 \leq i \leq 20$ *then* $f(m, 21)$ *depends only on* k, i *via a formula, given below, with 16 exceptions (we will note the exceptions).*

Case 0: $m = 21k + 0$ *with* $k \geq 1$. *Then* $f(21k, 21) = 1$.

Case 1: $m = 21k + 1$ *with* $k \geq 2$. *Then* $f(21k + 1, 21) = \frac{21k+1}{42k+21}$.*(Exception:* $f(22, 21) = 50/147$ *BM-exception.)*

Case 2: $m = 21k+2$ *with* $k \geq 2$. *Then* $f(21k+2, 21) = \frac{21k+2}{42k+21}$.*(Exception:* $f(23, 21) = 29/84$ *INT-exception.)*

Case 4: $m = 21k + 4$ *with* $k \geq 3$. *Then* $f(21k + 4, 21) = \frac{21k+4}{42k+21}$.*(Exception:* $f(25, 21) = 1/3$ *INT-exception,* $f(46, 21) = 73/168$ *INT-exception.)*

Case 5: $m = 21k+5$ *with* $k \geq 4$. *Then* $f(21k+5, 21) = \frac{21k+5}{42k+21}$.*(Exception:* $f(26, 21) = 22/63$ *INT-exception,* $f(47, 21) = 73/168$ *INT-exception,* $f(68, 21) = 115/252$ *INT-exception,* $f(89, 21) = 157/336$ *INT-exception.)*

Case 8: $m = 21k + 8$ *with* $k \geq 2$. *Then* $f(21k + 8, 21) = \frac{21k-8}{42k}$.*(Exception:* $f(29, 21) = 1/3$ *FC-exception.)*

Case 10: $m = 21k + 10$ *with* $k \geq 2$. *Then* $f(21k + 10, 21) = \frac{21k-10}{42k}$.*(Exception:* $f(31, 21) = 1/3$ *FC-exception.)*

Case 11: $m = 21k + 11$ *with* $k \geq 1$. *Then* $f(21k + 11, 21) = \frac{21k+11}{42k+42}$.

Case 13: $m = 21k + 13$ *with* $k \geq 1$. *Then* $f(21k + 13, 21) = \frac{21k+13}{42k+42}$.

Case 16: $m = 21k + 16$ *with* $k \geq 3$. *Then* $f(21k + 16, 21) = \frac{21k+5}{42k+21}$.*(Exception:* $f(37, 21) = 103/252$ *INT-exception,* $f(58, 21) = 169/378$ *INT-exception.)*

Case 17: $m = 21k + 17$ *with* $k \geq 1$. *Then* $f(21k + 17, 21) = \frac{21k+4}{42k+21}$.

Case 19: $m = 21k + 19$ *with* $k \geq 1$. *Then* $f(21k + 19, 21) = \frac{21k+2}{42k+21}$.

Case 20: $m = 21k + 20$ *with* $k \geq 1$. *Then* $f(21k + 20, 21) = \frac{21k+1}{42k+21}$.

Conjecture B.22 *If $m = 22k + i$ where $0 \le i \le 21$ then $f(m, 22)$ depends only on k, i via a formula, given below, with 14 exceptions (we will note the exceptions).*

Case 0: $m = 22k + 0$ *with* $k \ge 1$. *Then* $f(22k, 22) = 1$.

Case 1: $m = 22k+1$ *with* $k \ge 2$. *Then* $f(22k+1, 22) = \frac{22k+1}{44k+22}$.*(Exception:* $f(23, 22) = 15/44$ *BM-exception.)*

Case 3: $m = 22k+3$ *with* $k \ge 2$. *Then* $f(22k+3, 22) = \frac{22k+3}{44k+22}$.*(Exception:* $f(25, 22) = 23/66$ *BM-exception.)*

Case 5: $m = 22k + 5$ *with* $k \ge 4$. *Then* $f(22k + 5, 22) = \frac{22k+5}{44k+22}$.*(Exception:* $f(27, 22) = 4/11$ *INT-exception,* $f(49, 22) = 19/44$ *INT-exception,* $f(71, 22) = 5/11$ *INT-exception,* $f(93, 22) = 36/77$ *INT-exception.)*

Case 7: $m = 22k + 7$ *with* $k \ge 1$. *Then* $f(22k + 7, 22) = \frac{22k-7}{44k}$.

Case 9: $m = 22k + 9$ *with* $k \ge 2$. *Then* $f(22k + 9, 22) = \frac{22k-9}{44k}$.*(Exception:* $f(31, 22) = 1/3$ *FC-exception.)*

Case 13: $m = 22k + 13$ *with* $k \ge 1$. *Then* $f(22k + 13, 22) = \frac{22k+13}{44k+44}$.

Case 15: $m = 22k + 15$ *with* $k \ge 2$. *Then* $f(22k + 15, 22) = \frac{22k+15}{44k+44}$.*(Exception:* $f(37, 22) = 9/22$ *INT-exception.)*

Case 17: $m = 22k + 17$ *with* $k \ge 1$. *Then* $f(22k + 17, 22) = \frac{22k+5}{44k+22}$.

Case 19: $m = 22k + 19$ *with* $k \ge 1$. *Then* $f(22k + 19, 22) = \frac{22k+3}{44k+22}$.

Case 21: $m = 22k + 21$ *with* $k \ge 1$. *Then* $f(22k + 21, 22) = \frac{22k+1}{44k+22}$.

Conjecture B.23 *If $m = 23k + i$ where $0 \le i \le 22$ then $f(m, 23)$ depends only on k, i via a formula, given below, with 18 exceptions (we will note the exceptions).*

Case 0: $m = 23k + 0$ *with* $k \ge 1$. *Then* $f(23k, 23) = 1$.

Case 1: $m = 23k + 1$ *with* $k \ge 1$. *Then* $f(23k + 1, 23) = \frac{23k+1}{46k+23}$.

Case 2: $m = 23k + 2$ *with* $k \geq 2$*. Then* $f(23k + 2, 23) = \frac{23k+2}{46k+23}$*.(Exception:* $f(25, 23) = 1/3$ *INT-exception.)*

Case 3: $m = 23k+3$ *with* $k \geq 2$*. Then* $f(23k+3, 23) = \frac{23k+3}{46k+23}$*.(Exception:* $f(26, 23) = 8/23$ *INT-exception.)*

Case 4: $m = 23k + 4$ *with* $k \geq 2$*. Then* $f(23k + 4, 23) = \frac{23k+4}{46k+23}$*.(Exception:* $f(27, 23) = 1/3$ *INT-exception.)*

Case 5: $m = 23k + 5$ *with* $k \geq 4$*. Then* $f(23k + 5, 23) = \frac{23k+5}{46k+23}$*.(Exception:* $f(28, 23) =$ $33/92$ *INT-exception,* $f(51, 23) = 59/138$ *BM-exception,* $f(74, 23) = 21/46$ *INT-exception,* $f(97, 23) = 43/92$ *INT-exception.)*

Case 6: $m = 23k + 6$ *with* $k \geq 2$*. Then* $f(23k + 6, 23) = \frac{23k-6}{46k}$*.(Exception:* $f(29, 23) =$ $49/138$ *BM-exception.)*

Case 7: $m = 23k + 7$ *with* $k \geq 1$*. Then* $f(23k + 7, 23) = \frac{23k-7}{46k}$.

Case 8: $m = 23k + 8$ *with* $k \geq 2$*. Then* $f(23k + 8, 23) = \frac{23k-8}{46k}$*.(Exception:* $f(31, 23) = 1/3$ *FC-exception.)*

Case 9: $m = 23k + 9$ *with* $k \geq 2$*. Then* $f(23k + 9, 23) = \frac{23k-9}{46k}$*.(Exception:* $f(32, 23) = 1/3$ *FC-exception.)*

Case 10: $m = 23k + 10$ *with* $k \geq 2$*. Then* $f(23k + 10, 23) = \frac{23k-10}{46k}$*.(Exception:* $f(33, 23) =$ $1/3$ *FC-exception.)*

Case 11: $m = 23k + 11$ *with* $k \geq 2$*. Then* $f(23k + 11, 23) = \frac{23k-11}{46k}$*.(Exception:* $f(34, 23) =$ $1/3$ *FC-exception.)*

Case 12: $m = 23k + 12$ *with* $k \geq 1$*. Then* $f(23k + 12, 23) = \frac{23k+12}{46k+46}$.

Case 13: $m = 23k + 13$ *with* $k \geq 1$*. Then* $f(23k + 13, 23) = \frac{23k+13}{46k+46}$.

Case 14: $m = 23k + 14$ *with* $k \geq 1$*. Then* $f(23k + 14, 23) = \frac{23k+14}{46k+46}$.

Case 15: $m = 23k + 15$ *with* $k \geq 2$*. Then* $f(23k + 15, 23) = \frac{23k+15}{46k+46}$*.(Exception:* $f(38, 23) =$ $47/115$ *INT-exception.)*

Case 16: $m = 23k + 16$ *with* $k \geq 3$*. Then* $f(23k + 16, 23) = \frac{23k+16}{46k+46}$*.(Exception:* $f(39, 23) =$

37/92 INT-exception, $f(62,23) = 113/253$ INT-exception.)

Case 17: $m = 23k + 17$ *with* $k \geq 3$. *Then* $f(23k + 17, 23) = \frac{23k+17}{46k+46}$. *(Exception:* $f(40, 23) = 19/46$ *INT-exception,* $f(63, 23) = 103/230$ *INT-exception,* $f(86, 23) = 149/322$ *INT-exception.)*

Case 18: $m = 23k + 18$ *with* $k \geq 1$. *Then* $f(23k + 18, 23) = \frac{23k+5}{46k+23}$.

Case 19: $m = 23k + 19$ *with* $k \geq 1$. *Then* $f(23k + 19, 23) = \frac{23k+4}{46k+23}$.

Case 20: $m = 23k + 20$ *with* $k \geq 1$. *Then* $f(23k + 20, 23) = \frac{23k+3}{46k+23}$.

Case 21: $m = 23k + 21$ *with* $k \geq 1$. *Then* $f(23k + 21, 23) = \frac{23k+2}{46k+23}$.

Case 22: $m = 23k + 22$ *with* $k \geq 1$. *Then* $f(23k + 22, 23) = \frac{23k+1}{46k+23}$.

Conjecture B.24 *If* $m = 24k + i$ *where* $0 \leq i \leq 23$ *then* $f(m, 24)$ *depends only on* k, i *via a formula, given below, with 13 exceptions (we will note the exceptions).*

Case 0: $m = 24k + 0$ *with* $k \geq 1$. *Then* $f(24k, 24) = 1$.

Case 1: $m = 24k+1$ *with* $k \geq 2$. *Then* $f(24k+1, 24) = \frac{24k+1}{48k+24}$. *(Exception:* $f(25, 24) = 19/56$ *BM-exception.)*

Case 5: $m = 24k+5$ *with* $k \geq 3$. *Then* $f(24k+5, 24) = \frac{24k+5}{48k+24}$. *(Exception:* $f(29, 24) = 17/48$ *INT-exception,* $f(53, 24) = 31/72$ *INT-exception.)*

Case 7: $m = 24k+7$ *with* $k \geq 2$. *Then* $f(24k+7, 24) = \frac{24k-7}{48k}$. *(Exception:* $f(31, 24) = 25/72$ *BM-exception.)*

Case 11: $m = 24k + 11$ *with* $k \geq 2$. *Then* $f(24k + 11, 24) = \frac{24k-11}{48k}$. *(Exception:* $f(35, 24) = 1/3$ *FC-exception.)*

Case 13: $m = 24k + 13$ *with* $k \geq 1$. *Then* $f(24k + 13, 24) = \frac{24k+13}{48k+48}$.

Case 17: $m = 24k + 17$ *with* $k \geq 3$. *Then* $f(24k + 17, 24) = \frac{24k+17}{48k+48}$. *(Exception:* $f(41, 24) = 19/48$ *BM-exception,* $f(65, 24) = 43/96$ *INT-exception,* $f(89, 24) = 63/136$ *INT-exception.)*

Case 19: $m = 24k + 19$ *with* $k \geq 1$. *Then* $f(24k + 19, 24) = \frac{24k+5}{48k+24}$.

Case 23: $m = 24k + 23$ *with* $k \geq 1$. *Then* $f(24k + 23, 24) = \frac{24k+1}{48k+24}$.

Conjecture B.25 *If $m = 25k + i$ where $0 \le i \le 24$ then $f(m, 25)$ depends only on k, i via a formula, given below, with 20 exceptions (we will note the exceptions).*

Case 0: $m = 25k + 0$ *with* $k \ge 1$. *Then* $f(25k, 25) = 1$.

Case 1: $m = 25k+1$ *with* $k \ge 2$. *Then* $f(25k+1, 25) = \frac{25k+1}{50k+25}$.*(Exception:* $f(26, 25) = 17/50$ *BM-exception.)*

Case 2: $m = 25k+2$ *with* $k \ge 2$. *Then* $f(25k+2, 25) = \frac{25k+2}{50k+25}$.*(Exception:* $f(27, 25) = 17/50$ *INT-exception.)*

Case 3: $m = 25k + 3$ *with* $k \ge 2$. *Then* $f(25k + 3, 25) = \frac{25k+3}{50k+25}$.*(Exception:* $f(28, 25) = 1/3$ *INT-exception.)*

Case 4: $m = 25k+4$ *with* $k \ge 2$. *Then* $f(25k+4, 25) = \frac{25k+4}{50k+25}$.*(Exception:* $f(29, 25) = 17/50$ *INT-exception.)*

Case 6: $m = 25k+6$ *with* $k \ge 3$. *Then* $f(25k+6, 25) = \frac{25k+6}{50k+25}$.*(Exception:* $f(31, 25) = 26/75$ *INT-exception,* $f(56, 25) = 87/200$ *INT-exception,* $f(81, 25) = 137/300$ *INT-exception.)*

Case 7: $m = 25k + 7$ *with* $k \ge 2$. *Then* $f(25k + 7, 25) = \frac{25k-7}{50k}$.*(Exception:* $f(32, 25) = 44/125$ *BM-exception.)*

Case 8: $m = 25k + 8$ *with* $k \ge 1$. *Then* $f(25k + 8, 25) = \frac{25k-8}{50k}$.

Case 9: $m = 25k + 9$ *with* $k \ge 2$. *Then* $f(25k + 9, 25) = \frac{25k-9}{50k}$.*(Exception:* $f(34, 25) = 1/3$ *FC-exception.)*

Case 11: $m = 25k + 11$ *with* $k \ge 2$. *Then* $f(25k + 11, 25) = \frac{25k-11}{50k}$.*(Exception:* $f(36, 25) = 1/3$ *FC-exception.)*

Case 12: $m = 25k + 12$ *with* $k \ge 2$. *Then* $f(25k + 12, 25) = \frac{25k-12}{50k}$.*(Exception:* $f(37, 25) = 1/3$ *FC-exception.)*

Case 13: $m = 25k + 13$ *with* $k \ge 1$. *Then* $f(25k + 13, 25) = \frac{25k+13}{50k+50}$.

Case 14: $m = 25k + 14$ *with* $k \ge 1$. *Then* $f(25k + 14, 25) = \frac{25k+14}{50k+50}$.

Case 16: $m = 25k + 16$ *with* $k \geq 2$. *Then* $f(25k + 16, 25) = \frac{25k+16}{50k+50}$.*(Exception:* $f(41, 25) =$ $61/150$ *INT-exception.)*

Case 17: $m = 25k + 17$ *with* $k \geq 2$. *Then* $f(25k + 17, 25) = \frac{25k+17}{50k+50}$.*(Exception:* $f(42, 25) =$ $41/100$ *INT-exception.)*

Case 18: $m = 25k + 18$ *with* $k \geq 3$. *Then* $f(25k + 18, 25) = \frac{25k+18}{50k+50}$.*(Exception:* $f(43, 25) =$ $61/150$ *INT-exception,* $f(68, 25) = 89/200$ *INT-exception,* $f(93, 25) = 139/300$ *INT-exception.)*

Case 19: $m = 25k + 19$ *with* $k \geq 3$. *Then* $f(25k + 19, 25) = \frac{25k+6}{50k+25}$.*(Exception:* $f(44, 25) =$ $41/100$ *INT-exception,* $f(69, 25) = 67/150$ *INT-exception.)*

Case 21: $m = 25k + 21$ *with* $k \geq 1$. *Then* $f(25k + 21, 25) = \frac{25k+4}{50k+25}$.

Case 22: $m = 25k + 22$ *with* $k \geq 1$. *Then* $f(25k + 22, 25) = \frac{25k+3}{50k+25}$.

Case 23: $m = 25k + 23$ *with* $k \geq 1$. *Then* $f(25k + 23, 25) = \frac{25k+2}{50k+25}$.

Case 24: $m = 25k + 24$ *with* $k \geq 1$. *Then* $f(25k + 24, 25) = \frac{25k+1}{50k+25}$.

Conjecture B.26 *If* $m = 26k + i$ *where* $0 \leq i \leq 25$ *then* $f(m, 26)$ *depends only on* k, i *via a formula, given below, with 19 exceptions (we will note the exceptions).*

Case 0: $m = 26k + 0$ *with* $k \geq 1$. *Then* $f(26k, 26) = 1$.

Case 1: $m = 26k + 1$ *with* $k \geq 1$. *Then* $f(26k + 1, 26) = \frac{26k+1}{52k+26}$.

Case 3: $m = 26k + 3$ *with* $k \geq 2$. *Then* $f(26k + 3, 26) = \frac{26k+3}{52k+26}$.*(Exception:* $f(29, 26) = 1/3$ *INT-exception.)*

Case 5: $m = 26k + 5$ *with* $k \geq 3$. *Then* $f(26k + 5, 26) = \frac{26k+5}{52k+26}$.*(Exception:* $f(31, 26) = 1/3$ *INT-exception,* $f(57, 26) = 181/416$ *INT-exception.)*

Case 7: $m = 26k + 7$ *with* $k \geq 2$. *Then* $f(26k + 7, 26) = \frac{26k-7}{52k}$.*(Exception:* $f(33, 26) = 9/26$ *INT-exception.)*

Case 9: $m = 26k + 9$ *with* $k \geq 2$. *Then* $f(26k + 9, 26) = \frac{26k-9}{52k}$.*(Exception:* $f(35, 26) = 1/3$ *FC-exception.)*

Case 11: $m = 26k + 11$ *with* $k \geq 2$. *Then* $f(26k + 11, 26) = \frac{26k-11}{52k}$.*(Exception:* $f(37, 26) =$ $1/3$ *FC-exception.)*

Case 15: $m = 26k + 15$ *with* $k \geq 1$. *Then* $f(26k + 15, 26) = \frac{26k+15}{52k+52}$.

Case 17: $m = 26k + 17$ *with* $k \geq 2$. *Then* $f(26k + 17, 26) = \frac{26k+17}{52k+52}$.*(Exception:* $f(43, 26) =$ $53/130$ *INT-exception.)*

Case 19: $m = 26k + 19$ *with* $k \geq 3$. *Then* $f(26k + 19, 26) = \frac{26k+19}{52k+52}$.*(Exception:* $f(45, 26) =$ $16/39$ *INT-exception,* $f(71, 26) = 29/65$ *INT-exception,* $f(97, 26) = 6/13$ *INT-exception.)*

Case 21: $m = 26k + 21$ *with* $k \geq 1$. *Then* $f(26k + 21, 26) = \frac{26k+5}{52k+26}$.

Case 23: $m = 26k + 23$ *with* $k \geq 1$. *Then* $f(26k + 23, 26) = \frac{26k+3}{52k+26}$.

Case 25: $m = 26k + 25$ *with* $k \geq 1$. *Then* $f(26k + 25, 26) = \frac{26k+1}{52k+26}$.

Conjecture B.27 *If* $m = 27k + i$ *where* $0 \leq i \leq 26$ *then* $f(m, 27)$ *depends only on* k, i *via a formula, given below, with 21 exceptions (we will note the exceptions).*

Case 0: $m = 27k + 0$ *with* $k \geq 1$. *Then* $f(27k, 27) = 1$.

Case 1: $m = 27k + 1$ *with* $k \geq 2$. *Then* $f(27k + 1, 27) = \frac{27k+1}{54k+27}$.*(Exception:* $f(28, 27) =$ $64/189$ *BM-exception.)*

Case 2: $m = 27k + 2$ *with* $k \geq 2$. *Then* $f(27k + 2, 27) = \frac{27k+2}{54k+27}$.*(Exception:* $f(29, 27) =$ $37/108$ *BM-exception.)*

Case 4: $m = 27k + 4$ *with* $k \geq 2$. *Then* $f(27k + 4, 27) = \frac{27k+4}{54k+27}$.*(Exception:* $f(31, 27) =$ $37/108$ *BM-exception.)*

Case 5: $m = 27k + 5$ *with* $k \geq 3$. *Then* $f(27k + 5, 27) = \frac{27k+5}{54k+27}$.*(Exception:* $f(32, 27) = 1/3$ *INT-exception,* $f(59, 27) = 31/72$ *INT-exception.)*

Case 7: $m = 27k + 7$ *with* $k \geq 3$. *Then* $f(27k + 7, 27) = \frac{27k-7}{54k}$.*(Exception:* $f(34, 27) = 16/45$ *INT-exception,* $f(61, 27) = 281/648$ *INT-exception,* $f(88, 27) = 271/594$ *INT-exception.)*

Case 8: $m = 27k+8$ *with* $k \geq 2$. *Then* $f(27k+8, 27) = \frac{27k-8}{54k}$.*(Exception:* $f(35, 27) = 25/72$ *BM-exception.)*

Case 10: $m = 27k + 10$ *with* $k \geq 2$. *Then* $f(27k + 10, 27) = \frac{27k-10}{54k}$.*(Exception:* $f(37, 27) = 1/3$ *FC-exception.)*

Case 11: $m = 27k + 11$ *with* $k \geq 2$. *Then* $f(27k + 11, 27) = \frac{27k-11}{54k}$.*(Exception:* $f(38, 27) = 1/3$ *FC-exception.)*

Case 13: $m = 27k + 13$ *with* $k \geq 2$. *Then* $f(27k + 13, 27) = \frac{27k-13}{54k}$.*(Exception:* $f(40, 27) = 1/3$ *FC-exception.)*

Case 14: $m = 27k + 14$ *with* $k \geq 1$. *Then* $f(27k + 14, 27) = \frac{27k+14}{54k+54}$.

Case 16: $m = 27k + 16$ *with* $k \geq 1$. *Then* $f(27k + 16, 27) = \frac{27k+16}{54k+54}$.

Case 17: $m = 27k + 17$ *with* $k \geq 1$. *Then* $f(27k + 17, 27) = \frac{27k+17}{54k+54}$.

Case 19: $m = 27k + 19$ *with* $k \geq 3$. *Then* $f(27k + 19, 27) = \frac{27k+19}{54k+54}$.*(Exception:* $f(46, 27) = 43/108$ *BM-exception,* $f(73, 27) = 101/225$ *INT-exception.)*

Case 20: $m = 27k + 20$ *with* $k \geq 2$. *Then* $f(27k + 20, 27) = \frac{27k+20}{54k+54}$.*(Exception:* $f(47, 27) = 89/216$ *INT-exception,* $f(74, 27) = 121/270$ *INT-exception.)*

Case 22: $m = 27k + 22$ *with* $k \geq 1$. *Then* $f(27k + 22, 27) = \frac{27k+5}{54k+27}$.

Case 23: $m = 27k + 23$ *with* $k \geq 1$. *Then* $f(27k + 23, 27) = \frac{27k+4}{54k+27}$.

Case 25: $m = 27k + 25$ *with* $k \geq 1$. *Then* $f(27k + 25, 27) = \frac{27k+2}{54k+27}$.

Case 26: $m = 27k + 26$ *with* $k \geq 1$. *Then* $f(27k + 26, 27) = \frac{27k+1}{54k+27}$.

Conjecture B.28 *If* $m = 28k + i$ *where* $0 \leq i \leq 27$ *then* $f(m, 28)$ *depends only on* k, i *via a formula, given below, with 15 exceptions (we will note the exceptions).*

Case 0: $m = 28k + 0$ *with* $k \geq 1$. *Then* $f(28k, 28) = 1$.

Case 1: $m = 28k+1$ *with* $k \geq 2$. *Then* $f(28k+1, 28) = \frac{28k+1}{56k+28}$.*(Exception:* $f(29, 28) = 19/56$ *BM-exception.)*

Case 3: $m = 28k+3$ *with* $k \geq 2$. *Then* $f(28k+3, 28) = \frac{28k+3}{56k+28}$.*(Exception:* $f(31, 28) = 19/56$ *INT-exception.)*

Case 5: $m = 28k+5$ *with* $k \geq 3$. *Then* $f(28k+5, 28) = \frac{28k+5}{56k+28}$.*(Exception:* $f(33, 28) = 1/3$ *INT-exception,* $f(61, 28) = 97/224$ *INT-exception.)*

Case 9: $m = 28k+9$ *with* $k \geq 1$. *Then* $f(28k+9, 28) = \frac{28k-9}{56k}$.

Case 11: $m = 28k+11$ *with* $k \geq 2$. *Then* $f(28k+11, 28) = \frac{28k-11}{56k}$.*(Exception:* $f(39, 28) = 1/3$ *FC-exception.)*

Case 13: $m = 28k+13$ *with* $k \geq 2$. *Then* $f(28k+13, 28) = \frac{28k-13}{56k}$.*(Exception:* $f(41, 28) = 1/3$ *FC-exception.)*

Case 15: $m = 28k+15$ *with* $k \geq 1$. *Then* $f(28k+15, 28) = \frac{28k+15}{56k+56}$.

Case 17: $m = 28k+17$ *with* $k \geq 1$. *Then* $f(28k+17, 28) = \frac{28k+17}{56k+56}$.

Case 19: $m = 28k+19$ *with* $k \geq 2$. *Then* $f(28k+19, 28) = \frac{28k+19}{56k+56}$.*(Exception:* $f(47, 28) = 23/56$ *INT-exception.)*

Case 23: $m = 28k+23$ *with* $k \geq 1$. *Then* $f(28k+23, 28) = \frac{28k+5}{56k+28}$.

Case 25: $m = 28k+25$ *with* $k \geq 1$. *Then* $f(28k+25, 28) = \frac{28k+3}{56k+28}$.

Case 27: $m = 28k+27$ *with* $k \geq 1$. *Then* $f(28k+27, 28) = \frac{28k+1}{56k+28}$.

Conjecture B.29 *If* $m = 29k+i$ *where* $0 \leq i \leq 28$ *then* $f(m, 29)$ *depends only on* k, i *via a formula, given below, with 22 exceptions (we will note the exceptions).*

Case 0: $m = 29k+0$ *with* $k \geq 1$. *Then* $f(29k, 29) = 1$.

Case 1: $m = 29k+1$ *with* $k \geq 1$. *Then* $f(29k+1, 29) = \frac{29k+1}{58k+29}$.

Case 2: $m = 29k+2$ *with* $k \geq 2$. *Then* $f(29k+2, 29) = \frac{29k+2}{58k+29}$.*(Exception:* $f(31, 29) = 1/3$ *BM-exception.)*

Case 3: $m = 29k+3$ *with* $k \geq 2$. *Then* $f(29k+3, 29) = \frac{29k+3}{58k+29}$.*(Exception:* $f(32, 29) = 10/29$ *INT-exception.)*

Case 4: $m = 29k+4$ *with* $k \geq 2$. *Then* $f(29k+4, 29) = \frac{29k+4}{58k+29}$.*(Exception:* $f(33, 29) = 10/29$ *BM-exception.)*

Case 5: $m = 29k+5$ *with* $k \geq 3$. *Then* $f(29k+5, 29) = \frac{29k+5}{58k+29}$.*(Exception:* $f(34, 29) = 1/3$ *INT-exception,* $f(63, 29) = 25/58$ *INT-exception.)*

Case 6: $m = 29k+6$ *with* $k \geq 3$. *Then* $f(29k+6, 29) = \frac{29k+6}{58k+29}$.*(Exception:* $f(35, 29) = 41/116$ *INT-exception,* $f(64, 29) = 25/58$ *INT-exception,* $f(93, 29) = 185/406$ *INT-exception.)*

Case 7: $m = 29k+7$ *with* $k \geq 3$. *Then* $f(29k+7, 29) = \frac{29k+7}{58k+29}$.*(Exception:* $f(36, 29) = 10/29$ *INT-exception,* $f(65, 29) = 101/232$ *INT-exception,* $f(94, 29) = 53/116$ *INT-exception.)*

Case 8: $m = 29k+8$ *with* $k \geq 2$. *Then* $f(29k+8, 29) = \frac{29k-8}{58k}$.*(Exception:* $f(37, 29) = 61/174$ *INT-exception.)*

Case 9: $m = 29k+9$ *with* $k \geq 1$. *Then* $f(29k+9, 29) = \frac{29k-9}{58k}$.

Case 10: $m = 29k+10$ *with* $k \geq 2$. *Then* $f(29k+10, 29) = \frac{29k-10}{58k}$.*(Exception:* $f(39, 29) = 1/3$ *FC-exception.)*

Case 11: $m = 29k+11$ *with* $k \geq 2$. *Then* $f(29k+11, 29) = \frac{29k-11}{58k}$.*(Exception:* $f(40, 29) = 1/3$ *FC-exception.)*

Case 12: $m = 29k+12$ *with* $k \geq 2$. *Then* $f(29k+12, 29) = \frac{29k-12}{58k}$.*(Exception:* $f(41, 29) = 1/3$ *FC-exception.)*

Case 13: $m = 29k+13$ *with* $k \geq 2$. *Then* $f(29k+13, 29) = \frac{29k-13}{58k}$.*(Exception:* $f(42, 29) = 1/3$ *FC-exception.)*

Case 14: $m = 29k+14$ *with* $k \geq 2$. *Then* $f(29k+14, 29) = \frac{29k-14}{58k}$.*(Exception:* $f(43, 29) = 1/3$ *FC-exception.)*

Case 15: $m = 29k+15$ *with* $k \geq 1$. *Then* $f(29k+15, 29) = \frac{29k+15}{58k+58}$.

Case 16: $m = 29k+16$ *with* $k \geq 1$. *Then* $f(29k+16, 29) = \frac{29k+16}{58k+58}$.

Case 17: $m = 29k+17$ *with* $k \geq 1$. *Then* $f(29k+17, 29) = \frac{29k+17}{58k+58}$.

Case 18: $m = 29k+18$ *with* $k \geq 1$. *Then* $f(29k+18, 29) = \frac{29k+18}{58k+58}$.

Case 19: $m = 29k+19$ *with* $k \geq 2$. *Then* $f(29k+19, 29) = \frac{29k+19}{58k+58}$.*(Exception:* $f(48, 29) =$

59/145 *INT-exception.)*

Case 20: $m = 29k + 20$ *with* $k \geq 2$. *Then* $f(29k + 20, 29) = \frac{29k+20}{58k+58}$.*(Exception:* $f(49, 29) = 47/116$ *INT-exception.)*

Case 21: $m = 29k + 21$ *with* $k \geq 2$. *Then* $f(29k + 21, 29) = \frac{29k+21}{58k+58}$.*(Exception:* $f(50, 29) = 71/174$ *INT-exception,* $f(79, 29) = 4/9$ *INT-exception.)*

Case 22: $m = 29k + 22$ *with* $k \geq 2$. *Then* $f(29k + 22, 29) = \frac{29k+7}{58k+29}$.*(Exception:* $f(51, 29) = 95/232$ *INT-exception.)*

Case 23: $m = 29k + 23$ *with* $k \geq 1$. *Then* $f(29k + 23, 29) = \frac{29k+6}{58k+29}$.

Case 24: $m = 29k + 24$ *with* $k \geq 1$. *Then* $f(29k + 24, 29) = \frac{29k+5}{58k+29}$.

Case 25: $m = 29k + 25$ *with* $k \geq 1$. *Then* $f(29k + 25, 29) = \frac{29k+4}{58k+29}$.

Case 26: $m = 29k + 26$ *with* $k \geq 1$. *Then* $f(29k + 26, 29) = \frac{29k+3}{58k+29}$.

Case 27: $m = 29k + 27$ *with* $k \geq 1$. *Then* $f(29k + 27, 29) = \frac{29k+2}{58k+29}$.

Case 28: $m = 29k + 28$ *with* $k \geq 1$. *Then* $f(29k + 28, 29) = \frac{29k+1}{58k+29}$.

Conjecture B.30 *If* $m = 30k + i$ *where* $0 \leq i \leq 29$ *then* $f(m, 30)$ *depends only on* k, i *via a formula, given below, with 19 exceptions (we will note the exceptions).*

Case 0: $m = 30k + 0$ *with* $k \geq 1$. *Then* $f(30k, 30) = 1$.

Case 1: $m = 30k + 1$ *with* $k \geq 2$. *Then* $f(30k + 1, 30) = \frac{30k+1}{60k+30}$.*(Exception:* $f(31, 30) = 71/210$ *BM-exception.)*

Case 7: $m = 30k+7$ *with* $k \geq 3$. *Then* $f(30k+7, 30) = \frac{30k+7}{60k+30}$.*(Exception:* $f(37, 30) = 16/45$ *INT-exception,* $f(67, 30) = 13/30$ *INT-exception,* $f(97, 30) = 41/90$ *INT-exception.)*

Case 11: $m = 30k + 11$ *with* $k \geq 2$. *Then* $f(30k + 11, 30) = \frac{30k-11}{60k}$.*(Exception:* $f(41, 30) = 1/3$ *FC-exception.)*

Case 13: $m = 30k + 13$ *with* $k \geq 2$. *Then* $f(30k + 13, 30) = \frac{30k-13}{60k}$.*(Exception:* $f(43, 30) = 1/3$ *FC-exception.)*

Case 17: $m = 30k + 17$ *with* $k \geq 1$*. Then* $f(30k + 17, 30) = \frac{30k+17}{60k+60}$.

Case 19: $m = 30k + 19$ *with* $k \geq 1$*. Then* $f(30k + 19, 30) = \frac{30k+19}{60k+60}$.

Case 23: $m = 30k + 23$ *with* $k \geq 2$*. Then* $f(30k + 23, 30) = \frac{30k+7}{60k+30}$*.(Exception:* $f(53, 30) = 61/150$ *INT-exception.)*

Case 29: $m = 30k + 29$ *with* $k \geq 1$*. Then* $f(30k + 29, 30) = \frac{30k+1}{60k+30}$.

Conjecture B.31 *If* $m = 31k + i$ *where* $0 \leq i \leq 30$ *then* $f(m, 31)$ *depends only on* k, i *via a formula, given below, with 24 exceptions (we will note the exceptions).*

Case 0: $m = 31k + 0$ *with* $k \geq 1$*. Then* $f(31k, 31) = 1$.

Case 1: $m = 31k+1$ *with* $k \geq 2$*. Then* $f(31k+1, 31) = \frac{31k+1}{62k+31}$*.(Exception:* $f(32, 31) = 21/62$ *BM-exception.)*

Case 2: $m = 31k+2$ *with* $k \geq 2$*. Then* $f(31k+2, 31) = \frac{31k+2}{62k+31}$*.(Exception:* $f(33, 31) = 21/62$ *BM-exception.)*

Case 3: $m = 31k+3$ *with* $k \geq 2$*. Then* $f(31k+3, 31) = \frac{31k+3}{62k+31}$*.(Exception:* $f(34, 31) = 32/93$ *BM-exception.)*

Case 4: $m = 31k + 4$ *with* $k \geq 2$*. Then* $f(31k + 4, 31) = \frac{31k+4}{62k+31}$*.(Exception:* $f(35, 31) = 43/124$ *INT-exception.)*

Case 5: $m = 31k+5$ *with* $k \geq 2$*. Then* $f(31k+5, 31) = \frac{31k+5}{62k+31}$*.(Exception:* $f(36, 31) = 21/62$ *INT-exception.)*

Case 6: $m = 31k + 6$ *with* $k \geq 3$*. Then* $f(31k + 6, 31) = \frac{31k+6}{62k+31}$*.(Exception:* $f(37, 31) = 1/3$ *INT-exception,* $f(68, 31) = 27/62$ *INT-exception.)*

Case 7: $m = 31k+7$ *with* $k \geq 3$*. Then* $f(31k+7, 31) = \frac{31k+7}{62k+31}$*.(Exception:* $f(38, 31) = 11/31$ *BM-exception,* $f(69, 31) = 107/248$ *INT-exception,* $f(100, 31) = 141/310$ *INT-exception.)*

Case 8: $m = 31k+8$ *with* $k \geq 2$*. Then* $f(31k+8, 31) = \frac{31k-8}{62k}$*.(Exception:* $f(39, 31) = 11/31$ *INT-exception,* $f(70, 31) = 148/341$ *INT-exception.)*

Case 9: $m = 31k + 9$ *with* $k \geq 2$. *Then* $f(31k + 9, 31) = \frac{31k-9}{62k}$.*(Exception:* $f(40, 31) =$ 65/186 *BM-exception.)*

Case 10: $m = 31k + 10$ *with* $k \geq 1$. *Then* $f(31k + 10, 31) = \frac{31k-10}{62k}$.

Case 11: $m = 31k + 11$ *with* $k \geq 2$. *Then* $f(31k + 11, 31) = \frac{31k-11}{62k}$.*(Exception:* $f(42, 31) =$ 1/3 *FC-exception.)*

Case 12: $m = 31k + 12$ *with* $k \geq 2$. *Then* $f(31k + 12, 31) = \frac{31k-12}{62k}$.*(Exception:* $f(43, 31) =$ 1/3 *FC-exception.)*

Case 13: $m = 31k + 13$ *with* $k \geq 2$. *Then* $f(31k + 13, 31) = \frac{31k-13}{62k}$.*(Exception:* $f(44, 31) =$ 1/3 *FC-exception.)*

Case 14: $m = 31k + 14$ *with* $k \geq 2$. *Then* $f(31k + 14, 31) = \frac{31k-14}{62k}$.*(Exception:* $f(45, 31) =$ 1/3 *FC-exception.)*

Case 15: $m = 31k + 15$ *with* $k \geq 2$. *Then* $f(31k + 15, 31) = \frac{31k-15}{62k}$.*(Exception:* $f(46, 31) =$ 1/3 *FC-exception.)*

Case 16: $m = 31k + 16$ *with* $k \geq 1$. *Then* $f(31k + 16, 31) = \frac{31k+16}{62k+62}$.

Case 17: $m = 31k + 17$ *with* $k \geq 1$. *Then* $f(31k + 17, 31) = \frac{31k+17}{62k+62}$.

Case 18: $m = 31k + 18$ *with* $k \geq 1$. *Then* $f(31k + 18, 31) = \frac{31k+18}{62k+62}$.

Case 19: $m = 31k + 19$ *with* $k \geq 1$. *Then* $f(31k + 19, 31) = \frac{31k+19}{62k+62}$.

Case 20: $m = 31k + 20$ *with* $k \geq 2$. *Then* $f(31k + 20, 31) = \frac{31k+20}{62k+62}$.*(Exception:* $f(51, 31) =$ 25/62 *INT-exception.)*

Case 21: $m = 31k + 21$ *with* $k \geq 2$. *Then* $f(31k + 21, 31) = \frac{31k+21}{62k+62}$.*(Exception:* $f(52, 31) =$ 51/124 *INT-exception.)*

Case 22: $m = 31k + 22$ *with* $k \geq 2$. *Then* $f(31k + 22, 31) = \frac{31k+22}{62k+62}$.*(Exception:* $f(53, 31) =$ 49/124 *BM-exception,* $f(84, 31) = 111/248$ *INT-exception.)*

Case 23: $m = 31k + 23$ *with* $k \geq 2$. *Then* $f(31k + 23, 31) = \frac{31k+23}{62k+62}$.*(Exception:* $f(54, 31) =$ 51/124 *INT-exception,* $f(85, 31) = 139/310$ *INT-exception.)*

Case 24: $m = 31k + 24$ *with* $k \geq 1$. *Then* $f(31k + 24, 31) = \frac{31k+7}{62k+31}$.

Case 25: $m = 31k + 25$ *with* $k \geq 1$. *Then* $f(31k + 25, 31) = \frac{31k+6}{62k+31}$.

Case 26: $m = 31k + 26$ *with* $k \geq 1$. *Then* $f(31k + 26, 31) = \frac{31k+5}{62k+31}$.

Case 27: $m = 31k + 27$ *with* $k \geq 1$. *Then* $f(31k + 27, 31) = \frac{31k+4}{62k+31}$.

Case 28: $m = 31k + 28$ *with* $k \geq 1$. *Then* $f(31k + 28, 31) = \frac{31k+3}{62k+31}$.

Case 29: $m = 31k + 29$ *with* $k \geq 1$. *Then* $f(31k + 29, 31) = \frac{31k+2}{62k+31}$.

Case 30: $m = 31k + 30$ *with* $k \geq 1$. *Then* $f(31k + 30, 31) = \frac{31k+1}{62k+31}$.

Conjecture B.32 *If* $m = 32k + i$ *where* $0 \leq i \leq 31$ *then* $f(m, 32)$ *depends only on* k, i *via a formula, given below, with 17 exceptions (we will note the exceptions).*

Case 0: $m = 32k + 0$ *with* $k \geq 1$. *Then* $f(32k, 32) = 1$.

Case 1: $m = 32k + 1$ *with* $k \geq 1$. *Then* $f(32k + 1, 32) = \frac{32k+1}{64k+32}$.

Case 3: $m = 32k+3$ *with* $k \geq 2$. *Then* $f(32k+3, 32) = \frac{32k+3}{64k+32}$.*(Exception:* $f(35, 32) = 11/32$ *INT-exception.)*

Case 5: $m = 32k+5$ *with* $k \geq 2$. *Then* $f(32k+5, 32) = \frac{32k+5}{64k+32}$.*(Exception:* $f(37, 32) = 11/32$ *INT-exception.)*

Case 7: $m = 32k+7$ *with* $k \geq 2$. *Then* $f(32k+7, 32) = \frac{32k+7}{64k+32}$.*(Exception:* $f(39, 32) = 23/64$ *INT-exception,* $f(71, 32) = 41/96$ *BM-exception.)*

Case 9: $m = 32k + 9$ *with* $k \geq 2$. *Then* $f(32k + 9, 32) = \frac{32k-9}{64k}$.*(Exception:* $f(41, 32) = 7/20$ *BM-exception.)*

Case 11: $m = 32k + 11$ *with* $k \geq 2$. *Then* $f(32k + 11, 32) = \frac{32k-11}{64k}$.*(Exception:* $f(43, 32) = 1/3$ *FC-exception.)*

Case 13: $m = 32k + 13$ *with* $k \geq 2$. *Then* $f(32k + 13, 32) = \frac{32k-13}{64k}$.*(Exception:* $f(45, 32) = 1/3$ *FC-exception.)*

Case 15: $m = 32k + 15$ *with* $k \geq 2$. *Then* $f(32k + 15, 32) = \frac{32k-15}{64k}$.*(Exception:* $f(47, 32) = 1/3$ *FC-exception.)*

Case 17: $m = 32k + 17$ *with* $k \geq 1$. *Then* $f(32k + 17, 32) = \frac{32k+17}{64k+64}$.

Case 19: $m = 32k + 19$ *with* $k \geq 1$. *Then* $f(32k + 19, 32) = \frac{32k+19}{64k+64}$.

Case 21: $m = 32k + 21$ *with* $k \geq 2$. *Then* $f(32k + 21, 32) = \frac{32k+21}{64k+64}$.*(Exception:* $f(53, 32) = 13/32$ *INT-exception.)*

Case 23: $m = 32k + 23$ *with* $k \geq 2$. *Then* $f(32k + 23, 32) = \frac{32k+23}{64k+64}$.*(Exception:* $f(55, 32) = 13/32$ *INT-exception,* $f(87, 32) = 57/128$ *INT-exception.)*

Case 25: $m = 32k + 25$ *with* $k \geq 1$. *Then* $f(32k + 25, 32) = \frac{32k+7}{64k+32}$.

Case 27: $m = 32k + 27$ *with* $k \geq 1$. *Then* $f(32k + 27, 32) = \frac{32k+5}{64k+32}$.

Case 29: $m = 32k + 29$ *with* $k \geq 1$. *Then* $f(32k + 29, 32) = \frac{32k+3}{64k+32}$.

Case 31: $m = 32k + 31$ *with* $k \geq 1$. *Then* $f(32k + 31, 32) = \frac{32k+1}{64k+32}$.

Conjecture B.33 *If* $m = 33k + i$ *where* $0 \leq i \leq 32$ *then* $f(m, 33)$ *depends only on* k, i *via a formula, given below, with 21 exceptions (we will note the exceptions).*

Case 0: $m = 33k + 0$ *with* $k \geq 1$. *Then* $f(33k, 33) = 1$.

Case 1: $m = 33k+1$ *with* $k \geq 2$. *Then* $f(33k+1, 33) = \frac{33k+1}{66k+33}$.*(Exception:* $f(34, 33) = 26/77$ *BM-exception.)*

Case 2: $m = 33k+2$ *with* $k \geq 2$. *Then* $f(33k+2, 33) = \frac{33k+2}{66k+33}$.*(Exception:* $f(35, 33) = 15/44$ *BM-exception.)*

Case 4: $m = 33k + 4$ *with* $k \geq 2$. *Then* $f(33k + 4, 33) = \frac{33k+4}{66k+33}$.*(Exception:* $f(37, 33) = 1/3$ *INT-exception.)*

Case 5: $m = 33k+5$ *with* $k \geq 2$. *Then* $f(33k+5, 33) = \frac{33k+5}{66k+33}$.*(Exception:* $f(38, 33) = 34/99$ *BM-exception.)*

Case 7: $m = 33k + 7$ *with* $k \geq 2$. *Then* $f(33k + 7, 33) = \frac{33k+7}{66k+33}$.*(Exception:* $f(40, 33) = 47/132$ *INT-exception,* $f(73, 33) = 85/198$ *INT-exception.)*

Case 8: $m = 33k{+}8$ *with* $k \geq 2$. *Then* $f(33k{+}8, 33) = \frac{33k+8}{66k+33}$.*(Exception:* $f(41, 33) = 34/99$ *INT-exception,* $f(74, 33) = 115/264$ *INT-exception.)*

Case 10: $m = 33k + 10$ *with* $k \geq 1$. *Then* $f(33k + 10, 33) = \frac{33k-10}{66k}$.*(Exception:* $f(43, 33) = 91/264$ *ERIK-exception.)*

Case 13: $m = 33k + 13$ *with* $k \geq 2$. *Then* $f(33k + 13, 33) = \frac{33k-13}{66k}$.*(Exception:* $f(46, 33) = 1/3$ *FC-exception.)*

Case 14: $m = 33k + 14$ *with* $k \geq 2$. *Then* $f(33k + 14, 33) = \frac{33k-14}{66k}$.*(Exception:* $f(47, 33) = 1/3$ *FC-exception.)*

Case 16: $m = 33k + 16$ *with* $k \geq 2$. *Then* $f(33k + 16, 33) = \frac{33k-16}{66k}$.*(Exception:* $f(49, 33) = 1/3$ *FC-exception.)*

Case 17: $m = 33k + 17$ *with* $k \geq 1$. *Then* $f(33k + 17, 33) = \frac{33k+17}{66k+66}$.

Case 19: $m = 33k + 19$ *with* $k \geq 1$. *Then* $f(33k + 19, 33) = \frac{33k+19}{66k+66}$.

Case 20: $m = 33k + 20$ *with* $k \geq 1$. *Then* $f(33k + 20, 33) = \frac{33k+20}{66k+66}$.

Case 23: $m = 33k + 23$ *with* $k \geq 2$. *Then* $f(33k + 23, 33) = \frac{33k+23}{66k+66}$.*(Exception:* $f(56, 33) = 53/132$ *INT-exception,* $f(89, 33) = 54/121$ *INT-exception.)*

Case 25: $m = 33k + 25$ *with* $k \geq 2$. *Then* $f(33k + 25, 33) = \frac{33k+8}{66k+33}$.*(Exception:* $f(58, 33) = 9/22$ *INT-exception,* $f(91, 33) = 197/440$ *INT-exception.)*

Case 26: $m = 33k + 26$ *with* $k \geq 1$. *Then* $f(33k + 26, 33) = \frac{33k+7}{66k+33}$.

Case 28: $m = 33k + 28$ *with* $k \geq 1$. *Then* $f(33k + 28, 33) = \frac{33k+5}{66k+33}$.

Case 29: $m = 33k + 29$ *with* $k \geq 1$. *Then* $f(33k + 29, 33) = \frac{33k+4}{66k+33}$.

Case 31: $m = 33k + 31$ *with* $k \geq 1$. *Then* $f(33k + 31, 33) = \frac{33k+2}{66k+33}$.

Case 32: $m = 33k + 32$ *with* $k \geq 1$. *Then* $f(33k + 32, 33) = \frac{33k+1}{66k+33}$.

Conjecture B.34 *If* $m = 34k + i$ *where* $0 \leq i \leq 33$ *then* $f(m, 34)$ *depends only on* k, i *via a formula, given below, with 22 exceptions (we will note the exceptions).*

Case 0: $m = 34k + 0$ *with* $k \geq 1$*. Then* $f(34k, 34) = 1$.

Case 1: $m = 34k+1$ *with* $k \geq 2$*. Then* $f(34k+1, 34) = \frac{34k+1}{68k+34}$*.(Exception:* $f(35, 34) = 23/68$ *BM-exception.)*

Case 3: $m = 34k + 3$ *with* $k \geq 2$*. Then* $f(34k + 3, 34) = \frac{34k+3}{68k+34}$*.(Exception:* $f(37, 34) = 1/3$ *INT-exception.)*

Case 5: $m = 34k + 5$ *with* $k \geq 2$*. Then* $f(34k + 5, 34) = \frac{34k+5}{68k+34}$*.(Exception:* $f(39, 34) = 47/136$ *BM-exception.)*

Case 7: $m = 34k+7$ *with* $k \geq 2$*. Then* $f(34k+7, 34) = \frac{34k+7}{68k+34}$*.(Exception:* $f(41, 34) = 6/17$ *INT-exception,* $f(75, 34) = 22/51$ *INT-exception.)*

Case 9: $m = 34k + 9$ *with* $k \geq 2$*. Then* $f(34k + 9, 34) = \frac{34k-9}{68k}$*.(Exception:* $f(43, 34) = 6/17$ *INT-exception,* $f(77, 34) = 103/238$ *INT-exception.)*

Case 11: $m = 34k + 11$ *with* $k \geq 1$*. Then* $f(34k + 11, 34) = \frac{34k-11}{68k}$.

Case 13: $m = 34k + 13$ *with* $k \geq 2$*. Then* $f(34k + 13, 34) = \frac{34k-13}{68k}$*.(Exception:* $f(47, 34) = 1/3$ *FC-exception.)*

Case 15: $m = 34k + 15$ *with* $k \geq 2$*. Then* $f(34k + 15, 34) = \frac{34k-15}{68k}$*.(Exception:* $f(49, 34) = 1/3$ *FC-exception.)*

Case 19: $m = 34k + 19$ *with* $k \geq 1$*. Then* $f(34k + 19, 34) = \frac{34k+19}{68k+68}$.

Case 21: $m = 34k + 21$ *with* $k \geq 1$*. Then* $f(34k + 21, 34) = \frac{34k+21}{68k+68}$.

Case 23: $m = 34k + 23$ *with* $k \geq 2$*. Then* $f(34k + 23, 34) = \frac{34k+23}{68k+68}$*.(Exception:* $f(57, 34) = 7/17$ *INT-exception.)*

Case 25: $m = 34k + 25$ *with* $k \geq 2$*. Then* $f(34k + 25, 34) = \frac{34k+25}{68k+68}$*.(Exception:* $f(59, 34) = 7/17$ *INT-exception,* $f(93, 34) = 38/85$ *INT-exception.)*

Case 27: $m = 34k + 27$ *with* $k \geq 1$*. Then* $f(34k + 27, 34) = \frac{34k+7}{68k+34}$.

Case 29: $m = 34k + 29$ *with* $k \geq 1$*. Then* $f(34k + 29, 34) = \frac{34k+5}{68k+34}$.

Case 31: $m = 34k + 31$ *with* $k \geq 1$*. Then* $f(34k + 31, 34) = \frac{34k+3}{68k+34}$.

Case 33: $m = 34k + 33$ *with* $k \geq 1$*. Then* $f(34k + 33, 34) = \frac{34k+1}{68k+34}$.

Conjecture B.35 *If $m = 35k + i$ where $0 \le i \le 34$ then $f(m, 35)$ depends only on k, i via a formula, given below, with 24 exceptions (we will note the exceptions).*

Case 0: $m = 35k + 0$ *with* $k \ge 1$. *Then* $f(35k, 35) = 1$.

Case 1: $m = 35k + 1$ *with* $k \ge 1$. *Then* $f(35k + 1, 35) = \frac{35k+1}{70k+35}$.

Case 2: $m = 35k + 2$ *with* $k \ge 2$. *Then* $f(35k + 2, 35) = \frac{35k+2}{70k+35}$.*(Exception:* $f(37, 35) = 1/3$ *BM-exception.)*

Case 3: $m = 35k + 3$ *with* $k \ge 2$. *Then* $f(35k + 3, 35) = \frac{35k+3}{70k+35}$.*(Exception:* $f(38, 35) = 1/3$ *INT-exception.)*

Case 4: $m = 35k + 4$ *with* $k \ge 2$. *Then* $f(35k + 4, 35) = \frac{35k+4}{70k+35}$.*(Exception:* $f(39, 35) = 1/3$ *INT-exception.)*

Case 6: $m = 35k + 6$ *with* $k \ge 2$. *Then* $f(35k + 6, 35) = \frac{35k+6}{70k+35}$.*(Exception:* $f(41, 35) = 1/3$ *INT-exception,* $f(76, 35) = 151/350$ *INT-exception.)*

Case 8: $m = 35k+8$ *with* $k \ge 2$. *Then* $f(35k+8, 35) = \frac{35k+8}{70k+35}$.*(Exception:* $f(43, 35) = 5/14$ *BM-exception,* $f(78, 35) = 121/280$ *INT-exception.)*

Case 9: $m = 35k + 9$ *with* $k \ge 2$. *Then* $f(35k + 9, 35) = \frac{35k-9}{70k}$.*(Exception:* $f(44, 35) = 62/175$ *INT-exception,* $f(79, 35) = 167/385$ *INT-exception.)*

Case 11: $m = 35k + 11$ *with* $k \ge 1$. *Then* $f(35k + 11, 35) = \frac{35k-11}{70k}$.

Case 12: $m = 35k + 12$ *with* $k \ge 2$. *Then* $f(35k + 12, 35) = \frac{35k-12}{70k}$.*(Exception:* $f(47, 35) = 1/3$ *FC-exception.)*

Case 13: $m = 35k + 13$ *with* $k \ge 2$. *Then* $f(35k + 13, 35) = \frac{35k-13}{70k}$.*(Exception:* $f(48, 35) = 1/3$ *FC-exception.)*

Case 16: $m = 35k + 16$ *with* $k \ge 2$. *Then* $f(35k + 16, 35) = \frac{35k-16}{70k}$.*(Exception:* $f(51, 35) = 1/3$ *FC-exception.)*

Case 17: $m = 35k + 17$ *with* $k \ge 2$. *Then* $f(35k + 17, 35) = \frac{35k-17}{70k}$.*(Exception:* $f(52, 35) =$

1/3 *FC-exception.)*

Case 18: $m = 35k + 18$ *with* $k \geq 1$. *Then* $f(35k + 18, 35) = \frac{35k+18}{70k+70}$.

Case 19: $m = 35k + 19$ *with* $k \geq 1$. *Then* $f(35k + 19, 35) = \frac{35k+19}{70k+70}$.

Case 22: $m = 35k + 22$ *with* $k \geq 1$. *Then* $f(35k + 22, 35) = \frac{35k+22}{70k+70}$.

Case 23: $m = 35k + 23$ *with* $k \geq 2$. *Then* $f(35k + 23, 35) = \frac{35k+23}{70k+70}$.*(Exception:* $f(58, 35) =$ 71/175 *INT-exception.)*

Case 24: $m = 35k + 24$ *with* $k \geq 2$. *Then* $f(35k + 24, 35) = \frac{35k+24}{70k+70}$.*(Exception:* $f(59, 35) =$ 57/140 *INT-exception,* $f(94, 35) = 219/490$ *INT-exception.)*

Case 26: $m = 35k + 26$ *with* $k \geq 2$. *Then* $f(35k + 26, 35) = \frac{35k+26}{70k+70}$.*(Exception:* $f(61, 35) =$ 23/56 *INT-exception,* $f(96, 35) = 157/350$ *INT-exception.)*

Case 27: $m = 35k + 27$ *with* $k \geq 2$. *Then* $f(35k + 27, 35) = \frac{35k+8}{70k+35}$.*(Exception:* $f(62, 35) =$ 143/350 *INT-exception.)*

Case 29: $m = 35k + 29$ *with* $k \geq 1$. *Then* $f(35k + 29, 35) = \frac{35k+6}{70k+35}$.

Case 31: $m = 35k + 31$ *with* $k \geq 1$. *Then* $f(35k + 31, 35) = \frac{35k+4}{70k+35}$.

Case 32: $m = 35k + 32$ *with* $k \geq 1$. *Then* $f(35k + 32, 35) = \frac{35k+3}{70k+35}$.

Case 33: $m = 35k + 33$ *with* $k \geq 1$. *Then* $f(35k + 33, 35) = \frac{35k+2}{70k+35}$.

Case 34: $m = 35k + 34$ *with* $k \geq 1$. *Then* $f(35k + 34, 35) = \frac{35k+1}{70k+35}$.

Conjecture B.36 *If* $m = 36k + i$ *where* $0 \leq i \leq 35$ *then* $f(m, 36)$ *depends only on* k, i *via a formula, given below, with 21 exceptions (we will note the exceptions).*

Case 0: $m = 36k + 0$ *with* $k \geq 1$. *Then* $f(36k, 36) = 1$.

Case 1: $m = 36k + 1$ *with* $k \geq 2$. *Then* $f(36k + 1, 36) = \frac{36k+1}{72k+36}$.*(Exception:* $f(37, 36) =$ 85/252 *BM-exception.)*

Case 5: $m = 36k + 5$ *with* $k \geq 2$. *Then* $f(36k + 5, 36) = \frac{36k+5}{72k+36}$.*(Exception:* $f(41, 36) =$ 37/108 *BM-exception.)*

Case 7: $m = 36k + 7$ *with* $k \geq 2$. *Then* $f(36k + 7, 36) = \frac{36k+7}{72k+36}$.*(Exception:* $f(43, 36) = 1/3$ *INT-exception,* $f(79, 36) = 47/108$ *INT-exception.)*

Case 11: $m = 36k + 11$ *with* $k \geq 1$. *Then* $f(36k + 11, 36) = \frac{36k-11}{72k}$.

Case 13: $m = 36k + 13$ *with* $k \geq 2$. *Then* $f(36k + 13, 36) = \frac{36k-13}{72k}$.*(Exception:* $f(49, 36) = 1/3$ *FC-exception.)*

Case 17: $m = 36k + 17$ *with* $k \geq 2$. *Then* $f(36k + 17, 36) = \frac{36k-17}{72k}$.*(Exception:* $f(53, 36) = 1/3$ *FC-exception.)*

Case 19: $m = 36k + 19$ *with* $k \geq 1$. *Then* $f(36k + 19, 36) = \frac{36k+19}{72k+72}$.

Case 23: $m = 36k + 23$ *with* $k \geq 2$. *Then* $f(36k + 23, 36) = \frac{36k+23}{72k+72}$.*(Exception:* $f(59, 36) = 11/27$ *INT-exception.)*

Case 25: $m = 36k + 25$ *with* $k \geq 2$. *Then* $f(36k + 25, 36) = \frac{36k+25}{72k+72}$.*(Exception:* $f(61, 36) = 29/72$ *INT-exception,* $f(97, 36) = 59/132$ *INT-exception.)*

Case 29: $m = 36k + 29$ *with* $k \geq 1$. *Then* $f(36k + 29, 36) = \frac{36k+7}{72k+36}$.

Case 31: $m = 36k + 31$ *with* $k \geq 1$. *Then* $f(36k + 31, 36) = \frac{36k+5}{72k+36}$.

Case 35: $m = 36k + 35$ *with* $k \geq 1$. *Then* $f(36k + 35, 36) = \frac{36k+1}{72k+36}$.

Conjecture B.37 *If* $m = 37k + i$ *where* $0 \leq i \leq 36$ *then* $f(m, 37)$ *depends only on* k, i *via a formula, given below, with 26 exceptions (we will note the exceptions).*

Case 0: $m = 37k + 0$ *with* $k \geq 1$. *Then* $f(37k, 37) = 1$.

Case 1: $m = 37k+1$ *with* $k \geq 2$. *Then* $f(37k+1, 37) = \frac{37k+1}{74k+37}$.*(Exception:* $f(38, 37) = 25/74$ *BM-exception.)*

Case 2: $m = 37k+2$ *with* $k \geq 2$. *Then* $f(37k+2, 37) = \frac{37k+2}{74k+37}$.*(Exception:* $f(39, 37) = 25/74$ *BM-exception.)*

Case 3: $m = 37k+3$ *with* $k \geq 2$. *Then* $f(37k+3, 37) = \frac{37k+3}{74k+37}$.*(Exception:* $f(40, 37) = 25/74$ *INT-exception.)*

Case 4: $m = 37k{+}4$ *with* $k \geq 2$. *Then* $f(37k{+}4, 37) = \frac{37k+4}{74k+37}$.*(Exception:* $f(41, 37) = 25/74$ *INT-exception.)*

Case 5: $m = 37k{+}5$ *with* $k \geq 2$. *Then* $f(37k{+}5, 37) = \frac{37k+5}{74k+37}$.*(Exception:* $f(42, 37) = 13/37$ *INT-exception.)*

Case 6: $m = 37k{+}6$ *with* $k \geq 2$. *Then* $f(37k{+}6, 37) = \frac{37k+6}{74k+37}$.*(Exception:* $f(43, 37) = 25/74$ *INT-exception.)*

Case 7: $m = 37k + 7$ *with* $k \geq 2$. *Then* $f(37k + 7, 37) = \frac{37k+7}{74k+37}$.*(Exception:* $f(44, 37) = 1/3$ *INT-exception,* $f(81, 37) = 257/592$ *INT-exception.)*

Case 8: $m = 37k + 8$ *with* $k \geq 2$. *Then* $f(37k + 8, 37) = \frac{37k+8}{74k+37}$.*(Exception:* $f(45, 37) = 53/148$ *INT-exception,* $f(82, 37) = 95/222$ *BM-exception.)*

Case 9: $m = 37k + 9$ *with* $k \geq 2$. *Then* $f(37k + 9, 37) = \frac{37k+9}{74k+37}$.*(Exception:* $f(46, 37) = 38/111$ *INT-exception,* $f(83, 37) = 129/296$ *INT-exception.)*

Case 10: $m = 37k + 10$ *with* $k \geq 2$. *Then* $f(37k + 10, 37) = \frac{37k-10}{74k}$.*(Exception:* $f(47, 37) = 51/148$ *BM-exception.)*

Case 11: $m = 37k + 11$ *with* $k \geq 2$. *Then* $f(37k + 11, 37) = \frac{37k-11}{74k}$.*(Exception:* $102/296 \leq f(48, 37) < 103/296$ *BM-exception[OPEN].)*

Case 12: $m = 37k + 12$ *with* $k \geq 1$. *Then* $f(37k + 12, 37) = \frac{37k-12}{74k}$.

Case 13: $m = 37k + 13$ *with* $k \geq 2$. *Then* $f(37k + 13, 37) = \frac{37k-13}{74k}$.*(Exception:* $f(50, 37) = 1/3$ *FC-exception.)*

Case 14: $m = 37k + 14$ *with* $k \geq 2$. *Then* $f(37k + 14, 37) = \frac{37k-14}{74k}$.*(Exception:* $f(51, 37) = 1/3$ *FC-exception.)*

Case 15: $m = 37k + 15$ *with* $k \geq 2$. *Then* $f(37k + 15, 37) = \frac{37k-15}{74k}$.*(Exception:* $f(52, 37) = 1/3$ *FC-exception.)*

Case 16: $m = 37k + 16$ *with* $k \geq 2$. *Then* $f(37k + 16, 37) = \frac{37k-16}{74k}$.*(Exception:* $f(53, 37) = 1/3$ *FC-exception.)*

Case 17: $m = 37k + 17$ *with* $k \geq 2$. *Then* $f(37k + 17, 37) = \frac{37k-17}{74k}$.*(Exception:* $f(54, 37) =$

1/3 *FC-exception.)*

Case 18: $m = 37k + 18$ *with* $k \geq 2$. *Then* $f(37k + 18, 37) = \frac{37k-18}{74k}$.*(Exception:* $f(55, 37) =$ 1/3 *FC-exception.)*

Case 19: $m = 37k + 19$ *with* $k \geq 1$. *Then* $f(37k + 19, 37) = \frac{37k+19}{74k+74}$.

Case 20: $m = 37k + 20$ *with* $k \geq 1$. *Then* $f(37k + 20, 37) = \frac{37k+20}{74k+74}$.

Case 21: $m = 37k + 21$ *with* $k \geq 1$. *Then* $f(37k + 21, 37) = \frac{37k+21}{74k+74}$.

Case 22: $m = 37k + 22$ *with* $k \geq 1$. *Then* $f(37k + 22, 37) = \frac{37k+22}{74k+74}$.

Case 23: $m = 37k + 23$ *with* $k \geq 1$. *Then* $f(37k + 23, 37) = \frac{37k+23}{74k+74}$.

Case 24: $m = 37k + 24$ *with* $k \geq 2$. *Then* $f(37k + 24, 37) = \frac{37k+24}{74k+74}$.*(Exception:* $f(61, 37) =$ 181/444 *INT-exception.)*

Case 25: $m = 37k + 25$ *with* $k \geq 2$. *Then* $f(37k + 25, 37) = \frac{37k+25}{74k+74}$.*(Exception:* $f(62, 37) =$ 137/333 *INT-exception.)*

Case 26: $m = 37k + 26$ *with* $k \geq 2$. *Then* $f(37k + 26, 37) = \frac{37k+26}{74k+74}$.*(Exception:* $f(63, 37) =$ 59/148 *BM-exception,* $f(100, 37) = 83/185$ *INT-exception.)*

Case 27: $m = 37k + 27$ *with* $k \geq 1$. *Then* $f(37k + 27, 37) = \frac{37k+27}{74k+74}$.*(Exception:* $f(64, 37) =$ 91/222 *INT-exception.)*

Case 28: $m = 37k + 28$ *with* $k \geq 1$. *Then* $f(37k + 28, 37) = \frac{37k+9}{74k+37}$.*(Exception:* $f(65, 37) =$ 121/296 *INT-exception.)*

Case 29: $m = 37k + 29$ *with* $k \geq 1$. *Then* $f(37k + 29, 37) = \frac{37k+8}{74k+37}$.

Case 30: $m = 37k + 30$ *with* $k \geq 1$. *Then* $f(37k + 30, 37) = \frac{37k+7}{74k+37}$.

Case 31: $m = 37k + 31$ *with* $k \geq 1$. *Then* $f(37k + 31, 37) = \frac{37k+6}{74k+37}$.

Case 32: $m = 37k + 32$ *with* $k \geq 1$. *Then* $f(37k + 32, 37) = \frac{37k+5}{74k+37}$.

Case 33: $m = 37k + 33$ *with* $k \geq 1$. *Then* $f(37k + 33, 37) = \frac{37k+4}{74k+37}$.

Case 34: $m = 37k + 34$ *with* $k \geq 1$. *Then* $f(37k + 34, 37) = \frac{37k+3}{74k+37}$.

Case 35: $m = 37k + 35$ *with* $k \geq 1$. *Then* $f(37k + 35, 37) = \frac{37k+2}{74k+37}$.

Case 36: $m = 37k + 36$ *with* $k \geq 1$. *Then* $f(37k + 36, 37) = \frac{37k+1}{74k+37}$.

Conjecture B.38 *If $m = 38k + i$ where $0 \le i \le 37$ then $f(m, 38)$ depends only on k, i via a formula, given below, with 25 exceptions (we will note the exceptions).*

Case 0: $m = 38k + 0$ *with* $k \ge 1$. *Then* $f(38k, 38) = 1$.

Case 1: $m = 38k + 1$ *with* $k \ge 1$. *Then* $f(38k + 1, 38) = \frac{38k+1}{76k+38}$.

Case 3: $m = 38k+3$ *with* $k \ge 2$. *Then* $f(38k+3, 38) = \frac{38k+3}{76k+38}$.*(Exception:* $f(41, 38) = 13/38$ *INT-exception.)*

Case 5: $m = 38k + 5$ *with* $k \ge 2$. *Then* $f(38k + 5, 38) = \frac{38k+5}{76k+38}$.*(Exception:* $f(43, 38) = 53/152$ *INT-exception.)*

Case 7: $m = 38k + 7$ *with* $k \ge 2$. *Then* $f(38k + 7, 38) = \frac{38k+7}{76k+38}$.*(Exception:* $f(45, 38) = 1/3$ *INT-exception,* $f(83, 38) = 131/304$ *INT-exception.)*

Case 9: $m = 38k+9$ *with* $k \ge 2$. *Then* $f(38k+9, 38) = \frac{38k+9}{76k+38}$.*(Exception:* $f(47, 38) = 20/57$ *INT-exception,* $f(85, 38) = 33/76$ *INT-exception.)*

Case 11: $m = 38k + 11$ *with* $k \ge 2$. *Then* $f(38k + 11, 38) = \frac{38k-11}{76k}$.*(Exception:* $f(49, 38) = 93/266$ *BM-exception.)*

Case 13: $m = 38k + 13$ *with* $k \ge 2$. *Then* $f(38k + 13, 38) = \frac{38k-13}{76k}$.*(Exception:* $f(51, 38) = 1/3$ *FC-exception.)*

Case 15: $m = 38k + 15$ *with* $k \ge 2$. *Then* $f(38k + 15, 38) = \frac{38k-15}{76k}$.*(Exception:* $f(53, 38) = 1/3$ *FC-exception.)*

Case 17: $m = 38k + 17$ *with* $k \ge 2$. *Then* $f(38k + 17, 38) = \frac{38k-17}{76k}$.*(Exception:* $f(55, 38) = 1/3$ *FC-exception.)*

Case 21: $m = 38k + 21$ *with* $k \ge 1$. *Then* $f(38k + 21, 38) = \frac{38k+21}{76k+76}$.

Case 23: $m = 38k + 23$ *with* $k \ge 1$. *Then* $f(38k + 23, 38) = \frac{38k+23}{76k+76}$.

Case 25: $m = 38k + 25$ *with* $k \ge 1$. *Then* $f(38k + 25, 38) = \frac{38k+25}{76k+76}$.*(Exception:* $f(63, 38) = 77/190$ *INT-exception.)*

Case 27: $m = 38k + 27$ *with* $k \geq 1$. *Then* $f(38k + 27, 38) = \frac{38k+27}{76k+76}$.*(Exception:* $f(65, 38) = 15/38$ *BM-exception.)*

Case 29: $m = 38k + 29$ *with* $k \geq 1$. *Then* $f(38k + 29, 38) = \frac{38k+9}{76k+38}$.*(Exception:* $f(67, 38) = 31/76$ *INT-exception.)*

Case 31: $m = 38k + 31$ *with* $k \geq 1$. *Then* $f(38k + 31, 38) = \frac{38k+7}{76k+38}$.

Case 33: $m = 38k + 33$ *with* $k \geq 1$. *Then* $f(38k + 33, 38) = \frac{38k+5}{76k+38}$.

Case 35: $m = 38k + 35$ *with* $k \geq 1$. *Then* $f(38k + 35, 38) = \frac{38k+3}{76k+38}$.

Case 37: $m = 38k + 37$ *with* $k \geq 1$. *Then* $f(38k + 37, 38) = \frac{38k+1}{76k+38}$.

Conjecture B.39 *If* $m = 39k + i$ *where* $0 \leq i \leq 38$ *then* $f(m, 39)$ *depends only on* k, i *via a formula, given below, with 27 exceptions (we will note the exceptions).*

Case 0: $m = 39k + 0$ *with* $k \geq 1$. *Then* $f(39k, 39) = 1$.

Case 1: $m = 39k + 1$ *with* $k \geq 2$. *Then* $f(39k + 1, 39) = \frac{39k+1}{78k+39}$.*(Exception:* $f(40, 39) = 92/273$ *BM-exception.)*

Case 2: $m = 39k + 2$ *with* $k \geq 2$. *Then* $f(39k + 2, 39) = \frac{39k+2}{78k+39}$.*(Exception:* $f(41, 39) = 53/156$ *BM-exception.)*

Case 4: $m = 39k + 4$ *with* $k \geq 2$. *Then* $f(39k + 4, 39) = \frac{39k+4}{78k+39}$.*(Exception:* $f(43, 39) = 53/156$ *BM-exception.)*

Case 5: $m = 39k+5$ *with* $k \geq 2$. *Then* $f(39k+5, 39) = \frac{39k+5}{78k+39}$.*(Exception:* $f(44, 39) = 9/26$ *INT-exception.)*

Case 7: $m = 39k+7$ *with* $k \geq 2$. *Then* $f(39k+7, 39) = \frac{39k+7}{78k+39}$.*(Exception:* $f(46, 39) = 1/3$ *INT-exception,* $f(85, 39) = 45/104$ *INT-exception.)*

Case 8: $m = 39k + 8$ *with* $k \geq 2$. *Then* $f(39k + 8, 39) = \frac{39k+8}{78k+39}$.*(Exception:* $f(47, 39) = 55/156$ *INT-exception,* $f(86, 39) = 101/234$ *INT-exception.)*

Case 10: $m = 39k + 10$ *with* $k \geq 2$*. Then* $f(39k + 10, 39) = \frac{39k-10}{78k}$*.(Exception:* $f(49, 39) =$ $23/65$ *INT-exception,* $f(88, 39) = 62/143$ *INT-exception.)*

Case 11: $m = 39k + 11$ *with* $k \geq 2$*. Then* $f(39k + 11, 39) = \frac{39k-11}{78k}$*.(Exception:* $f(50, 39) =$ $68/195$ *BM-exception.)*

Case 14: $m = 39k + 14$ *with* $k \geq 2$*. Then* $f(39k + 14, 39) = \frac{39k-14}{78k}$*.(Exception:* $f(53, 39) =$ $1/3$ *FC-exception.)*

Case 16: $m = 39k + 16$ *with* $k \geq 2$*. Then* $f(39k + 16, 39) = \frac{39k-16}{78k}$*.(Exception:* $f(55, 39) =$ $1/3$ *FC-exception.)*

Case 17: $m = 39k + 17$ *with* $k \geq 2$*. Then* $f(39k + 17, 39) = \frac{39k-17}{78k}$*.(Exception:* $f(56, 39) =$ $1/3$ *FC-exception.)*

Case 19: $m = 39k + 19$ *with* $k \geq 2$*. Then* $f(39k + 19, 39) = \frac{39k-19}{78k}$*.(Exception:* $f(58, 39) =$ $1/3$ *FC-exception.)*

Case 20: $m = 39k + 20$ *with* $k \geq 1$*. Then* $f(39k + 20, 39) = \frac{39k+20}{78k+78}$.

Case 22: $m = 39k + 22$ *with* $k \geq 1$*. Then* $f(39k + 22, 39) = \frac{39k+22}{78k+78}$.

Case 23: $m = 39k + 23$ *with* $k \geq 1$*. Then* $f(39k + 23, 39) = \frac{39k+23}{78k+78}$.

Case 25: $m = 39k + 25$ *with* $k \geq 1$*. Then* $f(39k + 25, 39) = \frac{39k+25}{78k+78}$*.(Exception:* $f(64, 39) =$ $95/234$ *INT-exception.)*

Case 28: $m = 39k + 28$ *with* $k \geq 1$*. Then* $f(39k + 28, 39) = \frac{39k+28}{78k+78}$*.(Exception:* $f(67, 39) =$ $95/234$ *INT-exception.)*

Case 29: $m = 39k + 29$ *with* $k \geq 1$*. Then* $f(39k + 29, 39) = \frac{39k+29}{78k+78}$*.(Exception:* $f(68, 39) =$ $16/39$ *INT-exception.)*

Case 31: $m = 39k + 31$ *with* $k \geq 1$*. Then* $f(39k + 31, 39) = \frac{39k+8}{78k+39}$.

Case 32: $m = 39k + 32$ *with* $k \geq 1$*. Then* $f(39k + 32, 39) = \frac{39k+7}{78k+39}$.

Case 34: $m = 39k + 34$ *with* $k \geq 1$*. Then* $f(39k + 34, 39) = \frac{39k+5}{78k+39}$.

Case 35: $m = 39k + 35$ *with* $k \geq 1$*. Then* $f(39k + 35, 39) = \frac{39k+4}{78k+39}$.

Case 37: $m = 39k + 37$ *with* $k \geq 1$*. Then* $f(39k + 37, 39) = \frac{39k+2}{78k+39}$.

Case 38: $m = 39k + 38$ *with* $k \geq 1$*. Then* $f(39k + 38, 39) = \frac{39k+1}{78k+39}$.

Conjecture B.40 *If* $m = 40k + i$ *where* $0 \leq i \leq 39$ *then* $f(m, 40)$ *depends only on* k, i *via a formula, given below, with 20 exceptions (we will note the exceptions).*

Case 0: $m = 40k + 0$ *with* $k \geq 1$*. Then* $f(40k, 40) = 1$.

Case 1: $m = 40k+1$ *with* $k \geq 2$*. Then* $f(40k+1, 40) = \frac{40k+1}{80k+40}$*.(Exception:* $f(41, 40) = 27/80$ *BM-exception.)*

Case 3: $m = 40k + 3$ *with* $k \geq 2$*. Then* $f(40k + 3, 40) = \frac{40k+3}{80k+40}$*.(Exception:* $f(43, 40) = 41/120$ *BM-exception.)*

Case 7: $m = 40k + 7$ *with* $k \geq 2$*. Then* $f(40k + 7, 40) = \frac{40k+7}{80k+40}$*.(Exception:* $f(47, 40) = 1/3$ *INT-exception,* $f(87, 40) = 13/30$ *INT-exception.)*

Case 9: $m = 40k + 9$ *with* $k \geq 2$*. Then* $f(40k + 9, 40) = \frac{40k+9}{80k+40}$*.(Exception:* $f(49, 40) = 57/160$ *BM-exception,* $f(89, 40) = 69/160$ *INT-exception.)*

Case 11: $m = 40k + 11$ *with* $k \geq 2$*. Then* $f(40k + 11, 40) = \frac{40k-11}{80k}$*.(Exception:* $f(51, 40) = 7/20$ *INT-exception.)*

Case 13: $m = 40k + 13$ *with* $k \geq 1$*. Then* $f(40k + 13, 40) = \frac{40k-13}{80k}$.

Case 17: $m = 40k + 17$ *with* $k \geq 2$*. Then* $f(40k + 17, 40) = \frac{40k-17}{80k}$*.(Exception:* $f(57, 40) = 1/3$ *FC-exception.)*

Case 19: $m = 40k + 19$ *with* $k \geq 2$*. Then* $f(40k + 19, 40) = \frac{40k-19}{80k}$*.(Exception:* $f(59, 40) = 1/3$ *FC-exception.)*

Case 21: $m = 40k + 21$ *with* $k \geq 1$*. Then* $f(40k + 21, 40) = \frac{40k+21}{80k+80}$.

Case 23: $m = 40k + 23$ *with* $k \geq 1$*. Then* $f(40k + 23, 40) = \frac{40k+23}{80k+80}$.

Case 27: $m = 40k + 27$ *with* $k \geq 1$*. Then* $f(40k + 27, 40) = \frac{40k+27}{80k+80}$*.(Exception:* $f(67, 40) = 37/90$ *INT-exception.)*

Case 29: $m = 40k + 29$ *with* $k \geq 1$. *Then* $f(40k + 29, 40) = \frac{40k+29}{80k+80}$.*(Exception:* $f(69, 40) =$ $49/120$ *INT-exception.)*

Case 31: $m = 40k + 31$ *with* $k \geq 1$. *Then* $f(40k + 31, 40) = \frac{40k+9}{80k+40}$.

Case 33: $m = 40k + 33$ *with* $k \geq 1$. *Then* $f(40k + 33, 40) = \frac{40k+7}{80k+40}$.

Case 37: $m = 40k + 37$ *with* $k \geq 1$. *Then* $f(40k + 37, 40) = \frac{40k+3}{80k+40}$.

Case 39: $m = 40k + 39$ *with* $k \geq 1$. *Then* $f(40k + 39, 40) = \frac{40k+1}{80k+40}$.

Conjecture B.41 *If* $m = 41k + i$ *where* $0 \leq i \leq 40$ *then* $f(m, 41)$ *depends only on* k, i *via a formula, given below, with 27 exceptions (we will note the exceptions).*

Case 0: $m = 41k + 0$ *with* $k \geq 1$. *Then* $f(41k, 41) = 1$.

Case 1: $m = 41k + 1$ *with* $k \geq 1$. *Then* $f(41k + 1, 41) = \frac{41k+1}{82k+41}$.

Case 2: $m = 41k + 2$ *with* $k \geq 2$. *Then* $f(41k + 2, 41) = \frac{41k+2}{82k+41}$.*(Exception:* $f(43, 41) = 1/3$ *BM-exception.)*

Case 3: $m = 41k+3$ *with* $k \geq 2$. *Then* $f(41k+3, 41) = \frac{41k+3}{82k+41}$.*(Exception:* $f(44, 41) = 14/41$ *BM-exception.)*

Case 4: $m = 41k+4$ *with* $k \geq 2$. *Then* $f(41k+4, 41) = \frac{41k+4}{82k+41}$.*(Exception:* $f(45, 41) = 14/41$ *BM-exception.)*

Case 5: $m = 41k + 5$ *with* $k \geq 2$. *Then* $f(41k + 5, 41) = \frac{41k+5}{82k+41}$.*(Exception:* $f(46, 41) = 1/3$ *INT-exception.)*

Case 6: $m = 41k + 6$ *with* $k \geq 2$. *Then* $f(41k + 6, 41) = \frac{41k+6}{82k+41}$.*(Exception:* $f(47, 41) =$ $85/246$ *BM-exception.)*

Case 7: $m = 41k + 7$ *with* $k \geq 2$. *Then* $f(41k + 7, 41) = \frac{41k+7}{82k+41}$.*(Exception:* $f(48, 41) = 1/3$ *INT-exception,* $f(89, 41) = 177/410$ *INT-exception.)*

Case 8: $m = 41k + 8$ *with* $k \geq 2$. *Then* $f(41k + 8, 41) = \frac{41k+8}{82k+41}$.*(Exception:* $f(49, 41) = 1/3$ *INT-exception,* $f(90, 41) = 107/246$ *INT-exception.)*

Case 9: $m = 41k + 9$ *with* $k \geq 2$. *Then* $f(41k + 9, 41) = \frac{41k+9}{82k+41}$.*(Exception:* $58/164 \leq f(50, 41) < 59/164$ *INT-exception[OPEN],* $f(91, 41) = 35/82$ *BM-exception.)*

Case 10: $m = 41k + 10$ *with* $k \geq 2$. *Then* $f(41k + 10, 41) = \frac{41k+10}{82k+41}$.*(Exception:* $f(51, 41) = 14/41$ *INT-exception,* $f(92, 41) = 268/615$ *INT-exception.)*

Case 11: $m = 41k + 11$ *with* $k \geq 2$. *Then* $f(41k + 11, 41) = \frac{41k-11}{82k}$.*(Exception:* $f(52, 41) = 57/164$ *INT-exception.)*

Case 12: $m = 41k + 12$ *with* $k \geq 2$. *Then* $f(41k + 12, 41) = \frac{41k-12}{82k}$.*(Exception:* $f(53, 41) = 85/246$ *BM-exception.)*

Case 13: $m = 41k + 13$ *with* $k \geq 1$. *Then* $f(41k + 13, 41) = \frac{41k-13}{82k}$.

Case 14: $m = 41k + 14$ *with* $k \geq 2$. *Then* $f(41k + 14, 41) = \frac{41k-14}{82k}$.*(Exception:* $f(55, 41) = 1/3$ *FC-exception.)*

Case 15: $m = 41k + 15$ *with* $k \geq 2$. *Then* $f(41k + 15, 41) = \frac{41k-15}{82k}$.*(Exception:* $f(56, 41) = 1/3$ *FC-exception.)*

Case 16: $m = 41k + 16$ *with* $k \geq 2$. *Then* $f(41k + 16, 41) = \frac{41k-16}{82k}$.*(Exception:* $f(57, 41) = 1/3$ *FC-exception.)*

Case 17: $m = 41k + 17$ *with* $k \geq 2$. *Then* $f(41k + 17, 41) = \frac{41k-17}{82k}$.*(Exception:* $f(58, 41) = 1/3$ *FC-exception.)*

Case 18: $m = 41k + 18$ *with* $k \geq 2$. *Then* $f(41k + 18, 41) = \frac{41k-18}{82k}$.*(Exception:* $f(59, 41) = 1/3$ *FC-exception.)*

Case 19: $m = 41k + 19$ *with* $k \geq 1$. *Then* $f(41k + 19, 41) = \frac{41k-19}{82k}$.*(Exception:* $f(60, 41) = 1/3$ *FC-exception.)*

Case 20: $m = 41k + 20$ *with* $k \geq 1$. *Then* $f(41k + 20, 41) = \frac{41k-20}{82k}$.*(Exception:* $f(61, 41) = 1/3$ *FC-exception.)*

Case 21: $m = 41k + 21$ *with* $k \geq 1$. *Then* $f(41k + 21, 41) = \frac{41k+21}{82k+82}$.

Case 22: $m = 41k + 22$ *with* $k \geq 1$. *Then* $f(41k + 22, 41) = \frac{41k+22}{82k+82}$.

Case 23: $m = 41k + 23$ *with* $k \geq 1$. *Then* $f(41k + 23, 41) = \frac{41k+23}{82k+82}$.

Case 24: $m = 41k + 24$ *with* $k \geq 1$. *Then* $f(41k + 24, 41) = \frac{41k+24}{82k+82}$.

Case 25: $m = 41k + 25$ *with* $k \geq 1$. *Then* $f(41k + 25, 41) = \frac{41k+25}{82k+82}$.

Case 26: $m = 41k + 26$ *with* $k \geq 1$. *Then* $f(41k + 26, 41) = \frac{41k+26}{82k+82}$.

Case 27: $m = 41k + 27$ *with* $k \geq 1$. *Then* $f(41k + 27, 41) = \frac{41k+27}{82k+82}$.*(Exception:* $f(68, 41) = 83/205$ *INT-exception.)*

Case 28: $m = 41k + 28$ *with* $k \geq 1$. *Then* $f(41k + 28, 41) = \frac{41k+28}{82k+82}$.*(Exception:* $f(69, 41) = 67/164$ *INT-exception.)*

Case 29: $m = 41k + 29$ *with* $k \geq 1$. *Then* $f(41k + 29, 41) = \frac{41k+29}{82k+82}$.*(Exception:* $f(70, 41) = 65/164$ *BM-exception.)*

Case 30: $m = 41k + 30$ *with* $k \geq 1$. *Then* $f(41k + 30, 41) = \frac{41k+30}{82k+82}$.*(Exception:* $f(71, 41) = 101/246$ *INT-exception.)*

Case 31: $m = 41k + 31$ *with* $k \geq 1$. *Then* $f(41k + 31, 41) = \frac{41k+10}{82k+41}$.*(Exception:* $f(72, 41) = 67/164$ *INT-exception.)*

Case 32: $m = 41k + 32$ *with* $k \geq 1$. *Then* $f(41k + 32, 41) = \frac{41k+9}{82k+41}$.

Case 33: $m = 41k + 33$ *with* $k \geq 1$. *Then* $f(41k + 33, 41) = \frac{41k+8}{82k+41}$.

Case 34: $m = 41k + 34$ *with* $k \geq 1$. *Then* $f(41k + 34, 41) = \frac{41k+7}{82k+41}$.

Case 35: $m = 41k + 35$ *with* $k \geq 1$. *Then* $f(41k + 35, 41) = \frac{41k+6}{82k+41}$.

Case 36: $m = 41k + 36$ *with* $k \geq 1$. *Then* $f(41k + 36, 41) = \frac{41k+5}{82k+41}$.

Case 37: $m = 41k + 37$ *with* $k \geq 1$. *Then* $f(41k + 37, 41) = \frac{41k+4}{82k+41}$.

Case 38: $m = 41k + 38$ *with* $k \geq 1$. *Then* $f(41k + 38, 41) = \frac{41k+3}{82k+41}$.

Case 39: $m = 41k + 39$ *with* $k \geq 1$. *Then* $f(41k + 39, 41) = \frac{41k+2}{82k+41}$.

Case 40: $m = 41k + 40$ *with* $k \geq 1$. *Then* $f(41k + 40, 41) = \frac{41k+1}{82k+41}$.

Conjecture B.42 *If* $m = 42k + i$ *where* $0 \leq i \leq 41$ *then* $f(m, 42)$ *depends only on* k, i *via a formula, given below, with 25 exceptions (we will note the exceptions).*

Case 0: $m = 42k + 0$ *with* $k \geq 1$. *Then* $f(42k, 42) = 1$.

Case 1: $m = 42k+1$ *with* $k \geq 2$. *Then* $f(42k+1, 42) = \frac{42k+1}{84k+42}$.*(Exception:* $f(43, 42) = 33/98$ *BM-exception.)*

Case 5: $m = 42k + 5$ *with* $k \geq 2$. *Then* $f(42k + 5, 42) = \frac{42k+5}{84k+42}$.*(Exception:* $f(47, 42) = 1/3$ *INT-exception.)*

Case 11: $m = 42k + 11$ *with* $k \geq 2$. *Then* $f(42k + 11, 42) = \frac{42k-11}{84k}$.*(Exception:* $f(53, 42) = 5/14$ *INT-exception,* $f(95, 42) = 127/294$ *INT-exception.)*

Case 13: $m = 42k + 13$ *with* $k \geq 1$. *Then* $f(42k + 13, 42) = \frac{42k-13}{84k}$.

Case 17: $m = 42k + 17$ *with* $k \geq 1$. *Then* $f(42k + 17, 42) = \frac{42k-17}{84k}$.*(Exception:* $f(59, 42) = 1/3$ *FC-exception.)*

Case 19: $m = 42k + 19$ *with* $k \geq 1$. *Then* $f(42k + 19, 42) = \frac{42k-19}{84k}$.*(Exception:* $f(61, 42) = 1/3$ *FC-exception.)*

Case 23: $m = 42k + 23$ *with* $k \geq 1$. *Then* $f(42k + 23, 42) = \frac{42k+23}{84k+84}$.

Case 25: $m = 42k + 25$ *with* $k \geq 1$. *Then* $f(42k + 25, 42) = \frac{42k+25}{84k+84}$.

Case 29: $m = 42k + 29$ *with* $k \geq 1$. *Then* $f(42k + 29, 42) = \frac{42k+29}{84k+84}$.*(Exception:* $f(71, 42) = 17/42$ *INT-exception.)*

Case 31: $m = 42k + 31$ *with* $k \geq 1$. *Then* $f(42k + 31, 42) = \frac{42k+31}{84k+84}$.*(Exception:* $f(73, 42) = 26/63$ *INT-exception.)*

Case 37: $m = 42k + 37$ *with* $k \geq 1$. *Then* $f(42k + 37, 42) = \frac{42k+5}{84k+42}$.

Case 41: $m = 42k + 41$ *with* $k \geq 1$. *Then* $f(42k + 41, 42) = \frac{42k+1}{84k+42}$.

Conjecture B.43 *If* $m = 43k + i$ *where* $0 \leq i \leq 42$ *then* $f(m, 43)$ *depends only on* k, i *via a formula, given below, with 29 exceptions (we will note the exceptions).*

Case 0: $m = 43k + 0$ *with* $k \geq 1$. *Then* $f(43k, 43) = 1$.

Case 1: $m = 43k{+}1$ *with* $k \geq 2$. *Then* $f(43k{+}1, 43) = \frac{43k+1}{86k+43}$.*(Exception:* $f(44, 43) = 29/86$ *BM-exception.)*

Case 2: $m = 43k{+}2$ *with* $k \geq 2$. *Then* $f(43k{+}2, 43) = \frac{43k+2}{86k+43}$.*(Exception:* $f(45, 43) = 29/86$ *BM-exception.)*

Case 3: $m = 43k + 3$ *with* $k \geq 2$. *Then* $f(43k + 3, 43) = \frac{43k+3}{86k+43}$.*(Exception:* $f(46, 43) = 1/3$ *BM-exception.)*

Case 4: $m = 43k + 4$ *with* $k \geq 2$. *Then* $f(43k + 4, 43) = \frac{43k+4}{86k+43}$.*(Exception:* $f(47, 43) = 59/172$ *INT-exception.)*

Case 5: $m = 43k + 5$ *with* $k \geq 2$. *Then* $f(43k + 5, 43) = \frac{43k+5}{86k+43}$.*(Exception:* $f(48, 43) = 1/3$ *INT-exception.)*

Case 6: $m = 43k + 6$ *with* $k \geq 2$. *Then* $f(43k + 6, 43) = \frac{43k+6}{86k+43}$.*(Exception:* $f(49, 43) = 44/129$ *BM-exception.)*

Case 7: $m = 43k{+}7$ *with* $k \geq 2$. *Then* $f(43k{+}7, 43) = \frac{43k+7}{86k+43}$.*(Exception:* $f(50, 43) = 29/86$ *INT-exception.)*

Case 8: $m = 43k + 8$ *with* $k \geq 2$. *Then* $f(43k + 8, 43) = \frac{43k+8}{86k+43}$.*(Exception:* $f(51, 43) = 1/3$ *INT-exception,* $f(94, 43) = 37/86$ *INT-exception.)*

Case 9: $m = 43k + 9$ *with* $k \geq 2$. *Then* $f(43k + 9, 43) = \frac{43k+9}{86k+43}$.*(Exception:* $f(52, 43) = 61/172$ *INT-exception,* $f(95, 43) = 37/86$ *INT-exception.)*

Case 10: $m = 43k + 10$ *with* $k \geq 2$. *Then* $f(43k + 10, 43) = \frac{43k+10}{86k+43}$.*(Exception:* $f(53, 43) = 46/129$ *INT-exception,* $f(96, 43) = 149/344$ *INT-exception.)*

Case 11: $m = 43k + 11$ *with* $k \geq 2$. *Then* $f(43k + 11, 43) = \frac{43k-11}{86k}$.*(Exception:* $f(54, 43) = 76/215$ *INT-exception,* $f(97, 43) = 205/473$ *INT-exception.)*

Case 12: $m = 43k + 12$ *with* $k \geq 2$. *Then* $f(43k + 12, 43) = \frac{43k-12}{86k}$.*(Exception:* $f(55, 43) = 91/258$ *INT-exception.)*

Case 13: $m = 43k + 13$ *with* $k \geq 1$. *Then* $f(43k + 13, 43) = \frac{43k-13}{86k}$.

Case 14: $m = 43k + 14$ *with* $k \geq 1$. *Then* $f(43k + 14, 43) = \frac{43k-14}{86k}$.

Case 15: $m = 43k + 15$ *with* $k \geq 1$. *Then* $f(43k + 15, 43) = \frac{43k-15}{86k}$.*(Exception:* $f(58, 43) = 1/3$ *FC-exception.)*

Case 16: $m = 43k + 16$ *with* $k \geq 1$. *Then* $f(43k + 16, 43) = \frac{43k-16}{86k}$.*(Exception:* $f(59, 43) = 1/3$ *FC-exception.)*

Case 17: $m = 43k + 17$ *with* $k \geq 1$. *Then* $f(43k + 17, 43) = \frac{43k-17}{86k}$.*(Exception:* $f(60, 43) = 1/3$ *FC-exception.)*

Case 18: $m = 43k + 18$ *with* $k \geq 1$. *Then* $f(43k + 18, 43) = \frac{43k-18}{86k}$.*(Exception:* $f(61, 43) = 1/3$ *FC-exception.)*

Case 19: $m = 43k + 19$ *with* $k \geq 1$. *Then* $f(43k + 19, 43) = \frac{43k-19}{86k}$.*(Exception:* $f(62, 43) = 1/3$ *FC-exception.)*

Case 20: $m = 43k + 20$ *with* $k \geq 1$. *Then* $f(43k + 20, 43) = \frac{43k-20}{86k}$.*(Exception:* $f(63, 43) = 1/3$ *FC-exception.)*

Case 21: $m = 43k + 21$ *with* $k \geq 1$. *Then* $f(43k + 21, 43) = \frac{43k-21}{86k}$.*(Exception:* $f(64, 43) = 1/3$ *FC-exception.)*

Case 22: $m = 43k + 22$ *with* $k \geq 1$. *Then* $f(43k + 22, 43) = \frac{43k+22}{86k+86}$.

Case 23: $m = 43k + 23$ *with* $k \geq 1$. *Then* $f(43k + 23, 43) = \frac{43k+23}{86k+86}$.

Case 24: $m = 43k + 24$ *with* $k \geq 1$. *Then* $f(43k + 24, 43) = \frac{43k+24}{86k+86}$.

Case 25: $m = 43k + 25$ *with* $k \geq 1$. *Then* $f(43k + 25, 43) = \frac{43k+25}{86k+86}$.

Case 26: $m = 43k + 26$ *with* $k \geq 1$. *Then* $f(43k + 26, 43) = \frac{43k+26}{86k+86}$.

Case 27: $m = 43k + 27$ *with* $k \geq 1$. *Then* $f(43k + 27, 43) = \frac{43k+27}{86k+86}$.

Case 28: $m = 43k + 28$ *with* $k \geq 1$. *Then* $f(43k + 28, 43) = \frac{43k+28}{86k+86}$.*(Exception:* $f(71, 43) = 211/516$ *INT-exception.)*

Case 29: $m = 43k + 29$ *with* $k \geq 1$. *Then* $f(43k + 29, 43) = \frac{43k+29}{86k+86}$.*(Exception:* $f(72, 43) = 53/129$ *INT-exception.)*

Case 30: $m = 43k + 30$ *with* $k \geq 1$. *Then* $f(43k + 30, 43) = \frac{43k+30}{86k+86}$.*(Exception:* $f(73, 43) = 69/172$ *INT-exception.)*

Case 31: $m = 43k + 31$ *with* $k \geq 1$. *Then* $f(43k + 31, 43) = \frac{43k+31}{86k+86}$.*(Exception:* $f(74, 43) =$ $35/86$ *INT-exception.)*

Case 32: $m = 43k + 32$ *with* $k \geq 1$. *Then* $f(43k + 32, 43) = \frac{43k+32}{86k+86}$.*(Exception:* $f(75, 43) =$ $141/344$ *INT-exception.)*

Case 33: $m = 43k + 33$ *with* $k \geq 1$. *Then* $f(43k + 33, 43) = \frac{43k+10}{86k+43}$.*(Exception:* $f(76, 43) =$ $35/86$ *INT-exception.)*

Case 34: $m = 43k + 34$ *with* $k \geq 1$. *Then* $f(43k + 34, 43) = \frac{43k+9}{86k+43}$.

Case 35: $m = 43k + 35$ *with* $k \geq 1$. *Then* $f(43k + 35, 43) = \frac{43k+8}{86k+43}$.

Case 36: $m = 43k + 36$ *with* $k \geq 1$. *Then* $f(43k + 36, 43) = \frac{43k+7}{86k+43}$.

Case 37: $m = 43k + 37$ *with* $k \geq 1$. *Then* $f(43k + 37, 43) = \frac{43k+6}{86k+43}$.

Case 38: $m = 43k + 38$ *with* $k \geq 1$. *Then* $f(43k + 38, 43) = \frac{43k+5}{86k+43}$.

Case 39: $m = 43k + 39$ *with* $k \geq 1$. *Then* $f(43k + 39, 43) = \frac{43k+4}{86k+43}$.

Case 40: $m = 43k + 40$ *with* $k \geq 1$. *Then* $f(43k + 40, 43) = \frac{43k+3}{86k+43}$.

Case 41: $m = 43k + 41$ *with* $k \geq 1$. *Then* $f(43k + 41, 43) = \frac{43k+2}{86k+43}$.

Case 42: $m = 43k + 42$ *with* $k \geq 1$. *Then* $f(43k + 42, 43) = \frac{43k+1}{86k+43}$.

Conjecture B.44 *If* $m = 44k + i$ *where* $0 \leq i \leq 43$ *then* $f(m, 44)$ *depends only on* k, i *via a formula, given below, with 23 exceptions (we will note the exceptions).*

Case 0: $m = 44k + 0$ *with* $k \geq 1$. *Then* $f(44k, 44) = 1$.

Case 1: $m = 44k + 1$ *with* $k \geq 1$. *Then* $f(44k + 1, 44) = \frac{44k+1}{88k+44}$.

Case 3: $m = 44k + 3$ *with* $k \geq 2$. *Then* $f(44k + 3, 44) = \frac{44k+3}{88k+44}$.*(Exception:* $f(47, 44) = 1/3$ *BM-exception.)*

Case 5: $m = 44k + 5$ *with* $k \geq 2$. *Then* $f(44k + 5, 44) = \frac{44k+5}{88k+44}$.*(Exception:* $f(49, 44) = 1/3$ *INT-exception.)*

Case 7: $m = 44k{+}7$ *with* $k \geq 2$. *Then* $f(44k{+}7, 44) = \frac{44k+7}{88k+44}$.*(Exception:* $f(51, 44) = 15/44$ *INT-exception.)*

Case 9: $m = 44k{+}9$ *with* $k \geq 2$. *Then* $f(44k{+}9, 44) = \frac{44k+9}{88k+44}$.*(Exception:* $f(53, 44) = 31/88$ *INT-exception,* $f(97, 44) = 19/44$ *INT-exception.)*

Case 13: $m = 44k + 13$ *with* $k \geq 1$. *Then* $f(44k + 13, 44) = \frac{44k-13}{88k}$.*(Exception:* $f(57, 44) = 61/176$ *BM-exception.)*

Case 15: $m = 44k + 15$ *with* $k \geq 1$. *Then* $f(44k + 15, 44) = \frac{44k-15}{88k}$.*(Exception:* $f(59, 44) = 1/3$ *FC-exception.)*

Case 17: $m = 44k + 17$ *with* $k \geq 1$. *Then* $f(44k + 17, 44) = \frac{44k-17}{88k}$.*(Exception:* $f(61, 44) = 1/3$ *FC-exception.)*

Case 19: $m = 44k + 19$ *with* $k \geq 1$. *Then* $f(44k + 19, 44) = \frac{44k-19}{88k}$.*(Exception:* $f(63, 44) = 1/3$ *FC-exception.)*

Case 21: $m = 44k + 21$ *with* $k \geq 1$. *Then* $f(44k + 21, 44) = \frac{44k-21}{88k}$.*(Exception:* $f(65, 44) = 1/3$ *FC-exception.)*

Case 23: $m = 44k + 23$ *with* $k \geq 1$. *Then* $f(44k + 23, 44) = \frac{44k+23}{88k+88}$.

Case 25: $m = 44k + 25$ *with* $k \geq 1$. *Then* $f(44k + 25, 44) = \frac{44k+25}{88k+88}$.

Case 27: $m = 44k + 27$ *with* $k \geq 1$. *Then* $f(44k + 27, 44) = \frac{44k+27}{88k+88}$.

Case 29: $m = 44k + 29$ *with* $k \geq 1$. *Then* $f(44k + 29, 44) = \frac{44k+29}{88k+88}$.*(Exception:* $f(73, 44) = 89/220$ *INT-exception.)*

Case 31: $m = 44k + 31$ *with* $k \geq 1$. *Then* $f(44k + 31, 44) = \frac{44k+31}{88k+88}$.*(Exception:* $f(75, 44) = 35/88$ *BM-exception.)*

Case 35: $m = 44k + 35$ *with* $k \geq 1$. *Then* $f(44k + 35, 44) = \frac{44k+9}{88k+44}$.

Case 37: $m = 44k + 37$ *with* $k \geq 1$. *Then* $f(44k + 37, 44) = \frac{44k+7}{88k+44}$.

Case 39: $m = 44k + 39$ *with* $k \geq 1$. *Then* $f(44k + 39, 44) = \frac{44k+5}{88k+44}$.

Case 41: $m = 44k + 41$ *with* $k \geq 1$. *Then* $f(44k + 41, 44) = \frac{44k+3}{88k+44}$.

Case 43: $m = 44k + 43$ *with* $k \geq 1$. *Then* $f(44k + 43, 44) = \frac{44k+1}{88k+44}$.

Conjecture B.45 *If $m = 45k + i$ where $0 \le i \le 44$ then $f(m, 45)$ depends only on k, i via a formula, given below, with 27 exceptions (we will note the exceptions).*

Case 0: $m = 45k + 0$ *with* $k \ge 1$. *Then* $f(45k, 45) = 1$.

Case 1: $m = 45k + 1$ *with* $k \ge 2$. *Then* $f(45k + 1, 45) = \frac{45k+1}{90k+45}$.*(Exception:* $f(46, 45) = 106/315$ *BM-exception.)*

Case 2: $m = 45k + 2$ *with* $k \ge 2$. *Then* $f(45k + 2, 45) = \frac{45k+2}{90k+45}$.*(Exception:* $f(47, 45) = 61/180$ *BM-exception.)*

Case 4: $m = 45k + 4$ *with* $k \ge 2$. *Then* $f(45k + 4, 45) = \frac{45k+4}{90k+45}$.*(Exception:* $f(49, 45) = 1/3$ *INT-exception.)*

Case 7: $m = 45k + 7$ *with* $k \ge 2$. *Then* $f(45k + 7, 45) = \frac{45k+7}{90k+45}$.*(Exception:* $f(52, 45) = 46/135$ *BM-exception.)*

Case 8: $m = 45k + 8$ *with* $k \ge 2$. *Then* $f(45k + 8, 45) = \frac{45k+8}{90k+45}$.*(Exception:* $f(53, 45) = 1/3$ *INT-exception,* $f(98, 45) = 13/30$ *INT-exception.)*

Case 11: $m = 45k + 11$ *with* $k \ge 1$. *Then* $f(45k + 11, 45) = \frac{45k+11}{90k+45}$.*(Exception:* $f(56, 45) = 46/135$ *INT-exception.)*

Case 13: $m = 45k + 13$ *with* $k \ge 1$. *Then* $f(45k + 13, 45) = \frac{45k-13}{90k}$.*(Exception:* $f(58, 45) = 22/63$ *BM-exception.)*

Case 14: $m = 45k + 14$ *with* $k \ge 1$. *Then* $f(45k + 14, 45) = \frac{45k-14}{90k}$.

Case 16: $m = 45k + 16$ *with* $k \ge 1$. *Then* $f(45k + 16, 45) = \frac{45k-16}{90k}$.*(Exception:* $f(61, 45) = 1/3$ *FC-exception.)*

Case 17: $m = 45k + 17$ *with* $k \ge 1$. *Then* $f(45k + 17, 45) = \frac{45k-17}{90k}$.*(Exception:* $f(62, 45) = 1/3$ *FC-exception.)*

Case 19: $m = 45k + 19$ *with* $k \ge 1$. *Then* $f(45k + 19, 45) = \frac{45k-19}{90k}$.*(Exception:* $f(64, 45) = 1/3$ *FC-exception.)*

Case 22: $m = 45k + 22$ *with* $k \geq 1$. *Then* $f(45k + 22, 45) = \frac{45k-22}{90k}$.*(Exception:* $f(67, 45) =$ $1/3$ *FC-exception.)*

Case 23: $m = 45k + 23$ *with* $k \geq 1$. *Then* $f(45k + 23, 45) = \frac{45k+23}{90k+90}$.

Case 26: $m = 45k + 26$ *with* $k \geq 1$. *Then* $f(45k + 26, 45) = \frac{45k+26}{90k+90}$.

Case 28: $m = 45k + 28$ *with* $k \geq 1$. *Then* $f(45k + 28, 45) = \frac{45k+28}{90k+90}$.

Case 29: $m = 45k + 29$ *with* $k \geq 1$. *Then* $f(45k + 29, 45) = \frac{45k+29}{90k+90}$.*(Exception:* $f(74, 45) =$ $109/270$ *INT-exception.)*

Case 31: $m = 45k + 31$ *with* $k \geq 1$. *Then* $f(45k + 31, 45) = \frac{45k+31}{90k+90}$.*(Exception:* $f(76, 45) =$ $73/180$ *INT-exception.)*

Case 32: $m = 45k + 32$ *with* $k \geq 1$. *Then* $f(45k + 32, 45) = \frac{45k+32}{90k+90}$.*(Exception:* $f(77, 45) =$ $71/180$ *BM-exception.)*

Case 34: $m = 45k + 34$ *with* $k \geq 1$. *Then* $f(45k + 34, 45) = \frac{45k+11}{90k+45}$.*(Exception:* $f(79, 45) =$ $49/120$ *INT-exception.)*

Case 37: $m = 45k + 37$ *with* $k \geq 1$. *Then* $f(45k + 37, 45) = \frac{45k+8}{90k+45}$.

Case 38: $m = 45k + 38$ *with* $k \geq 1$. *Then* $f(45k + 38, 45) = \frac{45k+7}{90k+45}$.

Case 41: $m = 45k + 41$ *with* $k \geq 1$. *Then* $f(45k + 41, 45) = \frac{45k+4}{90k+45}$.

Case 43: $m = 45k + 43$ *with* $k \geq 1$. *Then* $f(45k + 43, 45) = \frac{45k+2}{90k+45}$.

Case 44: $m = 45k + 44$ *with* $k \geq 1$. *Then* $f(45k + 44, 45) = \frac{45k+1}{90k+45}$.

Conjecture B.46 *If* $m = 46k + i$ *where* $0 \leq i \leq 45$ *then* $f(m, 46)$ *depends only on* k, i *via a formula, given below, with 25 exceptions (we will note the exceptions).*

Case 0: $m = 46k + 0$ *with* $k \geq 1$. *Then* $f(46k, 46) = 1$.

Case 1: $m = 46k+1$ *with* $k \geq 2$. *Then* $f(46k+1, 46) = \frac{46k+1}{92k+46}$.*(Exception:* $f(47, 46) = 31/92$ *BM-exception.)*

Case 3: $m = 46k{+}3$ *with* $k \geq 2$. *Then* $f(46k{+}3, 46) = \frac{46k+3}{92k+46}$.*(Exception:* $f(49, 46) = 31/92$ *BM-exception.)*

Case 5: $m = 46k{+}5$ *with* $k \geq 2$. *Then* $f(46k{+}5, 46) = \frac{46k+5}{92k+46}$.*(Exception:* $f(51, 46) = 31/92$ *INT-exception.)*

Case 7: $m = 46k + 7$ *with* $k \geq 2$. *Then* $f(46k + 7, 46) = \frac{46k+7}{92k+46}$.*(Exception:* $f(53, 46) = 79/230$ *BM-exception.)*

Case 9: $m = 46k + 9$ *with* $k \geq 1$. *Then* $f(46k + 9, 46) = \frac{46k+9}{92k+46}$.*(Exception:* $f(55, 46) = 1/3$ *INT-exception.)*

Case 11: $m = 46k + 11$ *with* $k \geq 1$. *Then* $f(46k + 11, 46) = \frac{46k+11}{92k+46}$.*(Exception:* $f(57, 46) = 8/23$ *INT-exception.)*

Case 13: $m = 46k + 13$ *with* $k \geq 1$. *Then* $f(46k + 13, 46) = \frac{46k-13}{92k}$.*(Exception:* $f(59, 46) = 8/23$ *BM-exception.)*

Case 15: $m = 46k + 15$ *with* $k \geq 1$. *Then* $f(46k + 15, 46) = \frac{46k-15}{92k}$.

Case 17: $m = 46k + 17$ *with* $k \geq 1$. *Then* $f(46k + 17, 46) = \frac{46k-17}{92k}$.*(Exception:* $f(63, 46) = 1/3$ *FC-exception.)*

Case 19: $m = 46k + 19$ *with* $k \geq 1$. *Then* $f(46k + 19, 46) = \frac{46k-19}{92k}$.*(Exception:* $f(65, 46) = 1/3$ *FC-exception.)*

Case 21: $m = 46k + 21$ *with* $k \geq 1$. *Then* $f(46k + 21, 46) = \frac{46k-21}{92k}$.*(Exception:* $f(67, 46) = 1/3$ *FC-exception.)*

Case 25: $m = 46k + 25$ *with* $k \geq 1$. *Then* $f(46k + 25, 46) = \frac{46k+25}{92k+92}$.

Case 27: $m = 46k + 27$ *with* $k \geq 1$. *Then* $f(46k + 27, 46) = \frac{46k+27}{92k+92}$.

Case 29: $m = 46k + 29$ *with* $k \geq 1$. *Then* $f(46k + 29, 46) = \frac{46k+29}{92k+92}$.

Case 31: $m = 46k + 31$ *with* $k \geq 1$. *Then* $f(46k + 31, 46) = \frac{46k+31}{92k+92}$.*(Exception:* $f(77, 46) = 85/207$ *INT-exception.)*

Case 33: $m = 46k + 33$ *with* $k \geq 1$. *Then* $f(46k + 33, 46) = \frac{46k+33}{92k+92}$.*(Exception:* $f(79, 46) = 28/69$ *INT-exception.)*

Case 35: $m = 46k + 35$ *with* $k \geq 1$. *Then* $f(46k + 35, 46) = \frac{46k+11}{92k+46}$.*(Exception:* $f(81, 46) =$ 113/276 *INT-exception.)*

Case 37: $m = 46k + 37$ *with* $k \geq 1$. *Then* $f(46k + 37, 46) = \frac{46k+9}{92k+46}$.

Case 39: $m = 46k + 39$ *with* $k \geq 1$. *Then* $f(46k + 39, 46) = \frac{46k+7}{92k+46}$.

Case 41: $m = 46k + 41$ *with* $k \geq 1$. *Then* $f(46k + 41, 46) = \frac{46k+5}{92k+46}$.

Case 43: $m = 46k + 43$ *with* $k \geq 1$. *Then* $f(46k + 43, 46) = \frac{46k+3}{92k+46}$.

Case 45: $m = 46k + 45$ *with* $k \geq 1$. *Then* $f(46k + 45, 46) = \frac{46k+1}{92k+46}$.

Conjecture B.47 *If* $m = 47k + i$ *where* $0 \leq i \leq 46$ *then* $f(m, 47)$ *depends only on* k, i *via a formula, given below, with 28 exceptions (we will note the exceptions).*

Case 0: $m = 47k + 0$ *with* $k \geq 1$. *Then* $f(47k, 47) = 1$.

Case 1: $m = 47k + 1$ *with* $k \geq 1$. *Then* $f(47k + 1, 47) = \frac{47k+1}{94k+47}$.

Case 2: $m = 47k + 2$ *with* $k \geq 2$. *Then* $f(47k + 2, 47) = \frac{47k+2}{94k+47}$.*(Exception:* $f(49, 47) = 1/3$ *BM-exception.)*

Case 3: $m = 47k+3$ *with* $k \geq 2$. *Then* $f(47k+3, 47) = \frac{47k+3}{94k+47}$.*(Exception:* $f(50, 47) = 16/47$ *BM-exception.)*

Case 4: $m = 47k + 4$ *with* $k \geq 2$. *Then* $f(47k + 4, 47) = \frac{47k+4}{94k+47}$.*(Exception:* $f(51, 47) = 1/3$ *INT-exception.)*

Case 5: $m = 47k+5$ *with* $k \geq 2$. *Then* $f(47k+5, 47) = \frac{47k+5}{94k+47}$.*(Exception:* $f(52, 47) = 16/47$ *INT-exception.)*

Case 6: $m = 47k + 6$ *with* $k \geq 2$. *Then* $f(47k + 6, 47) = \frac{47k+6}{94k+47}$.*(Exception:* $f(53, 47) =$ 65/188 *INT-exception.)*

Case 7: $m = 47k+7$ *with* $k \geq 1$. *Then* $f(47k+7, 47) = \frac{47k+7}{94k+47}$.*(Exception:* $f(54, 47) = 16/47$ *BM-exception.)*

Case 8: $m = 47k + 8$ *with* $k \geq 1$. *Then* $f(47k + 8, 47) = \frac{47k+8}{94k+47}$.*(Exception:* $f(55, 47) = 1/3$ *INT-exception.)*

Case 9: $m = 47k + 9$ *with* $k \geq 1$. *Then* $f(47k + 9, 47) = \frac{47k+9}{94k+47}$.*(Exception:* $f(56, 47) = 1/3$ *INT-exception.)*

Case 10: $m = 47k + 10$ *with* $k \geq 1$. *Then* $f(47k + 10, 47) = \frac{47k+10}{94k+47}$.*(Exception:* $f(57, 47) = 67/188$ *INT-exception.)*

Case 11: $m = 47k + 11$ *with* $k \geq 1$. *Then* $f(47k + 11, 47) = \frac{47k+11}{94k+47}$.*(Exception:* $f(58, 47) = 50/141$ *INT-exception.)*

Case 12: $m = 47k + 12$ *with* $k \geq 1$. *Then* $f(47k + 12, 47) = \frac{47k-12}{94k}$.*(Exception:* $f(59, 47) = 83/235$ *INT-exception.)*

Case 13: $m = 47k + 13$ *with* $k \geq 1$. *Then* $f(47k + 13, 47) = \frac{47k-13}{94k}$.*(Exception:* $f(60, 47) = 33/94$ *INT-exception.)*

Case 14: $m = 47k + 14$ *with* $k \geq 1$. *Then* $f(47k + 14, 47) = \frac{47k-14}{94k}$.*(Exception:* $f(61, 47) = 131/376$ *BM-exception.)*

Case 15: $m = 47k + 15$ *with* $k \geq 1$. *Then* $f(47k + 15, 47) = \frac{47k-15}{94k}$.

Case 16: $m = 47k + 16$ *with* $k \geq 1$. *Then* $f(47k + 16, 47) = \frac{47k-16}{94k}$.*(Exception:* $f(63, 47) = 1/3$ *FC-exception.)*

Case 17: $m = 47k + 17$ *with* $k \geq 1$. *Then* $f(47k + 17, 47) = \frac{47k-17}{94k}$.*(Exception:* $f(64, 47) = 1/3$ *FC-exception.)*

Case 18: $m = 47k + 18$ *with* $k \geq 1$. *Then* $f(47k + 18, 47) = \frac{47k-18}{94k}$.*(Exception:* $f(65, 47) = 1/3$ *FC-exception.)*

Case 19: $m = 47k + 19$ *with* $k \geq 1$. *Then* $f(47k + 19, 47) = \frac{47k-19}{94k}$.*(Exception:* $f(66, 47) = 1/3$ *FC-exception.)*

Case 20: $m = 47k + 20$ *with* $k \geq 1$. *Then* $f(47k + 20, 47) = \frac{47k-20}{94k}$.*(Exception:* $f(67, 47) = 1/3$ *FC-exception.)*

Case 21: $m = 47k + 21$ *with* $k \geq 1$. *Then* $f(47k + 21, 47) = \frac{47k-21}{94k}$.*(Exception:* $f(68, 47) =$

1/3 *FC-exception.)*

Case 22: $m = 47k + 22$ *with* $k \geq 1$. *Then* $f(47k + 22, 47) = \frac{47k-22}{94k}$.*(Exception:* $f(69, 47) =$ 1/3 *FC-exception.)*

Case 23: $m = 47k + 23$ *with* $k \geq 1$. *Then* $f(47k + 23, 47) = \frac{47k-23}{94k}$.*(Exception:* $f(70, 47) =$ 1/3 *FC-exception.)*

Case 24: $m = 47k + 24$ *with* $k \geq 1$. *Then* $f(47k + 24, 47) = \frac{47k+24}{94k+94}$.

Case 25: $m = 47k + 25$ *with* $k \geq 1$. *Then* $f(47k + 25, 47) = \frac{47k+25}{94k+94}$.

Case 26: $m = 47k + 26$ *with* $k \geq 1$. *Then* $f(47k + 26, 47) = \frac{47k+26}{94k+94}$.

Case 27: $m = 47k + 27$ *with* $k \geq 1$. *Then* $f(47k + 27, 47) = \frac{47k+27}{94k+94}$.

Case 28: $m = 47k + 28$ *with* $k \geq 1$. *Then* $f(47k + 28, 47) = \frac{47k+28}{94k+94}$.

Case 29: $m = 47k + 29$ *with* $k \geq 1$. *Then* $f(47k + 29, 47) = \frac{47k+29}{94k+94}$.

Case 30: $m = 47k + 30$ *with* $k \geq 1$. *Then* $f(47k + 30, 47) = \frac{47k+30}{94k+94}$.*(Exception:* $f(77, 47) =$ 115/282 *INT-exception.)*

Case 31: $m = 47k + 31$ *with* $k \geq 1$. *Then* $f(47k + 31, 47) = \frac{47k+31}{94k+94}$.*(Exception:* $f(78, 47) =$ 19/47 *INT-exception.)*

Case 32: $m = 47k + 32$ *with* $k \geq 1$. *Then* $f(47k + 32, 47) = \frac{47k+32}{94k+94}$.*(Exception:* $f(79, 47) =$ 77/188 *INT-exception.)*

Case 33: $m = 47k + 33$ *with* $k \geq 1$. *Then* $f(47k + 33, 47) = \frac{47k+33}{94k+94}$.*(Exception:* $f(80, 47) =$ 75/188 *BM-exception.)*

Case 34: $m = 47k + 34$ *with* $k \geq 1$. *Then* $f(47k + 34, 47) = \frac{47k+34}{94k+94}$.*(Exception:* $f(81, 47) =$ 115/282 *INT-exception.)*

Case 35: $m = 47k + 35$ *with* $k \geq 1$. *Then* $f(47k + 35, 47) = \frac{47k+35}{94k+94}$.*(Exception:* $f(82, 47) =$ 77/188 *INT-exception.)*

Case 36: $m = 47k + 36$ *with* $k \geq 1$. *Then* $f(47k + 36, 47) = \frac{47k+11}{94k+47}$.*(Exception:* $f(83, 47) =$ 191/470 *INT-exception.)*

Case 37: $m = 47k + 37$ *with* $k \geq 1$. *Then* $f(47k + 37, 47) = \frac{47k+10}{94k+47}$.

Case 38: $m = 47k + 38$ *with* $k \geq 1$. *Then* $f(47k + 38, 47) = \frac{47k+9}{94k+47}$.

Case 39: $m = 47k + 39$ *with* $k \geq 1$. *Then* $f(47k + 39, 47) = \frac{47k+8}{94k+47}$.

Case 40: $m = 47k + 40$ *with* $k \geq 1$. *Then* $f(47k + 40, 47) = \frac{47k+7}{94k+47}$.

Case 41: $m = 47k + 41$ *with* $k \geq 1$. *Then* $f(47k + 41, 47) = \frac{47k+6}{94k+47}$.

Case 42: $m = 47k + 42$ *with* $k \geq 1$. *Then* $f(47k + 42, 47) = \frac{47k+5}{94k+47}$.

Case 43: $m = 47k + 43$ *with* $k \geq 1$. *Then* $f(47k + 43, 47) = \frac{47k+4}{94k+47}$.

Case 44: $m = 47k + 44$ *with* $k \geq 1$. *Then* $f(47k + 44, 47) = \frac{47k+3}{94k+47}$.

Case 45: $m = 47k + 45$ *with* $k \geq 1$. *Then* $f(47k + 45, 47) = \frac{47k+2}{94k+47}$.

Case 46: $m = 47k + 46$ *with* $k \geq 1$. *Then* $f(47k + 46, 47) = \frac{47k+1}{94k+47}$.

Conjecture B.48 *If* $m = 48k + i$ *where* $0 \leq i \leq 47$ *then* $f(m, 48)$ *depends only on* k, i *via a formula, given below, with 25 exceptions (we will note the exceptions).*

Case 0: $m = 48k + 0$ *with* $k \geq 1$. *Then* $f(48k, 48) = 1$.

Case 1: $m = 48k + 1$ *with* $k \geq 2$. *Then* $f(48k + 1, 48) = \frac{48k+1}{96k+48}$.*(Exception:* $f(49, 48) =$ $113/336$ *BM-exception.)*

Case 5: $m = 48k + 5$ *with* $k \geq 1$. *Then* $f(48k + 5, 48) = \frac{48k+5}{96k+48}$.*(Exception:* $f(53, 48) =$ $49/144$ *BM-exception.)*

Case 7: $m = 48k + 7$ *with* $k \geq 1$. *Then* $f(48k + 7, 48) = \frac{48k+7}{96k+48}$.*(Exception:* $f(55, 48) =$ $83/240$ *BM-exception.)*

Case 11: $m = 48k + 11$ *with* $k \geq 1$. *Then* $f(48k + 11, 48) = \frac{48k+11}{96k+48}$.*(Exception:* $f(59, 48) =$ $103/288$ *BM-exception.)*

Case 13: $m = 48k + 13$ *with* $k \geq 1$. *Then* $f(48k + 13, 48) = \frac{48k-13}{96k}$.*(Exception:* $f(61, 48) =$ $11/32$ *BM-exception.)*

Case 17: $m = 48k + 17$ *with* $k \geq 1$. *Then* $f(48k + 17, 48) = \frac{48k-17}{96k}$.*(Exception:* $f(65, 48) =$ $1/3$ *FC-exception.)*

Case 19: $m = 48k + 19$ *with* $k \geq 1$. *Then* $f(48k + 19, 48) = \frac{48k-19}{96k}$.*(Exception:* $f(67, 48) = 1/3$ *FC-exception.)*

Case 23: $m = 48k + 23$ *with* $k \geq 1$. *Then* $f(48k + 23, 48) = \frac{48k-23}{96k}$.*(Exception:* $f(71, 48) = 1/3$ *FC-exception.)*

Case 25: $m = 48k + 25$ *with* $k \geq 1$. *Then* $f(48k + 25, 48) = \frac{48k+25}{96k+96}$.

Case 29: $m = 48k + 29$ *with* $k \geq 1$. *Then* $f(48k + 29, 48) = \frac{48k+29}{96k+96}$.

Case 31: $m = 48k + 31$ *with* $k \geq 1$. *Then* $f(48k + 31, 48) = \frac{48k+31}{96k+96}$.*(Exception:* $f(79, 48) = 29/72$ *INT-exception.)*

Case 35: $m = 48k + 35$ *with* $k \geq 1$. *Then* $f(48k + 35, 48) = \frac{48k+35}{96k+96}$.*(Exception:* $f(83, 48) = 59/144$ *INT-exception.)*

Case 37: $m = 48k + 37$ *with* $k \geq 1$. *Then* $f(48k + 37, 48) = \frac{48k+11}{96k+48}$.*(Exception:* $f(85, 48) = 49/120$ *INT-exception.)*

Case 41: $m = 48k + 41$ *with* $k \geq 1$. *Then* $f(48k + 41, 48) = \frac{48k+7}{96k+48}$.

Case 43: $m = 48k + 43$ *with* $k \geq 1$. *Then* $f(48k + 43, 48) = \frac{48k+5}{96k+48}$.

Case 47: $m = 48k + 47$ *with* $k \geq 1$. *Then* $f(48k + 47, 48) = \frac{48k+1}{96k+48}$.

Conjecture B.49 *If* $m = 49k + i$ *where* $0 \leq i \leq 48$ *then* $f(m, 49)$ *depends only on* k, i *via a formula, given below, with 26 exceptions (we will note the exceptions).*

Case 0: $m = 49k + 0$ *with* $k \geq 1$. *Then* $f(49k, 49) = 1$.

Case 1: $m = 49k+1$ *with* $k \geq 2$. *Then* $f(49k+1, 49) = \frac{49k+1}{98k+49}$.*(Exception:* $f(50, 49) = 33/98$ *BM-exception.)*

Case 2: $m = 49k+2$ *with* $k \geq 2$. *Then* $f(49k+2, 49) = \frac{49k+2}{98k+49}$.*(Exception:* $f(51, 49) = 33/98$ *BM-exception.)*

Case 3: $m = 49k + 3$ *with* $k \geq 1$. *Then* $f(49k + 3, 49) = \frac{49k+3}{98k+49}$.*(Exception:* $f(52, 49) = 50/147$ *BM-exception.)*

Case 4: $m = 49k+4$ *with* $k \geq 1$. *Then* $f(49k+4, 49) = \frac{49k+4}{98k+49}$.*(Exception:* $f(53, 49) = 33/98$ *INT-exception.)*

Case 5: $m = 49k + 5$ *with* $k \geq 1$. *Then* $f(49k + 5, 49) = \frac{49k+5}{98k+49}$.*(Exception:* $f(54, 49) = 67/196$ *BM-exception.)*

Case 6: $m = 49k + 6$ *with* $k \geq 1$. *Then* $f(49k + 6, 49) = \frac{49k+6}{98k+49}$.*(Exception:* $f(55, 49) = 1/3$ *INT-exception.)*

Case 8: $m = 49k+8$ *with* $k \geq 1$. *Then* $f(49k+8, 49) = \frac{49k+8}{98k+49}$.*(Exception:* $f(57, 49) = 33/98$ *INT-exception.)*

Case 9: $m = 49k + 9$ *with* $k \geq 1$. *Then* $f(49k + 9, 49) = \frac{49k+9}{98k+49}$.*(Exception:* $f(58, 49) = 1/3$ *INT-exception.)*

Case 10: $m = 49k + 10$ *with* $k \geq 1$. *Then* $f(49k + 10, 49) = \frac{49k+10}{98k+49}$.*(Exception:* $f(59, 49) = 69/196$ *INT-exception.)*

Case 11: $m = 49k + 11$ *with* $k \geq 1$. *Then* $f(49k + 11, 49) = \frac{49k+11}{98k+49}$.*(Exception:* $f(60, 49) = 5/14$ *BM-exception.)*

Case 12: $m = 49k + 12$ *with* $k \geq 1$. *Then* $f(49k + 12, 49) = \frac{49k+12}{98k+49}$.*(Exception:* $f(61, 49) = 50/147$ *INT-exception.)*

Case 13: $m = 49k + 13$ *with* $k \geq 1$. *Then* $f(49k + 13, 49) = \frac{49k-13}{98k}$.*(Exception:* $f(62, 49) = 69/196$ *INT-exception.)*

Case 15: $m = 49k + 15$ *with* $k \geq 1$. *Then* $f(49k + 15, 49) = \frac{49k-15}{98k}$.

Case 16: $m = 49k + 16$ *with* $k \geq 1$. *Then* $f(49k + 16, 49) = \frac{49k-16}{98k}$.

Case 17: $m = 49k + 17$ *with* $k \geq 1$. *Then* $f(49k + 17, 49) = \frac{49k-17}{98k}$.*(Exception:* $f(66, 49) = 1/3$ *FC-exception.)*

Case 18: $m = 49k + 18$ *with* $k \geq 1$. *Then* $f(49k + 18, 49) = \frac{49k-18}{98k}$.*(Exception:* $f(67, 49) = 1/3$ *FC-exception.)*

Case 19: $m = 49k + 19$ *with* $k \geq 1$. *Then* $f(49k + 19, 49) = \frac{49k-19}{98k}$.*(Exception:* $f(68, 49) = 1/3$ *FC-exception.)*

Case 20: $m = 49k + 20$ *with* $k \geq 1$. *Then* $f(49k + 20, 49) = \frac{49k-20}{98k}$.*(Exception:* $f(69, 49) = 1/3$ *FC-exception.)*

Case 22: $m = 49k + 22$ *with* $k \geq 1$. *Then* $f(49k + 22, 49) = \frac{49k-22}{98k}$.*(Exception:* $f(71, 49) = 1/3$ *FC-exception.)*

Case 23: $m = 49k + 23$ *with* $k \geq 1$. *Then* $f(49k + 23, 49) = \frac{49k-23}{98k}$.*(Exception:* $f(72, 49) = 1/3$ *FC-exception.)*

Case 24: $m = 49k + 24$ *with* $k \geq 1$. *Then* $f(49k + 24, 49) = \frac{49k-24}{98k}$.*(Exception:* $f(73, 49) = 1/3$ *FC-exception.)*

Case 25: $m = 49k + 25$ *with* $k \geq 1$. *Then* $f(49k + 25, 49) = \frac{49k+25}{98k+98}$.

Case 26: $m = 49k + 26$ *with* $k \geq 1$. *Then* $f(49k + 26, 49) = \frac{49k+26}{98k+98}$.

Case 27: $m = 49k + 27$ *with* $k \geq 1$. *Then* $f(49k + 27, 49) = \frac{49k+27}{98k+98}$.

Case 29: $m = 49k + 29$ *with* $k \geq 1$. *Then* $f(49k + 29, 49) = \frac{49k+29}{98k+98}$.

Case 30: $m = 49k + 30$ *with* $k \geq 1$. *Then* $f(49k + 30, 49) = \frac{49k+30}{98k+98}$.

Case 31: $m = 49k + 31$ *with* $k \geq 1$. *Then* $f(49k + 31, 49) = \frac{49k+31}{98k+98}$.

Case 32: $m = 49k + 32$ *with* $k \geq 1$. *Then* $f(49k + 32, 49) = \frac{49k+32}{98k+98}$.*(Exception:* $f(81, 49) = 20/49$ *INT-exception.)*

Case 33: $m = 49k + 33$ *with* $k \geq 1$. *Then* $f(49k + 33, 49) = \frac{49k+33}{98k+98}$.*(Exception:* $f(82, 49) = 181/441$ *INT-exception.)*

Case 34: $m = 49k + 34$ *with* $k \geq 1$. *Then* $f(49k + 34, 49) = \frac{49k+34}{98k+98}$.*(Exception:* $f(83, 49) = 79/196$ *INT-exception.)*

Case 36: $m = 49k + 36$ *with* $k \geq 1$. *Then* $f(49k + 36, 49) = \frac{49k+36}{98k+98}$.*(Exception:* $f(85, 49) = 121/294$ *INT-exception.)*

Case 37: $m = 49k + 37$ *with* $k \geq 1$. *Then* $f(49k + 37, 49) = \frac{49k+12}{98k+49}$.*(Exception:* $f(86, 49) = 20/49$ *INT-exception.)*

Case 38: $m = 49k + 38$ *with* $k \geq 1$. *Then* $f(49k + 38, 49) = \frac{49k+11}{98k+49}$.

Case 39: $m = 49k + 39$ *with* $k \geq 1$. *Then* $f(49k + 39, 49) = \frac{49k+10}{98k+49}$.

Case 40: $m = 49k + 40$ *with* $k \geq 1$. *Then* $f(49k + 40, 49) = \frac{49k+9}{98k+49}$.

Case 41: $m = 49k + 41$ *with* $k \geq 1$. *Then* $f(49k + 41, 49) = \frac{49k+8}{98k+49}$.

Case 43: $m = 49k + 43$ *with* $k \geq 1$. *Then* $f(49k + 43, 49) = \frac{49k+6}{98k+49}$.

Case 44: $m = 49k + 44$ *with* $k \geq 1$. *Then* $f(49k + 44, 49) = \frac{49k+5}{98k+49}$.

Case 45: $m = 49k + 45$ *with* $k \geq 1$. *Then* $f(49k + 45, 49) = \frac{49k+4}{98k+49}$.

Case 46: $m = 49k + 46$ *with* $k \geq 1$. *Then* $f(49k + 46, 49) = \frac{49k+3}{98k+49}$.

Case 47: $m = 49k + 47$ *with* $k \geq 1$. *Then* $f(49k + 47, 49) = \frac{49k+2}{98k+49}$.

Case 48: $m = 49k + 48$ *with* $k \geq 1$. *Then* $f(49k + 48, 49) = \frac{49k+1}{98k+49}$.

Conjecture B.50 *If* $m = 50k + i$ *where* $0 \leq i \leq 49$ *then* $f(m, 50)$ *depends only on* k, i *via a formula, given below, with 27 exceptions (we will note the exceptions).*

Case 0: $m = 50k + 0$ *with* $k \geq 1$. *Then* $f(50k, 50) = 1$.

Case 1: $m = 50k + 1$ *with* $k \geq 1$. *Then* $f(50k + 1, 50) = \frac{50k+1}{100k+50}$.

Case 3: $m = 50k + 3$ *with* $k \geq 1$. *Then* $f(50k + 3, 50) = \frac{50k+3}{100k+50}$.*(Exception:* $f(53, 50) = 17/50$ *BM-exception.)*

Case 7: $m = 50k + 7$ *with* $k \geq 1$. *Then* $f(50k + 7, 50) = \frac{50k+7}{100k+50}$.*(Exception:* $f(57, 50) = 17/50$ *BM-exception.)*

Case 9: $m = 50k+9$ *with* $k \geq 1$. *Then* $f(50k+9, 50) = \frac{50k+9}{100k+50}$.*(Exception:* $f(59, 50) = 1/3$ *INT-exception.)*

Case 11: $m = 50k+11$ *with* $k \geq 1$. *Then* $f(50k+11, 50) = \frac{50k+11}{100k+50}$.*(Exception:* $f(61, 50) = 9/25$ *INT-exception.)*

Case 13: $m = 50k + 13$ *with* $k \geq 1$. *Then* $f(50k + 13, 50) = \frac{50k-13}{100k}$.*(Exception:* $f(63, 50) = 89/250$ *INT-exception.)*

Case 17: $m = 50k + 17$ *with* $k \geq 1$. *Then* $f(50k + 17, 50) = \frac{50k-17}{100k}$.*(Exception:* $f(67, 50) = 1/3$ *FC-exception.)*

Case 19: $m = 50k + 19$ *with* $k \geq 1$. *Then* $f(50k + 19, 50) = \frac{50k-19}{100k}$.*(Exception:* $f(69, 50) = 1/3$ *FC-exception.)*

Case 21: $m = 50k + 21$ *with* $k \geq 1$. *Then* $f(50k + 21, 50) = \frac{50k-21}{100k}$.*(Exception:* $f(71, 50) = 1/3$ *FC-exception.)*

Case 23: $m = 50k + 23$ *with* $k \geq 1$. *Then* $f(50k + 23, 50) = \frac{50k-23}{100k}$.*(Exception:* $f(73, 50) = 1/3$ *FC-exception.)*

Case 27: $m = 50k + 27$ *with* $k \geq 1$. *Then* $f(50k + 27, 50) = \frac{50k+27}{100k+100}$.

Case 29: $m = 50k + 29$ *with* $k \geq 1$. *Then* $f(50k + 29, 50) = \frac{50k+29}{100k+100}$.

Case 31: $m = 50k + 31$ *with* $k \geq 1$. *Then* $f(50k + 31, 50) = \frac{50k+31}{100k+100}$.

Case 33: $m = 50k+33$ *with* $k \geq 1$. *Then* $f(50k+33, 50) = \frac{50k+33}{100k+100}$.*(Exception:* $f(83, 50) = 101/250$ *INT-exception.)*

Case 37: $m = 50k+37$ *with* $k \geq 1$. *Then* $f(50k+37, 50) = \frac{50k+37}{100k+100}$.*(Exception:* $f(87, 50) = 33/80$ *INT-exception.)*

Case 39: $m = 50k + 39$ *with* $k \geq 1$. *Then* $f(50k + 39, 50) = \frac{50k+11}{100k+50}$.

Case 41: $m = 50k + 41$ *with* $k \geq 1$. *Then* $f(50k + 41, 50) = \frac{50k+9}{100k+50}$.

Case 43: $m = 50k + 43$ *with* $k \geq 1$. *Then* $f(50k + 43, 50) = \frac{50k+7}{100k+50}$.

Case 47: $m = 50k + 47$ *with* $k \geq 1$. *Then* $f(50k + 47, 50) = \frac{50k+3}{100k+50}$.

Case 49: $m = 50k + 49$ *with* $k \geq 1$. *Then* $f(50k + 49, 50) = \frac{50k+1}{100k+50}$.

C Appendix C: For $s = 1$ to 60, $m = s + 1$ to 59

M	S	UB	Method	Open?	LB
4	3	1/3	FC		
5	3	5/12	FC		
7	3	5/12	FC		
8	3	4/9	FC		
10	3	4/9	FC		
11	3	11/24	FC		
13	3	11/24	FC		
14	3	7/15	FC		
16	3	7/15	FC		
17	3	17/36	FC		
19	3	17/36	FC		
20	3	10/21	FC		
22	3	10/21	FC		
23	3	23/48	FC		
25	3	23/48	FC		
26	3	13/27	FC		
28	3	13/27	FC		
29	3	29/60	FC		
31	3	29/60	FC		
32	3	16/33	FC		
34	3	16/33	FC		
35	3	35/72	FC		

37	3	35/72	FC		
38	3	19/39	FC		
40	3	19/39	FC		
41	3	41/84	FC		
43	3	41/84	FC		
44	3	22/45	FC		
46	3	22/45	FC		
47	3	47/96	FC		
49	3	47/96	FC		
50	3	25/51	FC		
52	3	25/51	FC		
53	3	53/108	FC		
55	3	53/108	FC		
56	3	28/57	FC		
58	3	28/57	FC		
59	3	59/120	FC		
5	4	3/8	FC		
7	4	5/12	FC		
9	4	7/16	FC		
11	4	9/20	FC		
13	4	11/24	FC		
15	4	13/28	FC		
17	4	15/32	FC		
19	4	17/36	FC		

21	4	19/40	FC		
23	4	21/44	FC		
25	4	23/48	FC		
27	4	25/52	FC		
29	4	27/56	FC		
31	4	29/60	FC		
33	4	31/64	FC		
35	4	33/68	FC		
37	4	35/72	FC		
39	4	37/76	FC		
41	4	39/80	FC		
43	4	41/84	FC		
45	4	43/88	FC		
47	4	45/92	FC		
49	4	47/96	FC		
51	4	49/100	FC		
53	4	51/104	FC		
55	4	53/108	FC		
57	4	55/112	FC		
59	4	57/116	FC		
6	5	2/5	FC		
7	5	1/3	FC		
8	5	2/5	FC		
9	5	2/5	FC		

11	5	13/30	DK-TWO		
12	5	2/5	FC		
13	5	13/30	FC		
14	5	11/25	FC		
16	5	16/35	FC		
17	5	13/30	FC		
18	5	9/20	FC		
19	5	16/35	FC		
21	5	7/15	FC		
22	5	9/20	FC		
23	5	23/50	FC		
24	5	7/15	FC		
26	5	26/55	FC		
27	5	23/50	FC		
28	5	7/15	FC		
29	5	26/55	FC		
31	5	31/65	FC		
32	5	7/15	FC		
33	5	33/70	FC		
34	5	31/65	FC		
36	5	12/25	FC		
37	5	33/70	FC		
38	5	19/40	FC		
39	5	12/25	FC		

41	5	41/85	FC		
42	5	19/40	FC		
43	5	43/90	FC		
44	5	41/85	FC		
46	5	46/95	FC		
47	5	43/90	FC		
48	5	12/25	FC		
49	5	46/95	FC		
51	5	17/35	FC		
52	5	12/25	FC		
53	5	53/110	FC		
54	5	17/35	FC		
56	5	56/115	FC		
57	5	53/110	FC		
58	5	29/60	FC		
59	5	56/115	FC		
7	6	1/3	DK-TWO		
11	6	7/18	FC		
13	6	13/30	FC		
17	6	13/30	FC		
19	6	19/42	FC		
23	6	19/42	FC		
25	6	25/54	FC		
29	6	25/54	FC		

31	6	31/66	FC		
35	6	31/66	FC		
37	6	37/78	FC		
41	6	37/78	FC		
43	6	43/90	FC		
47	6	43/90	FC		
49	6	49/102	FC		
53	6	49/102	FC		
55	6	55/114	FC		
59	6	55/114	FC		
8	7	5/14	DK-TWO		
9	7	5/14	FC		
10	7	1/3	FC		
11	7	11/28	FC		
12	7	3/7	FC		
13	7	8/21	FC		
15	7	3/7	FC		
16	7	3/7	FC		
17	7	11/28	FC		
18	7	3/7	FC		
19	7	25/56	DK-TWO		
20	7	3/7	FC		
22	7	22/49	FC		
23	7	19/42	FC		

24	7	3/7	FC		
25	7	25/56	FC		
26	7	13/28	FC		
27	7	22/49	FC		
29	7	29/63	FC		
30	7	13/28	FC		
31	7	25/56	FC		
32	7	16/35	FC		
33	7	33/70	FC		
34	7	29/63	FC		
36	7	36/77	FC		
37	7	33/70	FC		
38	7	16/35	FC		
39	7	13/28	FC		
40	7	10/21	FC		
41	7	36/77	FC		
43	7	43/91	FC		
44	7	10/21	FC		
45	7	13/28	FC		
46	7	23/49	FC		
47	7	47/98	FC		
48	7	43/91	FC		
50	7	10/21	FC		
51	7	47/98	FC		

52	7	23/49	FC		
53	7	53/112	FC		
54	7	27/56	FC		
55	7	10/21	FC		
57	7	57/119	FC		
58	7	27/56	FC		
59	7	53/112	FC		
60	7	10/21	FC		
9	8	3/8	FC		
11	8	1/3	FC		
13	8	13/32	FC		
15	8	3/8	FC		
17	8	17/40	FC		
19	8	13/32	FC		
21	8	7/16	FC		
23	8	17/40	FC		
25	8	25/56	FC		
27	8	7/16	FC		
29	8	29/64	FC		
31	8	25/56	FC		
33	8	11/24	FC		
35	8	29/64	FC		
37	8	37/80	FC		
39	8	11/24	FC		

41	8	41/88	FC		
43	8	37/80	FC		
45	8	15/32	FC		
47	8	41/88	FC		
49	8	49/104	FC		
51	8	15/32	FC		
53	8	53/112	FC		
55	8	49/104	FC		
57	8	19/40	FC		
59	8	53/112	FC		
10	9	1/3	DK-TWO		
11	9	13/36	DK-ONE		
13	9	1/3	FC		
14	9	7/18	FC		
16	9	11/27	FC		
17	9	10/27	FC		
19	9	19/45	FC		
20	9	4/9	FC		
22	9	7/18	FC		
23	9	23/54	FC		
25	9	4/9	FC		
26	9	19/45	FC		
28	9	4/9	FC		
29	9	41/90	DK-TWO		

31	9	23/54	FC		
32	9	4/9	FC		
34	9	29/63	FC		
35	9	4/9	FC		
37	9	37/81	FC		
38	9	59/126	DK-TWO		
40	9	4/9	FC		
41	9	41/90	FC		
43	9	38/81	FC		
44	9	37/81	FC		
46	9	46/99	FC		
47	9	37/78	DKp-TWO		
49	9	41/90	FC		
50	9	25/54	FC		
52	9	47/99	FC		
53	9	46/99	FC		
55	9	55/117	FC		
56	9	56/117	FC		
58	9	25/54	FC		
59	9	59/126	FC		
11	10	7/20	DK-TWO		
13	10	7/20	FC		
17	10	2/5	DK-TWO		
19	10	11/30	FC		

21	10	21/50	FC		
23	10	17/40	FC		
27	10	9/20	FC		
29	10	21/50	FC		
31	10	31/70	FC		
33	10	9/20	FC		
37	10	37/80	FC		
39	10	31/70	FC		
41	10	41/90	FC		
43	10	37/80	FC		
47	10	47/100	FC		
49	10	41/90	FC		
51	10	51/110	FC		
53	10	47/100	FC		
57	10	19/40	FC		
59	10	51/110	FC		
12	11	4/11	FC		
13	11	1/3	DKp-ONE		
14	11	4/11	FC		
15	11	1/3	FC		
16	11	1/3	FC		
17	11	17/44	FC		
18	11	9/22	FC		
19	11	9/22	DK-ONE		

20	11	13/33	FC		
21	11	4/11	FC		
23	11	23/55	FC		
24	11	19/44	DK-TWO		
25	11	19/44	FC		
26	11	9/22	FC		
27	11	17/44	FC		
28	11	14/33	FC		
29	11	29/66	FC		
30	11	5/11	FC		
31	11	24/55	FC		
32	11	23/55	FC		
34	11	34/77	FC		
35	11	5/11	FC		
36	11	5/11	FC		
37	11	29/66	FC		
38	11	14/33	FC		
39	11	39/88	FC		
40	11	5/11	FC		
41	11	61/132	DK-TWO		
42	11	5/11	FC		
43	11	34/77	FC		
45	11	5/11	FC		
46	11	46/99	FC		

47	11	41/88	FC		
48	11	5/11	FC		
49	11	39/88	FC		
50	11	5/11	FC		
51	11	51/110	FC		
52	11	83/176	DK-TWO		
53	11	46/99	FC		
54	11	5/11	FC		
56	11	56/121	FC		
57	11	57/121	FC		
58	11	26/55	FC		
59	11	51/110	FC		
60	11	5/11	FC		
13	12	1/3	HALF-ONE		
17	12	1/3	FC		
19	12	19/48	FC		
23	12	13/36	FC		
25	12	5/12	FC		
29	12	19/48	FC		
31	12	31/72	FC		
35	12	5/12	FC		
37	12	37/84	FC		
41	12	31/72	FC		
43	12	43/96	FC		

47	12	37/84	FC		
49	12	49/108	FC		
53	12	43/96	FC		
55	12	11/24	FC		
59	12	49/108	FC		
14	13	9/26	HALF-ONE		
15	13	9/26	DK-TWO		
16	13	14/39	DK-ONE		
17	13	9/26	FC		
18	13	1/3	FC		
19	13	1/3	FC		
20	13	5/13	FC		
21	13	21/52	FC		
22	13	21/52	DK-TWO		
23	13	53/130	HALF-TWO		
24	13	5/13	FC		
25	13	14/39	FC		
27	13	27/65	FC		
28	13	28/65	FC		
29	13	45/104	DK-ONE		
30	13	11/26	FC		
31	13	21/52	FC		
32	13	5/13	FC		
33	13	11/26	FC		

34	13	17/39	FC		
35	13	64/143	DK-TWO		
36	13	29/65	FC		
37	13	28/65	FC		
38	13	27/65	FC		
40	13	40/91	FC		
41	13	41/91	FC		
42	13	6/13	FC		
43	13	35/78	FC		
44	13	17/39	FC		
45	13	11/26	FC		
46	13	23/52	FC		
47	13	47/104	FC		
48	13	6/13	FC		
49	13	6/13	FC		
50	13	41/91	FC		
51	13	40/91	FC		
53	13	53/117	FC		
54	13	6/13	FC		
55	13	85/182	DK-TWO		
56	13	6/13	FC		
57	13	47/104	FC		
58	13	23/52	FC		
59	13	59/130	FC		

60	13	6/13	FC		
15	14	5/14	FC		
17	14	5/14	DK-ONE		
19	14	1/3	FC		
23	14	17/42	HALF-ONE		
25	14	17/42	FC		
27	14	5/14	FC		
29	14	29/70	FC		
31	14	3/7	DK-TWO		
33	14	23/56	FC		
37	14	37/84	FC		
39	14	31/70	FC		
41	14	29/70	FC		
43	14	43/98	FC		
45	14	16/35	DK-TWO		
47	14	37/84	FC		
51	14	51/112	FC		
53	14	45/98	FC		
55	14	43/98	FC		
57	14	19/42	FC		
59	14	131/280	DK-TWO		
16	15	1/3	BM		
17	15	7/20	DK-TWO		
19	15	7/20	HALF-TWO		

22	15	1/3	FC		
23	15	23/60	FC		
26	15	37/90	DK-ONE		
28	15	17/45	FC		
29	15	16/45	FC		
31	15	31/75	FC		
32	15	32/75	FC		
34	15	13/30	FC		
37	15	23/60	FC		
38	15	19/45	FC		
41	15	67/150	DK-ONE		
43	15	32/75	FC		
44	15	31/75	FC		
46	15	46/105	FC		
47	15	47/105	FC		
49	15	41/90	FC		
52	15	19/45	FC		
53	15	53/120	FC		
56	15	7/15	FC		
58	15	47/105	FC		
59	15	46/105	FC		
17	16	11/32	BM		
19	16	1/3	DKp-ONE		
21	16	11/32	FC		

23	16	1/3	FC		
25	16	25/64	FC		
27	16	13/32	DK-TWO		
29	16	19/48	FC		
31	16	17/48	FC		
33	16	33/80	FC		
35	16	7/16	FC		
37	16	27/64	FC		
39	16	25/64	FC		
41	16	41/96	FC		
43	16	25/56	HALF-ONE		
45	16	7/16	FC		
47	16	33/80	FC		
49	16	7/16	FC		
51	16	51/112	FC		
53	16	43/96	FC		
55	16	41/96	FC		
57	16	57/128	FC		
59	16	59/128	FC		
18	17	6/17	FC		
19	17	1/3	DKp-TWO		
20	17	1/3	DKp-ONE		
21	17	6/17	DK-ONE		
22	17	6/17	FC		

23	17	1/3	FC		
24	17	1/3	FC		
25	17	1/3	FC		
26	17	13/34	FC		
27	17	27/68	FC		
28	17	7/17	FC		
29	17	27/68	BM		
30	17	7/17	FC		
31	17	20/51	FC		
32	17	19/51	FC		
33	17	6/17	FC		
35	17	7/17	FC		
36	17	36/85	FC		
37	17	59/136	DK-TWO		
38	17	59/136	DK-ONE		
39	17	29/68	FC		
40	17	7/17	FC		
41	17	27/68	FC		
42	17	13/34	FC		
43	17	43/102	FC		
44	17	22/51	FC		
45	17	15/34	FC		
46	17	61/136	DK-TWO		
47	17	91/204	ERIK		

48	17	37/85	FC		
49	17	36/85	FC		
50	17	7/17	FC		
52	17	52/119	FC		
53	17	53/119	FC		
54	17	54/119	FC		
55	17	31/68	DK-ONE		
56	17	23/51	FC		
57	17	15/34	FC		
58	17	22/51	FC		
59	17	43/102	FC		
60	17	15/34	FC		
19	18	1/3	BM		
23	18	19/54	HALF-TWO		
25	18	1/3	FC		
29	18	29/72	FC		
31	18	11/27	DK-ONE		
35	18	19/54	FC		
37	18	37/90	FC		
41	18	31/72	FC		
43	18	29/72	FC		
47	18	47/108	FC		
49	18	4/9	DK-TWO		
53	18	37/90	FC		

55	18	55/126	FC		
59	18	49/108	FC		
20	19	13/38	BM		
21	19	13/38	DK-TWO		
22	19	13/38	DK-TWO		
23	19	27/76	DK-ONE		
24	19	27/76	HALF-TWO		
25	19	13/38	FC		
26	19	1/3	FC		
27	19	1/3	FC		
28	19	1/3	FC		
29	19	29/76	FC		
30	19	15/38	FC		
31	19	54/133	ERIK		
32	19	31/76	DK-TWO		
33	19	47/114	DK-ONE		
34	19	23/57	FC		
35	19	22/57	FC		
36	19	7/19	FC		
37	19	20/57	FC		
39	19	39/95	FC		
40	19	8/19	FC		
41	19	983/2280	ERIK	Open	980/2280
42	19	49/114	DK-TWO		

43	19	115/266	HALF-TWO		
44	19	8/19	FC		
45	19	31/76	FC		
46	19	15/38	FC		
47	19	29/76	FC		
48	19	8/19	FC		
49	19	49/114	FC		
50	19	25/57	FC		
51	19	17/38	FC		
52	19	17/38	DK-ONE		
53	19	42/95	FC		
54	19	41/95	FC		
55	19	8/19	FC		
56	19	39/95	FC		
58	19	58/133	FC		
59	19	59/133	FC		
60	19	60/133	FC		
21	20	7/20	FC		
23	20	7/20	DK-TWO		
27	20	1/3	FC		
29	20	1/3	FC		
31	20	31/80	FC		
33	20	49/120	HALF-ONE		
37	20	23/60	FC		

39	20	7/20	FC		
41	20	41/100	FC		
43	20	43/100	FC		
47	20	33/80	FC		
49	20	31/80	FC		
51	20	17/40	FC		
53	20	53/120	FC		
57	20	43/100	FC		
59	20	41/100	FC		
22	21	1/3	BM		
23	21	29/84	HALF-ONE		
25	21	1/3	DKp-ONE		
26	21	22/63	DK-ONE		
29	21	1/3	FC		
31	21	1/3	FC		
32	21	8/21	FC		
34	21	17/42	FC		
37	21	103/252	HALF-TWO		
38	21	25/63	FC		
40	21	23/63	FC		
41	21	22/63	FC		
43	21	43/105	FC		
44	21	44/105	FC		
46	21	73/168	DK-TWO		

47	21	73/168	DK-ONE		
50	21	17/42	FC		
52	21	8/21	FC		
53	21	53/126	FC		
55	21	55/126	FC		
58	21	169/378	HALF-TWO		
59	21	46/105	FC		
23	22	15/44	BM		
25	22	23/66	BM		
27	22	4/11	DK-ONE		
29	22	15/44	FC		
31	22	1/3	FC		
35	22	35/88	FC		
37	22	9/22	DK-TWO		
39	22	9/22	FC		
41	22	25/66	FC		
43	22	23/66	FC		
45	22	9/22	FC		
47	22	47/110	FC		
49	22	19/44	DK-ONE		
51	22	37/88	FC		
53	22	35/88	FC		
57	22	19/44	FC		
59	22	59/132	FC	Open	58/132

24	23	8/23	FC		
25	23	1/3	HALF-ONE		
26	23	8/23	DK-TWO		
27	23	1/3	DKp-ONE		
28	23	33/92	DK-ONE		
29	23	49/138	BM		
30	23	8/23	FC		
31	23	1/3	FC		
32	23	1/3	FC		
33	23	1/3	FC		
34	23	1/3	FC		
35	23	35/92	FC		
36	23	9/23	FC		
37	23	37/92	FC		
38	23	47/115	DK-TWO		
39	23	37/92	DK-TWO		
40	23	19/46	DK-ONE		
41	23	28/69	FC	Open	27/69
42	23	9/23	FC		
43	23	26/69	FC		
44	23	25/69	FC		
45	23	8/23	FC		
47	23	47/115	FC		
48	23	48/115	FC		

49	23	49/115	FC		
50	23	10/23	FC		
51	23	59/138	BM		
52	23	10/23	FC		
53	23	39/92	FC		
54	23	19/46	FC		
55	23	37/92	FC		
56	23	9/23	FC		
57	23	35/92	FC		
58	23	29/69	FC		
59	23	59/138	FC		
60	23	10/23	FC		
25	24	1/3	BM		
29	24	17/48	DK-ONE		
31	24	25/72	BM		
35	24	1/3	FC		
37	24	37/96	FC		
41	24	19/48	BM		
43	24	29/72	FC		
47	24	25/72	FC		
49	24	49/120	FC		
53	24	31/72	DK-TWO		
55	24	41/96	FC		
59	24	37/96	FC		

26	25	17/50	BM		
27	25	17/50	HALF-ONE		
28	25	1/3	DKp-TWO		
29	25	17/50	DK-TWO		
31	25	26/75	DK-ONE		
32	25	44/125	BM		
33	25	17/50	FC		
34	25	1/3	FC		
36	25	1/3	FC		
37	25	1/3	FC		
38	25	19/50	FC		
39	25	39/100	FC		
41	25	61/150	HALF-ONE		
42	25	41/100	DK-TWO		
43	25	61/150	DK-ONE		
44	25	41/100	DK-ONE		
46	25	29/75	FC		
47	25	28/75	FC		
48	25	9/25	FC		
49	25	26/75	FC		
51	25	51/125	FC		
52	25	52/125	FC		
53	25	53/125	FC		
54	25	54/125	FC	Open	53/125

56	25	87/200	DK-ONE		
57	25	43/100	FC		
58	25	21/50	FC		
59	25	41/100	FC		
27	26	9/26	FC		
29	26	1/3	DKp-TWO		
31	26	1/3	DKp-ONE		
33	26	9/26	HALF-TWO		
35	26	1/3	FC		
37	26	1/3	FC		
41	26	41/104	FC		
43	26	53/130	DK-TWO		
45	26	16/39	DK-ONE		
47	26	31/78	FC		
49	26	29/78	FC		
51	26	9/26	FC		
53	26	53/130	FC		
55	26	11/26	FC		
57	26	181/416	DK-TWO		
59	26	45/104	FC	Open	44/104
28	27	1/3	BM		
29	27	37/108	BM		
31	27	37/108	BM		
32	27	1/3	DKp-ONE		

34	27	16/45	DK-ONE		
35	27	25/72	BM		
37	27	1/3	FC		
38	27	1/3	FC		
40	27	1/3	FC		
41	27	41/108	FC		
43	27	43/108	FC		
44	27	11/27	FC		
46	27	43/108	BM		
47	27	89/216	DK-ONE		
49	27	32/81	FC		
50	27	31/81	FC		
52	27	29/81	FC		
53	27	28/81	FC		
55	27	11/27	FC		
56	27	56/135	FC		
58	27	58/135	FC		
59	27	31/72	DKp-TWO		
29	28	19/56	BM		
31	28	19/56	DK-TWO		
33	28	1/3	DKp-ONE		
37	28	19/56	FC		
39	28	1/3	FC		
41	28	1/3	FC		

43	28	43/112	FC		
45	28	45/112	FC		
47	28	23/56	DK-TWO		
51	28	11/28	FC		
53	28	31/84	FC		
55	28	29/84	FC		
57	28	57/140	FC		
59	28	59/140	FC		
30	29	10/29	FC		
31	29	1/3	BM		
32	29	10/29	DK-TWO		
33	29	10/29	BM		
34	29	1/3	DKp-ONE		
35	29	41/116	DK-ONE		
36	29	10/29	DK-ONE		
37	29	61/174	HALF-TWO		
38	29	10/29	FC		
39	29	1/3	FC		
40	29	1/3	FC		
41	29	1/3	FC		
42	29	1/3	FC		
43	29	1/3	FC		
44	29	11/29	FC		
45	29	45/116	FC		

46	29	23/58	FC		
47	29	47/116	FC	Open	46/116
48	29	59/145	DK-TWO		
49	29	47/116	DK-TWO		
50	29	71/174	DK-ONE		
51	29	95/232	DK-ONE		
52	29	35/87	FC		
53	29	34/87	FC		
54	29	11/29	FC		
55	29	32/87	FC		
56	29	31/87	FC		
57	29	10/29	FC		
59	29	59/145	FC		
60	29	12/29	FC		
31	30	1/3	BM		
37	30	16/45	DK-ONE		
41	30	1/3	FC		
43	30	1/3	FC		
47	30	47/120	FC		
49	30	49/120	FC	Open	48/120
53	30	61/150	HALF-TWO		
59	30	31/90	FC		
32	31	21/62	BM		
33	31	21/62	BM		

34	31	32/93	BM		
35	31	43/124	DK-TWO		
36	31	21/62	DK-TWO		
37	31	1/3	DKp-ONE		
38	31	11/31	BM		
39	31	11/31	DK-ONE		
40	31	65/186	BM		
41	31	21/62	FC		
42	31	1/3	FC		
43	31	1/3	FC		
44	31	1/3	FC		
45	31	1/3	FC		
46	31	1/3	FC		
47	31	47/124	FC		
48	31	12/31	FC		
49	31	49/124	FC		
50	31	25/62	FC		
51	31	25/62	HALF-ONE		
52	31	51/124	DK-TWO	Open	50/124
53	31	49/124	BM		
54	31	51/124	DK-ONE		
55	31	38/93	FC	Open	37/93
56	31	37/93	FC		
57	31	12/31	FC		

58	31	35/93	FC		
59	31	34/93	FC		
60	31	11/31	FC		
33	32	11/32	FC		
35	32	11/32	HALF-ONE		
37	32	11/32	DK-TWO		
39	32	23/64	DK-ONE		
41	32	7/20	BM		
43	32	1/3	FC		
45	32	1/3	FC		
47	32	1/3	FC		
49	32	49/128	FC		
51	32	51/128	FC		
53	32	13/32	DK-TWO		
55	32	13/32	DK-ONE		
57	32	13/32	FC		
59	32	37/96	FC		
34	33	1/3	BM		
35	33	15/44	BM		
37	33	1/3	DKp-TWO		
38	33	34/99	BM		
40	33	47/132	DK-ONE		
41	33	34/99	DKp-ONE		
43	33	91/264	ERIK		

46	33	1/3	FC		
47	33	1/3	FC		
49	33	1/3	FC		
50	33	25/66	FC		
52	33	13/33	FC		
53	33	53/132	FC		
56	33	53/132	DK-TWO		
58	33	9/22	DK-ONE		
59	33	40/99	FC	Open	39/99
35	34	23/68	BM		
37	34	1/3	HALF-ONE		
39	34	47/136	BM		
41	34	6/17	DK-ONE		
43	34	6/17	HALF-TWO		
45	34	23/68	FC		
47	34	1/3	FC		
49	34	1/3	FC		
53	34	53/136	FC		
55	34	55/136	FC	Open	54/136
57	34	7/17	DK-TWO		
59	34	7/17	DK-ONE		
36	35	12/35	FC		
37	35	1/3	BM		
38	35	1/3	HALF-ONE		

39	35	1/3	DKp-TWO		
41	35	1/3	DKp-ONE		
43	35	5/14	BM		
44	35	62/175	DK-ONE		
46	35	12/35	FC		
47	35	1/3	FC		
48	35	1/3	FC		
51	35	1/3	FC		
52	35	1/3	FC		
53	35	53/140	FC		
54	35	27/70	FC		
57	35	57/140	FC	Open	56/140
58	35	71/175	DK-TWO		
59	35	57/140	DK-TWO		
37	36	1/3	BM		
41	36	37/108	BM		
43	36	1/3	DKp-ONE		
47	36	25/72	FC	Open	24/72
49	36	1/3	FC		
53	36	1/3	FC		
55	36	55/144	FC		
59	36	11/27	HALF-ONE		
38	37	25/74	BM		
39	37	25/74	BM		

40	37	25/74	HALF-ONE		
41	37	25/74	DK-TWO		
42	37	13/37	DK-TWO		
43	37	25/74	DK-TWO		
44	37	1/3	DKp-ONE		
45	37	53/148	DK-ONE		
46	37	38/111	DKp-ONE		
47	37	51/148	BM		
48	37	103/296	BM	Open	102/296
49	37	25/74	FC		
50	37	1/3	FC		
51	37	1/3	FC		
52	37	1/3	FC		
53	37	1/3	FC		
54	37	1/3	FC		
55	37	1/3	FC		
56	37	14/37	FC		
57	37	57/148	FC		
58	37	29/74	FC		
59	37	59/148	FC		
60	37	15/37	FC		
39	38	13/38	FC		
41	38	13/38	HALF-ONE		
43	38	53/152	DK-TWO		

45	38	1/3	DKp-ONE		
47	38	20/57	DK-ONE		
49	38	93/266	BM		
51	38	1/3	FC		
53	38	1/3	FC		
55	38	1/3	FC		
59	38	59/152	FC		
40	39	1/3	BM		
41	39	53/156	BM		
43	39	53/156	BM		
44	39	9/26	DK-TWO		
46	39	1/3	DKp-ONE		
47	39	55/156	DK-ONE		
49	39	23/65	DK-ONE		
50	39	68/195	BM		
53	39	1/3	FC		
55	39	1/3	FC		
56	39	1/3	FC		
58	39	1/3	FC		
59	39	59/156	FC		
41	40	27/80	BM		
43	40	41/120	BM		
47	40	1/3	DKp-ONE		
49	40	57/160	BM		

51	40	7/20	HALF-TWO		
53	40	27/80	FC		
57	40	1/3	FC		
59	40	1/3	FC		
42	41	14/41	FC		
43	41	1/3	BM		
44	41	14/41	BM		
45	41	14/41	BM		
46	41	1/3	DKp-TWO		
47	41	85/246	BM		
48	41	1/3	DKp-ONE		
49	41	1/3	DKp-ONE		
50	41	59/164	DK-ONE	Open	58/164
51	41	14/41	DKp-ONE		
52	41	57/164	HALF-TWO		
53	41	85/246	BM		
54	41	14/41	FC		
55	41	1/3	FC		
56	41	1/3	FC		
57	41	1/3	FC		
58	41	1/3	FC		
59	41	1/3	FC		
60	41	1/3	FC		
43	42	1/3	BM		

47	42	1/3	DKp-TWO		
53	42	5/14	DK-ONE		
55	42	29/84	FC	Open	28/84
59	42	1/3	FC		
44	43	29/86	BM		
45	43	29/86	BM		
46	43	1/3	BM		
47	43	59/172	HALF-ONE		
48	43	1/3	DKp-TWO		
49	43	44/129	BM		
50	43	29/86	DK-TWO		
51	43	1/3	DKp-ONE		
52	43	61/172	DK-ONE		
53	43	46/129	DK-ONE	Open	45/129
54	43	76/215	DK-ONE		
55	43	91/258	HALF-TWO	Open	90/258
56	43	15/43	FC	Open	14/43
57	43	29/86	FC		
58	43	1/3	FC		
59	43	1/3	FC		
60	43	1/3	FC		
45	44	15/44	FC		
47	44	1/3	BM		
49	44	1/3	DKp-TWO		

51	44	15/44	DK-TWO		
53	44	31/88	DK-ONE		
57	44	61/176	BM		
59	44	1/3	FC		
46	45	1/3	BM		
47	45	61/180	BM		
49	45	1/3	HALF-ONE		
52	45	46/135	BM		
53	45	1/3	DKp-ONE		
56	45	46/135	DKp-ONE		
58	45	22/63	BM		
59	45	31/90	FC	Open	30/90
47	46	31/92	BM		
49	46	31/92	BM		
51	46	31/92	DK-TWO		
53	46	79/230	BM		
55	46	1/3	DKp-ONE		
57	46	8/23	DK-ONE		
59	46	8/23	BM		
48	47	16/47	FC		
49	47	1/3	BM		
50	47	16/47	BM		
51	47	1/3	HALF-ONE		
52	47	16/47	DK-TWO		

53	47	65/188	DK-TWO		
54	47	16/47	BM		
55	47	1/3	DKp-ONE		
56	47	1/3	DKp-ONE		
57	47	67/188	DK-ONE		
58	47	50/141	DK-ONE		
59	47	83/235	DK-ONE		
60	47	33/94	HALF-TWO		
49	48	1/3	BM		
53	48	49/144	BM		
55	48	83/240	BM		
59	48	103/288	BM		
50	49	33/98	BM		
51	49	33/98	BM		
52	49	50/147	BM		
53	49	33/98	HALF-ONE		
54	49	67/196	BM		
55	49	1/3	DKp-TWO		
57	49	33/98	DK-TWO		
58	49	1/3	DKp-ONE		
59	49	69/196	DK-ONE		
60	49	5/14	BM		
51	50	17/50	FC		
53	50	17/50	BM		

57	50	17/50	BM		
59	50	1/3	DKp-ONE		

References

[1] H. Aziz and S. Mackenzie. A discrete and bounded envy-free cake cutting protocol for any number of agents. In *FOCS16*, pages 416–427. IEEE Computer Society Press, 2016. `https://arxiv.org/abs/1604.03655`.

[2] S. Brams and A. Taylor. *Fair division: From cake-cutting do dispute resolution*. Cambridge University Press, 1996.

[3] S. J. Brams and A. D. Taylor. An envy-free cake division protocol. *The American Mathematical Monthly*, 102:9–18, 1995.

[4] J. Edmonds and K. Pruhs. Cake cutting really is not a piece of cake. In *Seventeenth Symposium on Discrete Algorithms: Proceedings of SODA '06*, 2006.

[5] S. Even and A. Paz. A note on cake cutting. *Discrete Applied Mathematics*, 7:285–296, 1984.

[6] A. Frank. The muffin problem, 2013. Described to Jeremy Copeland and in the New York Times Numberplay Online Blog `wordplay.blogs.nytimes.com/2013/08/19/cake`.

[7] J. Robertson and W. Webb. *Cake cutting algorithms: Be fair if you can*. A.K. Peters, 1998.