The Muffin Problem

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How it Began

A Recreational Math Conference
(Gathering for Gardner)
May 2016

I found a pamphlet:
The Julia Robinson Mathematics Festival:
A Sample of Mathematical Puzzles
Compiled by Nancy Blachman

which had this problem, proposed by Alan Frank:

How can you divide and distribute 5 muffins to 3 students so that every student gets \( \frac{5}{3} \) where nobody gets a tiny sliver?
Five Muffins, Three Students, Proc by Picture

<table>
<thead>
<tr>
<th>Person</th>
<th>Color</th>
<th>What they Get</th>
</tr>
</thead>
<tbody>
<tr>
<td>Alice</td>
<td>RED</td>
<td>$1 + \frac{2}{3} = \frac{5}{3}$</td>
</tr>
<tr>
<td>Bob</td>
<td>BLUE</td>
<td>$1 + \frac{2}{3} = \frac{5}{3}$</td>
</tr>
<tr>
<td>Carol</td>
<td>GREEN</td>
<td>$1 + \frac{1}{3} + \frac{1}{3} = \frac{5}{3}$</td>
</tr>
</tbody>
</table>

Smallest Piece: $\frac{1}{3}$
Can We Do Better?

The smallest piece in the above solution is $\frac{1}{3}$.

Is there a procedure with a larger smallest piece?

VOTE
Can We Do Better?

The smallest piece in the above solution is $\frac{1}{3}$.

Is there a procedure with a larger smallest piece?

VOTE

- YES
- NO
Can We Do Better?

The smallest piece in the above solution is $\frac{1}{3}$.

Is there a procedure with a larger smallest piece?

VOTE

- YES
- NO

YES WE CAN!

We use ! since we are excited that we can!
Five Muffins, Three People–Proc by Picture

<table>
<thead>
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<tbody>
<tr>
<td>Alice</td>
<td>RED</td>
<td>$\frac{6}{12} + \frac{7}{12} + \frac{7}{12}$</td>
</tr>
<tr>
<td>Bob</td>
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<td>$\frac{6}{12} + \frac{7}{12} + \frac{7}{12}$</td>
</tr>
<tr>
<td>Carol</td>
<td>GREEN</td>
<td>$\frac{5}{12} + \frac{5}{12} + \frac{5}{12} + \frac{5}{12}$</td>
</tr>
</tbody>
</table>

Smallest Piece: $\frac{5}{12}$
Can We Do Better?

The smallest piece in the above solution is \( \frac{5}{12} \).

**Is there a procedure with a larger smallest piece?**

**VOTE**

- **YES**
- **NO**
Can We Do Better?

The smallest piece in the above solution is \( \frac{5}{12} \).

Is there a procedure with a larger smallest piece?

VOTE

- YES
- NO

NO WE CAN’T!

We use ! since we are excited to prove we can’t do better!
Five Muffins, Three People—Can’t Do Better Than $\frac{5}{12}$

There is a procedure for 5 muffins, 3 students where each student gets $\frac{5}{3}$ muffins, smallest piece $N$. We want $N \leq \frac{5}{12}$.

**Case 0:** Some muffin is uncut. Cut it $(\frac{1}{2}, \frac{1}{2})$ and give both $\frac{1}{2}$-sized pieces to whoever got the uncut muffin. (Note $\frac{1}{2} > \frac{5}{12}$.) Reduces to other cases.

**Henceforth:** All muffins are cut into $\geq 2$ pieces.)

**Case 1:** Some muffin is cut into $\geq 3$ pieces. Then $N \leq \frac{1}{3} < \frac{5}{12}$.

**Henceforth:** All muffins are cut into 2 pieces.)

**Case 2:** All muffins are cut into 2 pieces. 10 pieces, 3 students: **Someone** gets $\geq 4$ pieces. He has some piece

$$\leq \frac{5}{3} \times \frac{1}{4} = \frac{5}{12}$$

Great to see $\frac{5}{12}$
1. Procedure for 5 muffins, 3 people, smallest piece \( \frac{5}{12} \).
2. NO Procedure for 5 muffins, 3 people, smallest piece > \( \frac{5}{12} \).

Amazing That Have Exact Result!
Be Amazed Now! And Later!

1. Procedure for 5 muffins, 3 people, smallest piece \( \frac{5}{12} \).
2. NO Procedure for 5 muffins, 3 people, smallest piece \( > \frac{5}{12} \).

Amazing That Have Exact Result!

Prepare To Be More Amazed! On Next Page!
Amazing Results!

1. Procedure for 47 muffins, 9 people, smallest piece $\frac{111}{234}$.
2. NO Procedure for 47 muffins, 9 people, smallest piece $\frac{111}{234}$. 

All done by hand, no use of a computer.

Co-author Erik Metz is a muffin savant.
Amazing Results!

1. Procedure for 47 muffins, 9 people, smallest piece $\frac{111}{234}$.
2. NO Procedure for 47 muffins, 9 people, smallest piece $\geq \frac{111}{234}$.

1. Procedure for 52 muffins, 11 people, smallest piece $\frac{83}{176}$.
2. NO Procedure for 52 muffins, 11 people, smallest piece $\geq \frac{83}{176}$.

All done by hand, no use of a computer.

Co-author Erik Metz is a muffin savant.
Amazing Results!

1. Procedure for 47 muffins, 9 people, smallest piece $\frac{111}{234}$.
2. NO Procedure for 47 muffins, 9 people, smallest piece $> \frac{111}{234}$.

1. Procedure for 52 muffins, 11 people, smallest piece $\frac{83}{176}$.
2. NO Procedure for 52 muffins, 11 people, smallest piece $> \frac{83}{176}$.

1. Procedure for 35 muffins, 13 people, smallest piece $\frac{64}{143}$.
2. NO Procedure for 35 muffins, 13 people, smallest piece $> \frac{64}{143}$. 

Co-author Erik Metz is a muffin savant.
Amazing Results!

1. Procedure for 47 muffins, 9 people, smallest piece $\frac{111}{234}$.
2. NO Procedure for 47 muffins, 9 people, smallest piece $\geq \frac{111}{234}$.

1. Procedure for 52 muffins, 11 people, smallest piece $\frac{83}{176}$.
2. NO Procedure for 52 muffins, 11 people, smallest piece $\geq \frac{83}{176}$.

1. Procedure for 35 muffins, 13 people, smallest piece $\frac{64}{143}$.
2. NO Procedure for 35 muffins, 13 people, smallest piece $\geq \frac{64}{143}$.

All done by hand, no use of a computer
Amazing Results!

1. Procedure for 47 muffins, 9 people, smallest piece $\frac{111}{234}$.
2. NO Procedure for 47 muffins, 9 people, smallest piece $\frac{111}{234}$.

1. Procedure for 52 muffins, 11 people, smallest piece $\frac{83}{176}$.
2. NO Procedure for 52 muffins, 11 people, smallest piece $\frac{83}{176}$.

1. Procedure for 35 muffins, 13 people, smallest piece $\frac{64}{143}$.
2. NO Procedure for 35 muffins, 13 people, smallest piece $\frac{64}{143}$.

All done by hand, no use of a computer

Co-author Erik Metz is a muffin savant
General Problem

How can you divide and distribute \( m \) muffins to \( s \) students so that each students gets \( \frac{m}{s} \) AND the MIN piece is MAXIMIZED?

An \((m, s)\)-procedure is a way to divide and distribute \( m \) muffins to \( s \) students so that each student gets \( \frac{m}{s} \) muffins.

An \((m, s)\)-procedure is **optimal** if it has the largest smallest piece of any procedure.

\( f(m, s) \) be the smallest piece in an optimal \((m, s)\)-procedure.

We have shown \( f(5, 3) = \frac{5}{12} \).

**Note:** \( f(m, s) \geq \frac{1}{s} \): divide each muffin into \( s \) pieces of size \( \frac{1}{s} \) and give each student \( m \) of them.
$f(3, 5) \geq ?$

Clearly $f(3, 5) \geq \frac{1}{5}$. Can we get $f(3, 5) > \frac{1}{5}$? Think about it at your desk.
Clearly $f(3, 5) \geq \frac{1}{5}$. Can we get $f(3, 5) > \frac{1}{5}$?

Think about it at your desk.

$f(3, 5) \geq \frac{1}{4}$

1. Divide 2 muffin $[\frac{6}{20}, \frac{7}{20}, \frac{7}{20}]$
2. Divide 1 muffin $[\frac{5}{20}, \frac{5}{20}, \frac{5}{20}, \frac{5}{20}]$
3. Give 4 students $\left(\frac{5}{20}, \frac{7}{20}\right)$
4. Give 1 students $\left(\frac{6}{20}, \frac{6}{20}\right)$

Can we do better? Vote: YES NO UNKNOWN TO SCIENCE NO

Proof on next slide.
$f(3, 5) \geq \frac{1}{5}$\?

Clearly $f(3, 5) \geq \frac{1}{5}$. Can we get $f(3, 5) > \frac{1}{5}$?

Think about it at your desk.

$f(3, 5) \geq \frac{1}{4}$

1. Divide 2 muffin $[\frac{6}{20}, \frac{7}{20}, \frac{7}{20}]$
2. Divide 1 muffin $[\frac{5}{20}, \frac{5}{20}, \frac{5}{20}, \frac{5}{20}]$
3. Give 4 students $\left(\frac{5}{20}, \frac{7}{20}\right)$
4. Give 1 students $\left(\frac{6}{20}, \frac{6}{20}\right)$

Can we do better? Vote:
Clearly $f(3, 5) \geq \frac{1}{5}$. Can we get $f(3, 5) > \frac{1}{5}$?

Think about it at your desk.

$f(3, 5) \geq \frac{1}{4}$

1. Divide 2 muffin $[\frac{6}{20}, \frac{7}{20}, \frac{7}{20}]$
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3. Give 4 students $\left(\frac{5}{20}, \frac{7}{20}\right)$
4. Give 1 students $\left(\frac{6}{20}, \frac{6}{20}\right)$

Can we do better? Vote:

YES

NO

UNKNOWN TO SCIENCE
Clearly \( f(3, 5) \geq \frac{1}{5} \). Can we get \( f(3, 5) > \frac{1}{5} \)?

Think about it at your desk.

\[ f(3, 5) \geq \frac{1}{4} \]

1. Divide 2 muffin \( [\frac{6}{20}, \frac{7}{20}, \frac{7}{20}] \)
2. Divide 1 muffin \( [\frac{5}{20}, \frac{5}{20}, \frac{5}{20}, \frac{5}{20}] \)
3. Give 4 students \( (\frac{5}{20}, \frac{7}{20}) \)
4. Give 1 students \( (\frac{6}{20}, \frac{6}{20}) \)

Can we do better? Vote:

YES
NO
UNKNOWN TO SCIENCE
NO Proof on next slide.
There is a procedure for 3 muffins, 5 students where each student gets \( \frac{3}{5} \) muffins, smallest piece \( N \). We want \( N \leq \frac{1}{4} \).

**Case 0:** Some student gets 1 piece, so size \( \frac{3}{5} \). Cut that piece in half and give both \( \frac{3}{10} \)-sized pieces to that student. (Note \( \frac{3}{10} > \frac{1}{4} \).) Reduces to other cases. (Henceforth: All students get \( \geq 2 \) pieces.)

**Case 1:** Some student gets \( \geq 3 \) pieces. Then \( N \leq \frac{3}{5} \times \frac{1}{3} = \frac{1}{5} < \frac{1}{4} \). (Henceforth: All students get 2 pieces.)

**Case 2:** All students get 2 pieces. 5 students, so 10 pieces. **Some muffin** gets cut into \( \geq 4 \) pieces. Some piece \( \leq \frac{1}{4} \).
3 People, 5 Muffins VS 5 People, 3 Muffins

\[ f(5, 3) \geq \frac{5}{12} \]

1. Divide 4 muffins \([\frac{5}{12}, \frac{7}{12}]\)
2. Divide 1 muffin \([\frac{6}{12}, \frac{6}{12}]\)
3. Give 2 students \((\frac{6}{12}, \frac{7}{12}, \frac{7}{12})\)
4. Give 1 students \((\frac{5}{12}, \frac{5}{12}, \frac{5}{12}, \frac{5}{12})\)
3 People, 5 Muffins VS 5 People, 3 Muffins

\[ f(5, 3) \geq \frac{5}{12} \]

1. Divide 4 muffins \([\frac{5}{12}, \frac{7}{12}]\)
2. Divide 1 muffin \([\frac{6}{12}, \frac{6}{12}]\)
3. Give 2 students \((\frac{6}{12}, \frac{7}{12}, \frac{7}{12})\)
4. Give 1 students \((\frac{5}{12}, \frac{5}{12}, \frac{5}{12}, \frac{5}{12})\)

\[ f(3, 5) \geq \frac{1}{4} \]

1. Divide 2 muffin \([\frac{6}{20}, \frac{7}{20}, \frac{7}{20}]\)
2. Divide 1 muffin \([\frac{5}{20}, \frac{5}{20}, \frac{5}{20}, \frac{5}{20}]\)
3. Give 4 students \((\frac{5}{20}, \frac{7}{20})\)
4. Give 1 students \((\frac{6}{20}, \frac{6}{20})\)
3 People, 5 Muffins VS 5 People, 3 Muffins

\[ f(5, 3) \geq \frac{5}{12} \]

1. Divide 4 muffins \([\frac{5}{12}, \frac{7}{12}]\)
2. Divide 1 muffin \([\frac{6}{12}, \frac{6}{12}]\)
3. Give 2 students \((\frac{6}{12}, \frac{7}{12}, \frac{7}{12})\)
4. Give 1 students \((\frac{5}{12}, \frac{5}{12}, \frac{5}{12}, \frac{5}{12})\)

\[ f(3, 5) \geq \frac{1}{4} \]

1. Divide 2 muffin \([\frac{6}{20}, \frac{7}{20}, \frac{7}{20}]\)
2. Divide 1 muffin \([\frac{5}{20}, \frac{5}{20}, \frac{5}{20}, \frac{5}{20}]\)
3. Give 4 students \((\frac{5}{20}, \frac{7}{20})\)
4. Give 1 students \((\frac{6}{20}, \frac{6}{20})\)

\[ f(3, 5) \text{ proc is } f(5, 3) \text{ proc but swap Divide/Give and mult by 3/5.} \]
3 People, 5 Muffins VS 5 People, 3 Muffins

\[ f(5, 3) \geq \frac{5}{12} \]

1. Divide 4 muffins \([\frac{5}{12}, \frac{7}{12}]\)
2. Divide 1 muffin \([\frac{6}{12}, \frac{6}{12}]\)
3. Give 2 students \((\frac{6}{12}, \frac{7}{12}, \frac{7}{12})\)
4. Give 1 students \((\frac{5}{12}, \frac{5}{12}, \frac{5}{12}, \frac{5}{12})\)

\[ f(3, 5) \geq \frac{1}{4} \]

1. Divide 2 muffin \([\frac{6}{20}, \frac{7}{20}, \frac{7}{20}]\)
2. Divide 1 muffin \([\frac{5}{20}, \frac{5}{20}, \frac{5}{20}, \frac{5}{20}]\)
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4. Give 1 students \((\frac{6}{20}, \frac{6}{20})\)

\( f(3, 5) \) proc is \( f(5, 3) \) proc but swap Divide/Give and mult by 3/5.

**Theorem:** \( f(m, s) = \frac{m}{s} f(s, m) \).
Floor-Ceiling Theorem (Generalize $f(5, 3) \leq \frac{5}{12}$)

$$f(m, s) \leq \max\left\{\frac{1}{3}, \min\left\{\frac{m}{s\lceil 2m/s \rceil}, 1 - \frac{m}{s\lceil 2m/s \rceil}\right\}\right\}.$$  

**Case 0:** Some muffin is uncut. Cut it ($\frac{1}{2}, \frac{1}{2}$) and give both halves to whoever got the uncut muffin, so reduces to other cases.

**Case 1:** Some muffin is cut into $\geq 3$ pieces. Some piece $\leq \frac{1}{3}$.

**Case 2:** Every muffin is cut into 2 pieces, so $2m$ pieces.

**Someone** gets $\geq \lceil \frac{2m}{s} \rceil$ pieces. $\exists$ piece $\leq \frac{m}{s} \times \frac{1}{\lceil 2m/s \rceil} = \frac{m}{s\lceil 2m/s \rceil}$.

**Someone** gets $\leq \lfloor \frac{2m}{s} \rfloor$ pieces. $\exists$ piece $\geq \frac{m}{s} \lfloor 2m/s \rfloor = \frac{m}{s\lfloor 2m/s \rfloor}$.

The other piece from that muffin is of size $\leq 1 - \frac{m}{s\lfloor 2m/s \rfloor}$.
THREE Students

CLEVERNESS, COMP PROGS for the procedure.

Floor-Ceiling Theorem for optimality.

\[ f(1, 3) = \frac{1}{3} \]

\[ f(3k, 3) = 1. \]

\[ f(3k + 1, 3) = \frac{3k-1}{6k}, \quad k \geq 1. \]

\[ f(3k + 2, 3) = \frac{3k+2}{6k+6}. \]
FOUR Students

CLEVERNESS, COMP PROGS for procedures.

Floor-Ceiling Theorem for optimality.

\[ f(4k, 4) = 1 \text{ (easy)} \]
\[ f(1, 4) = \frac{1}{4} \text{ (easy)} \]
\[ f(4k + 1, 4) = \frac{4k-1}{8k}, \text{ } k \geq 1. \]
\[ f(4k + 2, 4) = \frac{1}{2}. \]
\[ f(4k + 3, 4) = \frac{4k+1}{8k+4}. \]

Is FIVE student case a Mod 5 pattern?
VOTE YES or NO
FOUR Students

CLEVERNESS, COMP PROGS for procedures.

Floor-Ceiling Theorem for optimality.

\[ f(4k, 4) = 1 \text{ (easy)} \]
\[ f(1, 4) = \frac{1}{4} \text{ (easy)} \]
\[ f(4k + 1, 4) = \frac{4k - 1}{8k}, \quad k \geq 1. \]
\[ f(4k + 2, 4) = \frac{1}{2}. \]
\[ f(4k + 3, 4) = \frac{4k + 1}{8k + 4}. \]

Is FIVE student case a Mod 5 pattern?
VOTE YES or NO
YES but with some exceptions
FIVE Students, \( m = 1, \ldots, 11 \)

\[
f(1, 5) = \frac{1}{5} \quad (\text{easy or use } f(1, 5) = \frac{5}{1} f(5, 1).)
\]

\[
f(2, 5) = \frac{1}{5} \quad (\text{easy or use } f(2, 5) = \frac{5}{2} f(5, 2).)
\]

\[
f(3, 5) = \frac{1}{4} \quad (\text{use } f(3, 5) = \frac{3}{5} f(5, 3).)
\]

\[
f(4, 5) = \frac{3}{10} \quad (\text{use } f(4, 5) = \frac{4}{5} f(5, 4).)
\]

\[
f(5, 5) = 1 \quad (\text{Easy and fits pattern})
\]

\[
f(6, 5) = \frac{2}{5} \quad (\text{Use Floor-Ceiling Thm}, \text{ fits pattern})
\]

\[
f(7, 5) = \frac{1}{3} \quad (\text{Use Floor-Ceiling Thm}, \text{ NOT pattern})
\]

\[
f(8, 5) = \frac{2}{5} \quad (\text{Use Floor-Ceiling Thm}, \text{ fits pattern})
\]

\[
f(9, 5) = \frac{2}{5} \quad (\text{Use Floor-Ceiling Thm}, \text{ fits pattern})
\]

\[
f(10, 5) = 1 \quad (\text{Easy and fits pattern})
\]

\[
f(11, 5) = (\text{Will come back to this later})
\]
CLEVERNESS, COMP PROGS for procedures.

Floor-Ceiling Theorem for optimality.

For \( k \geq 1 \), \( f(5k, 5) = 1 \).

For \( k = 1 \) and \( k \geq 3 \), \( f(5k + 1, 5) = \frac{5k+1}{10k+5} \)

For \( k \geq 2 \), \( f(5k + 2, 5) = \frac{5k-2}{10k} \)

For \( k \geq 1 \), \( f(5k + 3, 5) = \frac{5k+3}{10k+10} \)

For \( k \geq 1 \), \( f(5k + 4, 5) = \frac{5k+1}{10k+5} \)
What About FIVE students, ELEVEN muffins?

\[ f(11, 5) \geq \frac{13}{30}. \]

**Procedure:**

1. Divide 8 muffins into \( \left( \frac{13}{30}, \frac{17}{30} \right) \).
2. Divide 2 muffins into \( \left( \frac{14}{30}, \frac{16}{30} \right) \).
3. Divide 1 muffin into \( \left( \frac{15}{30}, \frac{15}{30} \right) \).
4. Give 2 students \( \left[ \frac{14}{30}, \frac{13}{30}, \frac{13}{30}, \frac{13}{20}, \frac{13}{20} \right] \).
5. Give 1 student \( \left[ \frac{17}{30}, \frac{17}{30}, \frac{16}{30}, \frac{16}{20} \right] \).
6. Give 2 students \( \left[ \frac{17}{30}, \frac{17}{30}, \frac{17}{30}, \frac{15}{30} \right] \).
What About FIVE students, ELEVEN muffins?

\[
f(m, s) \leq \max\left\{ \frac{1}{3}, \min\left\{ \frac{m}{s \lceil 2m/s \rceil}, 1 - \frac{m}{s \lfloor 2m/s \rfloor} \right\} \right\} \leq 0.44.\]

So

\[
\frac{13}{30} \leq f(11, 5) \leq \frac{11}{25} \quad \text{Diff} = 0.006666\ldots
\]
What About FIVE students, ELEVEN muffins?

\[
f(m, s) \leq \max \left\{ \frac{1}{3}, \min \left\{ \frac{m}{s \lceil 2m/s \rceil}, 1 - \frac{m}{s \lfloor 2m/s \rfloor} \right\} \right\} \leq 0.44.
\]

So
\[
\frac{13}{30} \leq f(11, 5) \leq \frac{11}{25} \quad \text{Diff} = 0.006666\ldots
\]

VOTE:
1. \( f(11, 5) = \frac{13}{30} \): Needs NEW methods to bound \( f(m, s) \).
2. \( f(11, 5) = \frac{11}{25} \): Needs NEW better procedure.
3. \( f(11, 5) = \alpha \) where \( \frac{13}{30} < \alpha < \frac{11}{25} \). Needs both:
4. **UNKNOWN TO SCIENCE!**
What About FIVE students, ELEVEN muffins?

\[ f(m, s) \leq \max \left\{ \frac{1}{3}, \min \left\{ \frac{m}{s \lceil 2m/s \rceil}, 1 - \frac{m}{s \lfloor 2m/s \rfloor} \right\} \right\} \leq 0.44. \]

So

\[ \frac{13}{30} \leq f(11, 5) \leq \frac{11}{25} \quad \text{Diff} = 0.006666... \]

VOTE:

1. \( f(11, 5) = \frac{13}{30} \): Needs NEW methods to bound \( f(m, s) \).
2. \( f(11, 5) = \frac{11}{25} \): Needs NEW better procedure.
3. \( f(11, 5) = \alpha \) where \( \frac{13}{30} < \alpha < \frac{11}{25} \). Needs both:
4. **UNKNOWN TO SCIENCE!**

KNOWN: \( f(11, 5) = \frac{13}{30} \)

HAPPY: New opt tech more interesting than new proc.
There is a procedure for 11 muffins, 5 students where each student gets \( \frac{11}{5} \) muffins, smallest piece \( N \). We want \( N \leq \frac{13}{30} \).

**Case 0:** Some muffin is uncut. Cut it \( (\frac{1}{2}, \frac{1}{2}) \) and give both halves to whoever got the uncut muffin. Reduces to other cases.

**Case 1:** Some muffin is cut into \( \geq 3 \) pieces. \( N \leq \frac{1}{3} < \frac{13}{30} \).

*(Negation of Case 0 and Case 1: All muffins cut into 2 pieces.)*
$f(11, 5) = \frac{13}{30}$, Easy Case Based on Students

**Case 2:** Some student gets $\geq 6$ pieces.

\[ N \leq \frac{11}{5} \times \frac{1}{6} = \frac{11}{30} < \frac{13}{30}. \]

**Case 3:** Some student gets $\leq 3$ pieces.

One of the pieces is

\[ \geq \frac{11}{5} \times \frac{1}{3} = \frac{11}{15}. \]

Look at the muffin it came from to find a piece that is

\[ \leq 1 - \frac{11}{15} = \frac{4}{15} < \frac{13}{30}. \]

(****Negation of Cases 2 and 3: Every student gets 4 or 5 pieces.****)
Case 4: Every muffin is cut in 2 pieces, every student gets 4 or 5 pieces. Number of pieces: 22. Note \( \leq 11 \) pieces are \( > \frac{1}{2} \).

- \( s_4 \) is number of students who get 4 pieces
- \( s_5 \) is number of students who get 5 pieces

\[
4s_4 + 5s_5 = 22 \\
\quad s_4 + s_5 = 5
\]

\( s_4 = 3 \): There are 3 students who have 4 pieces.
\( s_5 = 2 \): There are 2 students who have 5 pieces.
\[ f(11, 5) = \frac{13}{30}, \text{ Fun Cases} \]

\[ \begin{align*}
\Diamond & \; \Diamond & \; \Diamond & \; \Diamond & \; \Diamond & \; \Diamond & \; (\text{Sums to } 11/5) \\
\Diamond & \; \Diamond & \; \Diamond & \; \Diamond & \; \Diamond & \; \Diamond & \; (\text{Sums to } 11/5) \\
\Diamond & \; \Diamond & \; \Diamond & \; \Diamond & \; \Diamond & \; \Diamond & \; (\text{Sums to } 11/5) \\
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\Diamond & \; \Diamond & \; \Diamond & \; \Diamond & \; \Diamond & \; \Diamond & \; (\text{Sums to } 11/5) \\
\end{align*} \]

**Case 4.1:** One of (say)

\[ \begin{align*}
\Diamond & \; \Diamond & \; \Diamond & \; \Diamond & \; \Diamond & \; \Diamond & \; (\text{Sums to } 11/5) \\
\Diamond & \; \Diamond & \; \Diamond & \; \Diamond & \; \Diamond & \; \Diamond & \; (\text{Sums to } 11/5) \\
\Diamond & \; \Diamond & \; \Diamond & \; \Diamond & \; \Diamond & \; \Diamond & \; (\text{Sums to } 11/5) \\
\Diamond & \; \Diamond & \; \Diamond & \; \Diamond & \; \Diamond & \; \Diamond & \; (\text{Sums to } 11/5) \\
\Diamond & \; \Diamond & \; \Diamond & \; \Diamond & \; \Diamond & \; \Diamond & \; (\text{Sums to } 11/5) \\
\Diamond & \; \Diamond & \; \Diamond & \; \Diamond & \; \Diamond & \; \Diamond & \; (\text{Sums to } 11/5) \\
\Diamond & \; \Diamond & \; \Diamond & \; \Diamond & \; \Diamond & \; \Diamond & \; (\text{Sums to } 11/5) \\
\Diamond & \; \Diamond & \; \Diamond & \; \Diamond & \; \Diamond & \; \Diamond & \; (\text{Sums to } 11/5) \\
\end{align*} \]

is \( \leq \frac{1}{2} \). Then there is a piece

\[ \geq \frac{(11/5) - (1/2)}{3} = \frac{17}{30}. \]

The other piece from the muffin is

\[ \leq 1 - \frac{17}{30} = \frac{13}{30} \quad \text{Great to see } \frac{13}{30}. \]
\( f(11, 5) = \frac{13}{30} \), Fun Cases

**Case 4.2:** All

\[ \circ \circ \circ \circ \circ \] (Sums to 11/5)
\[ \circ \circ \circ \circ \circ \] (Sums to 11/5)
\[ \circ \circ \circ \circ \circ \] (Sums to 11/5)

are > \( \frac{1}{2} \).
There are \( \geq 12 \) pieces > \( \frac{1}{2} \). Can't occur.
The technique for $f(11, 5) \leq \frac{13}{30}$ has a generalization with a bakers dozen subcases. We do one concrete example:

**Definition:** Assume we have a protocol where all muffins are cut into two pieces. If $x$ is a piece then the other piece in the muffin it came from is its **buddy**. Note that $B(x) = 1 - x$. 
Theorem: \( f(24, 11) \leq \frac{19}{44} \) (≥ also known)

Assume \((24, 11)\)-procedure with smallest piece > \(\frac{19}{44}\).
Can assume all muffin cut in two and all student gets ≥ 2 shares.
We show that there is a piece ≤ \(\frac{19}{44}\).

Case 1: A student gets ≥ 6 shares. Some piece ≤ \(\frac{24}{11 \times 6} < \frac{19}{44}\).

Case 2: A student gets ≤ 3 shares. Some piece ≥ \(\frac{24}{11 \times 3} = \frac{8}{11}\).
Buddy of that piece ≤ \(1 - \frac{8}{11} \leq \frac{3}{11} < \frac{19}{44}\).

Case 3: Every muffin is cut in 2 pieces and every student gets either 4 or 5 shares. Total number of shares is 48.
How many students get 4? 5? Where are the Shares?

Let $s_4$ ($s_5$) be the number of 4-students (5-students).

$$4s_4 + 5s_5 = 48$$
$$s_4 + s_5 = 11$$

Get $s_4 = 7$ and $s_5 = 4$

**Case 3.1:** ($\exists$) 4-sh $\leq \frac{21}{44}$. Rm. Now: 3 shares $\geq \frac{24}{11} - \frac{21}{44}$. ($\exists$) share

$$\geq \frac{(24/11) - (21/44)}{3} = \frac{25}{44}.$$  

Buddy is

$$\leq 1 - \frac{25}{44} = \frac{19}{44}.$$  

SO can assume all 4-shares are $> \frac{21}{44}$.

By similar reasoning:

**Case 3.2:** 4-shares in $\left( \frac{21}{44}, \frac{25}{44} \right)$, 5-shares in $\left( \frac{19}{44}, \frac{20}{44} \right)$.

$$\begin{pmatrix} 19/44 & 20/44 \end{pmatrix} \begin{pmatrix} 0 \end{pmatrix} \begin{pmatrix} 21/44 & 25/44 \end{pmatrix}$$
More Refined Picture of What is Going On

\[
\begin{pmatrix}
(20 \text{ 5-shs})
\end{pmatrix}[0 \text{ shs}]
(28 \text{ 4-shs})
\]
\[
\begin{pmatrix}
\frac{19}{44} & \frac{20}{44} & \frac{21}{44} & \frac{25}{44}
\end{pmatrix}
\]

**Claim 1:** There are no shares \( x \in \left[\frac{23}{44}, \frac{24}{44}\right] \).

If there was such a share then \( B(x) \in \left[\frac{20}{44}, \frac{21}{44}\right] \).

The following picture captures what we know so far.

\[
\begin{pmatrix}
(20 \text{ 5-shs})
\end{pmatrix}[8 \text{ S4-shs}][0 \text{ shs}]
(20 \text{ L4-shs})
\]
\[
\begin{pmatrix}
\frac{19}{44} & \frac{20}{44} & \frac{21}{44} & \frac{23}{44} & \frac{24}{44} & \frac{25}{44}
\end{pmatrix}
\]
Claim 2: Every 4-student has at least 3 L4 shares.

If a 4-student had \( \leq 2 \) L4 shares then he has

\[
< 2 \times \left( \frac{23}{44} \right) + 2 \times \left( \frac{25}{44} \right) = \frac{24}{11}.
\]

Contradiction: There are at least \( 3 \times s_4 = 3 \times 7 = 21 \) L4 shares. But there are only 20.
What Else do we Have -Concrete

1. Formulas for $f(m, 6)$ and $f(m, 7)$.
2. For $s = 8, \ldots, 100$ conjectures for $f(m, s)$. $f(m, s)$ seems to be a mod $s$ pattern.
3. Formulas for $f(s + 1, s)$, $f(s + 2, s)$, $f(s + 3, s)$, $f(s + 4, s)$. $f(s + d, s)$ seems to have a mod $3d$ pattern.
4. A computer program that, on input $m, s$ uses our theorems to find $\alpha$ with $f(m, s) \leq \alpha$ and then tries to prove $f(m, s) \geq \alpha$ using linear algebra.
5. For $1 \leq m, s \leq 50$ have all $f(m, s)$ (Need to check that.)
6. Mixed integer program that always solves the problem but it is slow and has not been that useful.
1. For fixed $s$, for $m \geq \frac{s^3 + 2s^2 + s}{2}$ $f(m, s)$ matches the Floor-ceiling bound.

2. $f(m, s)$ always exists and is rational. Provable by compactness argument OR a large number of Linear Programs, OR one MIP. The last two proofs also give that $f(m, s)$ is computable. Nice synergy – applied math tools helping us prove theorems in pure math!
Consider:
Given $m, s$ in binary, compute $f(m, s)$.

1. Is the problem in P? We keep on finding techniques that we think cover all cases (so it would be in P) but then finding a case not covered.
2. Is it in NP? The procedure might be very large compared to the input.
3. Is it NP-complete or NP-hard?
4. Given $m, s$ is there a bound on the denominators of the sizes of shares used?
Open Problems-Misc

1. Show that for all $m \geq s$, $f(m, s) \geq \frac{1}{3}$.
2. Prove that we ALWAYS get mod $s$ pattern for $f(m, s)$ (true for large enough $m$).
3. Prove that we ALWAYS get mod $3d$ pattern for $f(s + d, s)$. 
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