The Muffin Problem

Guangi Cui - Montgomery Blair HS
John Dickerson- University of MD
Naveen Durvasula - Montgomery Blair HS
William Gasarch - University of MD
Erik Metz - University of MD
Naveen Raman - Richard Montgomery HS
Sung Hyun Yoo - Bergen County Academies (in NJ)

Five Muffins, Three Students

At

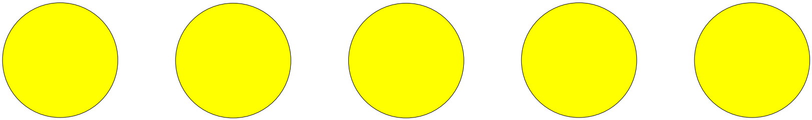
A Recreational Math Conference
(Gathering for Gardner)
May 2016

I found a pamphlet advertising

The Julia Robinson Mathematics Festival

which had this problem, proposed by Alan Frank:

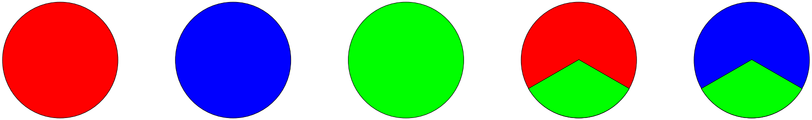
How can you divide and distribute 5 muffins to 3 students so that every student gets $\frac{5}{3}$ *where nobody gets a tiny sliver?*



Five Muffins, Three Students, Proc by Picture

Person	Color	What they Get
Alice	**RED**	$1 + \frac{2}{3} = \frac{5}{3}$
Bob	**BLUE**	$1 + \frac{2}{3} = \frac{5}{3}$
Carol	GREEN	$1 + \frac{1}{3} + \frac{1}{3} = \frac{5}{3}$

Smallest Piece: $\frac{1}{3}$



Can We Do Better?

The smallest piece in the above solution is $\frac{1}{3}$.

Is there a procedure with a larger smallest piece?

VOTE

Can We Do Better?

The smallest piece in the above solution is $\frac{1}{3}$.

Is there a procedure with a larger smallest piece?

VOTE

- **YES**
- **NO**

Can We Do Better?

The smallest piece in the above solution is $\frac{1}{3}$.

Is there a procedure with a larger smallest piece?

VOTE

- **YES**
- **NO**

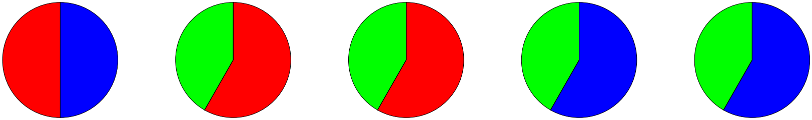
YES WE CAN!

We use **!** since we are excited that we can!

Five Muffins, Three People–Proc by Picture

Person	Color	What they Get
Alice	**RED**	$\frac{6}{12}+\frac{7}{12}+\frac{7}{12}$
Bob	**BLUE**	$\frac{6}{12}+\frac{7}{12}+\frac{7}{12}$
Carol	GREEN	$\frac{5}{12}+\frac{5}{12}+\frac{5}{12}+\frac{5}{12}$

Smallest Piece: $\frac{5}{12}$



Can We Do Better?

The smallest piece in the above solution is $\frac{5}{12}$.

Is there a procedure with a larger smallest piece?

VOTE

- **YES**
- **NO**

Can We Do Better?

The smallest piece in the above solution is $\frac{5}{12}$.
Is there a procedure with a larger smallest piece?
VOTE

- **YES**
- **NO**

NO WE CAN'T!

We use **!** since we are excited to prove we can't do better!

Five Muffins, Three People–Can't Do Better Than $\frac{5}{12}$

There is a procedure for 5 muffins,3 students where each student gets $\frac{5}{3}$ muffins, smallest piece N. We want $N \le \frac{5}{12}$.

Case 0: Some muffin is uncut. Cut it $(\frac{1}{2}, \frac{1}{2})$ and give both $\frac{1}{2}$-sized pieces to whoever got the uncut muffin. (Note $\frac{1}{2} > \frac{5}{12}$.) Reduces to other cases.
(**Henceforth:** All muffins are cut into ≥ 2 pieces.)

Case 1: Some muffin is cut into ≥ 3 pieces. Then $N \le \frac{1}{3} < \frac{5}{12}$.
(**Henceforth:** All muffins are cut into 2 pieces.)

Case 2: All muffins are cut into 2 pieces. 10 pieces, 3 students: **Someone** gets ≥ 4 pieces. He has some piece

$$\le \frac{5}{3} \times \frac{1}{4} = \frac{5}{12} \qquad \text{Great to see } \frac{5}{12}$$

Be Amazed Now! And Later!

1. Procedure for 5 muffins, 3 people, smallest piece $\frac{5}{12}$.
2. NO Procedure for 5 muffins, 3 people, smallest piece$> \frac{5}{12}$.

Amazing That Have Exact Result!

Be Amazed Now! And Later!

1. Procedure for 5 muffins, 3 people, smallest piece $\frac{5}{12}$.
2. NO Procedure for 5 muffins, 3 people, smallest piece$> \frac{5}{12}$.

Amazing That Have Exact Result!

Prepare To Be More Amazed! On Next Page!

Amazing Results!

1. Procedure for 47 muffins, 9 people, smallest piece $\frac{111}{234}$.
2. NO Procedure for 47 muffins, 9 people, smallest piece$> \frac{111}{234}$.

Amazing Results!

1. Procedure for 47 muffins, 9 people, smallest piece $\frac{111}{234}$.
2. NO Procedure for 47 muffins, 9 people, smallest piece$> \frac{111}{234}$.

1. Procedure for 52 muffins, 11 people, smallest piece $\frac{83}{176}$.
2. NO Procedure for 52 muffins, 11 people, smallest piece$> \frac{83}{176}$.

Amazing Results!

1. Procedure for 47 muffins, 9 people, smallest piece $\frac{111}{234}$.
2. NO Procedure for 47 muffins, 9 people, smallest piece$> \frac{111}{234}$.

1. Procedure for 52 muffins, 11 people, smallest piece $\frac{83}{176}$.
2. NO Procedure for 52 muffins, 11 people, smallest piece$> \frac{83}{176}$.

1. Procedure for 35 muffins, 13 people, smallest piece $\frac{64}{143}$.
2. NO Procedure for 35 muffins, 13 people, smallest piece$> \frac{64}{143}$.

Amazing Results!

1. Procedure for 47 muffins, 9 people, smallest piece $\frac{111}{234}$.
2. NO Procedure for 47 muffins, 9 people, smallest piece $> \frac{111}{234}$.

1. Procedure for 52 muffins, 11 people, smallest piece $\frac{83}{176}$.
2. NO Procedure for 52 muffins, 11 people, smallest piece $> \frac{83}{176}$.

1. Procedure for 35 muffins, 13 people, smallest piece $\frac{64}{143}$.
2. NO Procedure for 35 muffins, 13 people, smallest piece $> \frac{64}{143}$.

All done by hand, no use of a computer

Amazing Results!

1. Procedure for 47 muffins, 9 people, smallest piece $\frac{111}{234}$.
2. NO Procedure for 47 muffins, 9 people, smallest piece$> \frac{111}{234}$.

1. Procedure for 52 muffins, 11 people, smallest piece $\frac{83}{176}$.
2. NO Procedure for 52 muffins, 11 people, smallest piece$> \frac{83}{176}$.

1. Procedure for 35 muffins, 13 people, smallest piece $\frac{64}{143}$.
2. NO Procedure for 35 muffins, 13 people, smallest piece$> \frac{64}{143}$.

All done by hand, no use of a computer

Co-author Erik Metz is a *muffin savant*

General Problem

How can you divide and distribute m muffins to s students so that each students gets $\frac{m}{s}$ AND the MIN piece is MAXIMIZED?

An *(m, s)-procedure* is a way to divide and distribute m muffins to s students so that each student gets $\frac{m}{s}$ muffins.

An (m, s)-procedure is *optimal* if it has the largest smallest piece of any procedure.

$f(m, s)$ be the smallest piece in an optimal (m, s)-procedure.

We have shown $f(5, 3) = \frac{5}{12}$.

Note: $f(m, s) \geq \frac{1}{s}$: divide each M into s pieces of size $\frac{1}{s}$ and give each S m of them.

$f(3,5) \geq ?$

Clearly $f(3,5) \geq \frac{1}{5}$. Can we get $f(3,5) > \frac{1}{5}$?
Think about it at your desk.

$f(3,5) \geq ?$

Clearly $f(3,5) \geq \frac{1}{5}$. Can we get $f(3,5) > \frac{1}{5}$?
Think about it at your desk.
$f(3,5) \geq \frac{1}{4}$

1. Divide 2 muffin $[\frac{6}{20}, \frac{7}{20}, \frac{7}{20}]$
2. Divide 1 muffin $[\frac{5}{20}, \frac{5}{20}, \frac{5}{20}, \frac{5}{20}]$
3. Give 4 students $(\frac{5}{20}, \frac{7}{20})$
4. Give 1 students $(\frac{6}{20}, \frac{6}{20})$

$f(3,5) \geq ?$

Clearly $f(3,5) \geq \frac{1}{5}$. Can we get $f(3,5) > \frac{1}{5}$?
Think about it at your desk.
$f(3,5) \geq \frac{1}{4}$

1. Divide 2 muffin $[\frac{6}{20}, \frac{7}{20}, \frac{7}{20}]$
2. Divide 1 muffin $[\frac{5}{20}, \frac{5}{20}, \frac{5}{20}, \frac{5}{20}]$
3. Give 4 students $(\frac{5}{20}, \frac{7}{20})$
4. Give 1 students $(\frac{6}{20}, \frac{6}{20})$

Can we do better? Vote!
YES
NO
UNKNOWN TO SCIENCE

$f(3,5) \geq ?$

Clearly $f(3,5) \geq \frac{1}{5}$. Can we get $f(3,5) > \frac{1}{5}$?
Think about it at your desk.
$f(3,5) \geq \frac{1}{4}$

1. Divide 2 muffin $[\frac{6}{20}, \frac{7}{20}, \frac{7}{20}]$
2. Divide 1 muffin $[\frac{5}{20}, \frac{5}{20}, \frac{5}{20}, \frac{5}{20}]$
3. Give 4 students $(\frac{5}{20}, \frac{7}{20})$
4. Give 1 students $(\frac{6}{20}, \frac{6}{20})$

Can we do better? Vote!
YES
NO
UNKNOWN TO SCIENCE
NO Proof on next slide.

$f(3,5) \leq \frac{1}{4}$

There is a procedure for 3 muffins,5 students where each student gets $\frac{3}{5}$ muffins, smallest piece N. We want $N \leq \frac{1}{4}$.

Case 0: Some student gets 1 piece, so size $\frac{3}{5}$. Cut that piece in half and give both $\frac{3}{10}$-sized pieces to that student. (Note $\frac{3}{10} > \frac{1}{4}$.) Reduces to other cases.
(**Henceforth:** All students get ≥ 2 pieces.)

Case 1: Some student gets ≥ 3 pieces. Then $N \leq \frac{3}{5} \times \frac{1}{3} = \frac{1}{5} < \frac{1}{4}$.
(**Henceforth:** All students get 2 pieces.)

Case 2: All students get 2 pieces. 5 students, so 10 pieces.
Some muffin gets cut into ≥ 4 pieces. Some piece $\leq \frac{1}{4}$.

3 People, 5 Muffins VS 5 People, 3 Muffins

$f(5,3) \geq \frac{5}{12}$

1. Divide 4 muffins $[\frac{5}{12}, \frac{7}{12}]$
2. Divide 1 muffin $[\frac{6}{12}, \frac{6}{12}]$
3. Give 2 students $(\frac{6}{12}, \frac{7}{12}, \frac{7}{12})$
4. Give 1 students $(\frac{5}{12}, \frac{5}{12}, \frac{5}{12}, \frac{5}{12})$

3 People, 5 Muffins VS 5 People, 3 Muffins

$f(5,3) \geq \frac{5}{12}$

1. Divide 4 muffins $[\frac{5}{12}, \frac{7}{12}]$
2. Divide 1 muffin $[\frac{6}{12}, \frac{6}{12}]$
3. Give 2 students $(\frac{6}{12}, \frac{7}{12}, \frac{7}{12})$
4. Give 1 students $(\frac{5}{12}, \frac{5}{12}, \frac{5}{12}, \frac{5}{12})$

$f(3,5) \geq \frac{1}{4}$

1. Divide 2 muffin $[\frac{6}{20}, \frac{7}{20}, \frac{7}{20}]$
2. Divide 1 muffin $[\frac{5}{20}, \frac{5}{20}, \frac{5}{20}, \frac{5}{20}]$
3. Give 4 students $(\frac{5}{20}, \frac{7}{20})$
4. Give 1 students $(\frac{6}{20}, \frac{6}{20})$

3 People, 5 Muffins VS 5 People, 3 Muffins

$f(5,3) \geq \frac{5}{12}$

1. Divide 4 muffins $[\frac{5}{12}, \frac{7}{12}]$
2. Divide 1 muffin $[\frac{6}{12}, \frac{6}{12}]$
3. Give 2 students $(\frac{6}{12}, \frac{7}{12}, \frac{7}{12})$
4. Give 1 students $(\frac{5}{12}, \frac{5}{12}, \frac{5}{12}, \frac{5}{12})$

$f(3,5) \geq \frac{1}{4}$

1. Divide 2 muffin $[\frac{6}{20}, \frac{7}{20}, \frac{7}{20}]$
2. Divide 1 muffin $[\frac{5}{20}, \frac{5}{20}, \frac{5}{20}, \frac{5}{20}]$
3. Give 4 students $(\frac{5}{20}, \frac{7}{20})$
4. Give 1 students $(\frac{6}{20}, \frac{6}{20})$

$f(3,5)$ proc is $f(5,3)$ proc but swap Divide/Give and mult by $3/5$.

3 People, 5 Muffins VS 5 People, 3 Muffins

$f(5,3) \geq \frac{5}{12}$

1. Divide 4 muffins $[\frac{5}{12}, \frac{7}{12}]$
2. Divide 1 muffin $[\frac{6}{12}, \frac{6}{12}]$
3. Give 2 students $(\frac{6}{12}, \frac{7}{12}, \frac{7}{12})$
4. Give 1 students $(\frac{5}{12}, \frac{5}{12}, \frac{5}{12}, \frac{5}{12})$

$f(3,5) \geq \frac{1}{4}$

1. Divide 2 muffin $[\frac{6}{20}, \frac{7}{20}, \frac{7}{20}]$
2. Divide 1 muffin $[\frac{5}{20}, \frac{5}{20}, \frac{5}{20}, \frac{5}{20}]$
3. Give 4 students $(\frac{5}{20}, \frac{7}{20})$
4. Give 1 students $(\frac{6}{20}, \frac{6}{20})$

$f(3,5)$ proc is $f(5,3)$ proc but swap Divide/Give and mult by 3/5.
Theorem: $f(m,s) = \frac{m}{s} f(s,m)$.

Floor-Ceiling Theorem (Generalize $f(5,3) \le \frac{5}{12}$)

$$f(m,s) \le \max\left\{\frac{1}{3}, \min\left\{\frac{m}{s\lceil 2m/s\rceil}, 1-\frac{m}{s\lfloor 2m/s\rfloor}\right\}\right\}.$$

Case 0: Some muffin is uncut. Cut it $(\frac{1}{2},\frac{1}{2})$ and give both halves to whoever got the uncut muffin, so reduces to other cases.

Case 1: Some muffin is cut into ≥ 3 pieces. Some piece $\le \frac{1}{3}$.

Case 2: Every muffin is cut into 2 pieces, so $2m$ pieces.

Someone gets $\ge \lceil \frac{2m}{s} \rceil$ pieces. $\exists$ piece $\le \frac{m}{s} \times \frac{1}{\lceil 2m/s\rceil} = \frac{\mathbf{m}}{\mathbf{s}\lceil \mathbf{2m/s}\rceil}$.

Someone gets $\le \lfloor \frac{2m}{s} \rfloor$ pieces. $\exists$ piece $\ge \frac{m}{s}\frac{1}{\lfloor 2m/s\rfloor} = \frac{m}{s\lfloor 2m/s\rfloor}$.

The other piece from that muffin is of size $\le 1 - \frac{m}{s\lfloor 2m/s\rfloor}$.

THREE Students

CLEVERNESS, COMP PROGS for the procedure.

Floor-Ceiling Theorem for optimality.

$f(1,3) = \frac{1}{3}$

$f(3k,3) = 1.$

$f(3k+1,3) = \frac{3k-1}{6k}$, $k \geq 1.$

$f(3k+2,3) = \frac{3k+2}{6k+6}.$

FOUR Students

CLEVERNESS, COMP PROGS for procedures.

Floor-Ceiling Theorem for optimality.

$f(4k,4) = 1$ (easy)

$f(1,4) = \frac{1}{4}$ (easy)

$f(4k+1,4) = \frac{4k-1}{8k}$, $k \geq 1$.

$f(4k+2,4) = \frac{1}{2}$.

$f(4k+3,4) = \frac{4k+1}{8k+4}$.

Is FIVE student case a Mod 5 pattern?
VOTE YES or NO

FOUR Students

CLEVERNESS, COMP PROGS for procedures.

Floor-Ceiling Theorem for optimality.

$f(4k, 4) = 1$ (easy)

$f(1, 4) = \frac{1}{4}$ (easy)

$f(4k+1, 4) = \frac{4k-1}{8k}$, $k \geq 1$.

$f(4k+2, 4) = \frac{1}{2}$.

$f(4k+3, 4) = \frac{4k+1}{8k+4}$.

Is FIVE student case a Mod 5 pattern?
VOTE YES or NO
YES but with some exceptions

FIVE Students, $m = 1, \ldots, 11$

$f(1,5) = \frac{1}{5}$ (**easy** or use $f(1,5) = \frac{5}{1} f(5,1)$.)

$f(2,5) = \frac{1}{5}$ (**easy** or use $f(2,5) = \frac{5}{2} f(5,2)$.)

$f(3,5) = \frac{1}{4}$ (use $f(3,5) = \frac{3}{5} f(5,3)$.)

$f(4,5) = \frac{3}{10}$ (use $f(4,5) = \frac{4}{5} f(5,4)$.)

$f(5,5) = 1$ (Easy and fits pattern)

$f(6,5) = \frac{2}{5}$ (Use **Floor-Ceiling Thm**, fits pattern)

$f(7,5) = \frac{1}{3}$ (Use **Floor-Ceiling Thm**, NOT pattern)

$f(8,5) = \frac{2}{5}$ (Use **Floor-Ceiling Thm**, fits pattern)

$f(9,5) = \frac{2}{5}$ (Use **Floor-Ceiling Thm**, fits pattern)

$f(10,5) = 1$ (Easy and fits pattern)

$f(11,5) =$ (Will come back to this **later**)

FIVE Students

CLEVERNESS, COMP PROGS for procedures.

Floor-Ceiling Theorem for optimality.

For $k \geq 1$, $f(5k,5) = 1$.

For $k = 1$ and $k \geq 3$, $f(5k+1,5) = \frac{5k+1}{10k+5}$

For $k \geq 2$, $f(5k+2,5) = \frac{5k-2}{10k}$

For $k \geq 1$, $f(5k+3,5) = \frac{5k+3}{10k+10}$

For $k \geq 1$, $f(5k+4,5) = \frac{5k+1}{10k+5}$

What About FIVE students, ELEVEN muffins?

Procedure:

Divide the Muffins in to Pieces:

1. Divide 6 muffins into $(\frac{13}{30}, \frac{17}{30})$.
2. Divide 4 muffins into $(\frac{9}{20}, \frac{11}{20})$.
3. Divide 1 muffin into $(\frac{1}{2}, \frac{1}{2})$.

Distribute the Shares to Students:

1. Give 2 students $[\frac{17}{30}, \frac{17}{30}, \frac{17}{30}, \frac{1}{2}]$.
2. Give 2 students $[\frac{13}{30}, \frac{13}{30}, \frac{13}{30}, \frac{9}{20}, \frac{9}{20}]$
3. Give 1 student $[\frac{11}{20}, \frac{11}{20}, \frac{11}{20}, \frac{11}{20}]$

So

$$f(11, 5) \geq \frac{13}{30} \sim 0.43333.$$

What About FIVE students, ELEVEN muffins? Opt

Recall: **Floor-Ceiling Theorem:**

$$f(m,s) \le \max\left\{\frac{1}{3}, \min\left\{\frac{m}{s\lceil 2m/s\rceil}, 1-\frac{m}{s\lfloor 2m/s\rfloor}\right\}\right\}.$$

$$f(11,5) \le \max\left\{\frac{1}{3}, \min\left\{\frac{11}{5\lceil 22/5\rceil}, 1-\frac{11}{5\lfloor 22/5\rfloor}\right\}\right\}.$$

$$f(11,5) \le \max\left\{\frac{1}{3}, \min\left\{\frac{11}{5\times 5}, 1-\frac{11}{5\times 4}\right\}\right\}.$$

$$f(11,5) \le \max\left\{\frac{1}{3}, \min\left\{\frac{11}{25}, \frac{9}{20}\right\}\right\}.$$

$$f(11,5) \le \max\left\{\frac{1}{3}, \frac{11}{25}\right\} = \frac{11}{25} = 0.44.$$

Where Are We On FIVE students, ELEVEN muffins?

- By **Procedure** $\frac{13}{30} \sim 0.43333 \leq f(11,5)$
- By **Floor-Ceiling** $f(11,5) \leq \frac{11}{25} \sim .44$

So

$$\frac{13}{30} \leq f(11,5) \leq \frac{11}{25} \qquad \text{Diff} = 0.006666\ldots$$

Where Are We On FIVE students, ELEVEN muffins?

- By **Procedure** $\frac{13}{30} \sim 0.43333 \le f(11,5)$
- By **Floor-Ceiling** $f(11,5) \le \frac{11}{25} \sim .44$

So

$$\frac{13}{30} \le f(11,5) \le \frac{11}{25} \qquad \text{Diff} = 0.006666\ldots$$

Darling: 0.0066666 close enough ?

Where Are We On FIVE students, ELEVEN muffins?

- By **Procedure** $\frac{13}{30} \sim 0.43333 \le f(11,5)$
- By **Floor-Ceiling** $f(11,5) \le \frac{11}{25} \sim .44$

So

$$\frac{13}{30} \le f(11,5) \le \frac{11}{25} \quad \text{Diff} = 0.006666\ldots$$

Darling: 0.0066666 close enough ?

VOTE:

1. $f(11,5) = \frac{13}{30}$: Needs NEW technique to show limits on procedures.
2. $f(11,5) = \frac{11}{25}$: Needs NEW better procedure.
3. $f(11,5) = \alpha$ where $\frac{13}{30} < \alpha < \frac{11}{25}$. Needs both:
4. **UNKNOWN TO SCIENCE!**

Where Are We On FIVE students, ELEVEN muffins?

- By **Procedure** $\frac{13}{30} \sim 0.43333 \le f(11,5)$
- By **Floor-Ceiling** $f(11,5) \le \frac{11}{25} \sim .44$

So

$$\frac{13}{30} \le f(11,5) \le \frac{11}{25} \quad \text{Diff} = 0.006666\ldots$$

Darling: 0.0066666 close enough ?

VOTE:

1. $f(11,5) = \frac{13}{30}$: Needs NEW technique to show limits on procedures.
2. $f(11,5) = \frac{11}{25}$: Needs NEW better procedure.
3. $f(11,5) = \alpha$ where $\frac{13}{30} < \alpha < \frac{11}{25}$. Needs both:
4. **UNKNOWN TO SCIENCE!**

$$\text{KNOWN: } \mathbf{f(11,5) = \frac{13}{30}}$$

HAPPY: New opt tech more interesting than new proc.

$f(11,5) = \frac{13}{30}$, Easy Case Based on Muffins

There is a procedure for 11 muffins, 5 students where each student gets $\frac{11}{5}$ muffins, smallest piece N. We want $N \le \frac{13}{30}$.

Case 0: Some muffin is uncut. Cut it $(\frac{1}{2}, \frac{1}{2})$ and give both halves to whoever got the uncut muffin. Reduces to other cases.

Case 1: Some muffin is cut into ≥ 3 pieces. $N \le \frac{1}{3} < \frac{13}{30}$.

(**Negation of Case 0 and Case 1:** All muffins cut into 2 pieces.)

$f(11,5)=\frac{13}{30}$, Easy Case Based on Students

Case 2: Some student gets ≥ 6 pieces.

$$N \leq \frac{11}{5} \times \frac{1}{6} = \frac{11}{30} < \frac{13}{30}.$$

Case 3: Some student gets ≤ 3 pieces.
One of the pieces is

$$\geq \frac{11}{5} \times \frac{1}{3} = \frac{11}{15}.$$

Look at the muffin it came from to find a piece that is

$$\leq 1 - \frac{11}{15} = \frac{4}{15} < \frac{13}{30}.$$

(**Negation of Cases 2 and 3:** Every student gets 4 or 5 pieces.)

$f(11,5)=\frac{13}{30}$, Fun Cases

Case 4: Every muffin is cut in 2 pieces, every student gets 4 or 5 pieces. Number of pieces: 22. Note ≤ 11 pieces are $> \frac{1}{2}$.

- s_4 is number of students who get 4 pieces
- s_5 is number of students who get 5 pieces

$$\begin{aligned} 4s_4 + 5s_5 &= 22 \\ s_4 + s_5 &= 5 \end{aligned}$$

$s_4 = 3$: There are 3 students who have 4 pieces.
$s_5 = 2$: There are 2 students who have 5 pieces.

$f(11,5) = \frac{13}{30}$, Fun Cases

$$\diamond \quad \diamond \quad \diamond \quad \diamond \quad \diamond \qquad \text{(Sums to } 11/5\text{)}$$
$$\diamond \quad \diamond \quad \diamond \quad \Diamond \quad \Diamond \qquad \text{(Sums to } 11/5\text{)}$$

$$\circ \quad \circ \quad \circ \quad \circ \qquad \text{(Sums to } 11/5\text{)}$$
$$\circ \quad \circ \quad \circ \quad \bigcirc \qquad \text{(Sums to } 11/5\text{)}$$
$$\circ \quad \circ \quad \circ \quad \bigcirc \qquad \text{(Sums to } 11/5\text{)}$$

Case 4.1: One of (say)

$$\circ \quad \circ \quad \circ \quad \bigcirc \qquad \text{(Sums to } 11/5\text{)}$$

is $\le \frac{1}{2}$. Then there is a piece

$$\ge \frac{(11/5) - (1/2)}{3} = \frac{17}{30}.$$

The other piece from the muffin is

$$\le 1 - \frac{17}{30} = \frac{13}{30} \qquad \text{Great to see } \frac{13}{30}.$$

$f(11,5) = \frac{13}{30}$, Fun Cases

Case 4.2: All

$\circ$	$\circ$	$\circ$	$\circ$	(Sums to $11/5$)
$\circ$	$\circ$	$\circ$	$\bigcirc$	(Sums to $11/5$)
$\circ$	$\circ$	$\circ$	$\bigcirc$	(Sums to $11/5$)

are $> \frac{1}{2}$.
There are ≥ 12 pieces $> \frac{1}{2}$. Can't occur.

The Techniques Generalizes!

Good News!
The technique used to get $f(11,5) \le \frac{13}{30}$ lead to a theorem that apply to other cases! We call it **The Interval Theorem**

Bad News!
Interval Theorem is hard to state, so you don't **get** to see it.

Good News!
Interval Theorem is hard to state, so you don't **have** to see it.

For Fixed Num of Students s do get a Mod Pattern?

Known: (Empirical) For $1 \le s \le 100$, $f(m,s)$ has mod-s pattern with a finite number of exceptions.

Exceptions!

1. $f(s+1,s)$
2. $f(m,s) = \frac{1}{3}$
3. $f(m,s)$ used Interval Theorem

The Number of Exceptions (1-10)

s	$(s+1,s)$/excep	$\frac{1}{3}$/excep	INT/excep
1	1/0	0/0	0/0
2	1/0	0/0	0/0
3	1/0	0/0	0/0
4	1/0	0/0	0/0
5	1/0	1/1	1/1
6	1/1	0/0	0/0
7	1/1	1/1	1/1
8	1/0	1/1	0/0
9	1/1	1/1	4/4
10	1/1	0/0	0/0

The Number of Exceptions (11-20)

s	$(s+1,s)$/excep	$\frac{1}{3}$/excep	INT/excep
11	1/0	2/2	5/5
12	1/1	1/1	0/0
13	1/1	2/2	9/9
14	1/1	1/1	3/3
15	1/0	1/1	8/8
16	1/1	1/1	2/2
17	1/1	3/3	12/12
18	1/0	1/1	2/2
19	1/1	3/3	15/15
20	1/1	2/2	2/2

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Plausible:

1. There is a protocol showing $f(m, s) \geq \frac{1}{5}$
2. There is a protocol showing $f(m, s) \geq \frac{1}{5} + \frac{1}{5^2}$
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But NO protocol shows $f(m, s) \geq \frac{1}{5} + \frac{1}{5^2} + \cdots = \frac{1}{4}$.

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Plausible: $f(m,s)$ is not computable.
But **no**. Will show $f(m,s)$ is computable.

$f(m, s)$ Exist, Rational, Computable

Let x_{ij} be the fraction of Muffin i that Student j gets.
Each Muffin adds to 1:

$$(\forall i)[\sum_{j=1}^{s} x_{ij} = 1].$$

Each Student gets $\frac{m}{s}$:

$$(\forall j)[\sum_{i=1}^{m} x_{ij} = \frac{m}{s}].$$

Each Piece is of size between 0 and 1:

$$(\forall i, j)[0 \le x_{ij} \le 1].$$

$$\text{Maximize} \min_{1 \le i \le m, 1 \le j \le s} x_{ij}$$

relative to the constraints above.

Rephrase the Problem

Maximize z

Relative to constraints:

$$(\forall i)[\sum_{j=1}^{s} x_{ij} = 1]$$

$$(\forall j)[\sum_{i=1}^{m} x_{ij} = \frac{m}{s}]$$

$$(\forall i,j)[z \leq x_{ij} \leq 1]$$

This is a standard **Linear Programming Problem!**

There are very fast **packages** for it!

And Linear Programming is in P.

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Does not work. Could have some $x_{ij} = 0$.
If NONE of Muffin 1's goes to Student 3, so $x_{13} = 0$.
Get $z = 0$. Not what we want.

Plan for Correct Version of the Problem

For each i, j introduce variable $y_{ij} \in \{0, 1\}$ (0 OR 1).

Plan:

1. Will ensure that $x_{ij} = 0 \implies y_{ij} = 1$
2. Will ensure that $x_{ij} > 0 \implies y_{ij} = 0$
3. Will constrain z by $z \leq x_{ij} + y_{ij}$
   3.1 If $x_{ij} = 0$ then constraint is $z \leq 1$, NO EFFECT.
   3.2 If $x_{ij} > 0$ then constraint is $z \leq x_{ij}$. WHAT WE WANT.

Correct Version of he Problem

Add to the constraints:

1. Add variable y_{ij} which is in $\{0, 1\}$.
2. Add the constraint $x_{ij} + y_{ij} \leq 1$. Note that
   - $x_{ij} = 0 \implies x_{ij} + y_{ij} \leq 1$ (no constraint on y_{ij})
   - $x_{ij} > 0 \implies y_{ij} < 1 \implies y_{ij} = 0$
3. Add the constraint $x_{ij} + y_{ij} \geq \frac{1}{s}$. Note that
   - $x_{ij} = 0 \implies y_{ij} \geq \frac{1}{s} \implies y_{ij} = 1 \implies x_{ij} + y_{ij} = 1$
   - $x_{ij} > 0 \implies x_{ij} \geq \frac{1}{s} \implies x_{ij} + y_{ij} \geq \frac{1}{s}$ (no constraint on y_{ij})
4. Replace the constraint $z \leq x_{ij}$ with $z \leq x_{ij} + y_{ij}$.

$f(m,s)$ Rational! $f(m,s)$ Computable!

Definition: A **Mixed Integer Problem** is defined by

1. linear constraints on the variables,
2. want to maximize (or minimize) a linear function,
3. some of the variables are constrained to be integers, the rest reals.

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Known:

1. All MIP's with integer coefficients have rational solutions.
2. There is an algorithm to FIND the solutions to an MIP.
3. The problem is NP-complete (so thought to be hard to compute).

We have an MIP for $f(m,s)$ hence $f(m,s)$ is **exists!**, **rational! computable!**

Not Just Theoretical

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Good News: There are packages for MIP's that are ... okay.
Bad News: There is no more bad news which breaks the symmetry of good/bad/good/bad.
Good News: We HAVE coded it up and we HAVE gotten some results this way.

The Synergy Between Fields

One often hears:
Pure Math done without an application in mind often ends up being Applied!
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Pure Math, **Applied Math**, **Computer Science**, **Physics**, all play off each other! None of the four has moral superiority!

How Research Works

1. Obtain particular results.
2. Prove a general theorem based on those results.
3. Run into a case we cannot solve (e.g., (11,5) and (35,13)).
4. Lather, Rinse, Repeat.

What Else Have We Accomplished?

1. A formula for $f(s+1, s)$.
2. A computer program that helps us get procedures- used MIP
3. For $1 \le s \le 12$, for all m, know $f(m, s)$. Follows Mod Pattern.
4. Fix s. For large m $f(m, s)$ is Floor-Ceiling bound. (Proven June 21, 2017).

Conjectures I

Conjecture: For all s, $f(m,s)$ has a mod pattern. For $s=5$ mod is 30, for $s=6$ mod is 18, for all all other s, mod is s.

If Conjecture is true then:

$$\text{Computing } f(m,s) \text{ NP-hard} \implies \Sigma_2^p = \Pi_2^p$$

Hence: We do not think that $f(m,s)$ is NP-hard.

Conjectures II

$FC(m, s)$ is the upper bound provided by Floor-Ceiling Thm.
$IN(m, s)$ is the upper bound provided by INterval Thm.
$SP(s+1, s)$ is the exact answer provided by $f(s+1, s)$ Thm.

Conjectures II

$FC(m,s)$ is the upper bound provided by Floor-Ceiling Thm.
$IN(m,s)$ is the upper bound provided by INterval Thm.
$SP(s+1,s)$ is the exact answer provided by $f(s+1,s)$ Thm.
Conjecture: The following program computes $f(m,s)$ for $m > s$.

- If $d = gcd(m,s) \neq 1$ then call $f(m/d, s/d)$.
- If $m = s+1$ output $SP(s+1,s)$.
- If $s = 1$ then output 1.
- Otherwise output the MIN of $FC(m,s)$ and $IN(m,s)$ (Also conjecture that for fixed s, $IN(m,s)$ will be the answer only finitely often.)

Empirically true for $1 \le s \le 20$, $1 \le m \le 100$.
If True: Then computing $f(m,s)$ would be in P and would not need MIP to do so.

Accomplishment I Am Most Proud of

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Convinced

- 4 High School students (Guang, Naveen, Naveen, Sunny)
- 1 college student (Erik)
- 1 professor (John D.)

that the most important field of Mathematics is **Muffinry.**