The Muffin Problem

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Five Muffins, Three Students

At

A Recreational Math Conference
(Gathering for Gardner)

I found a pamphlet advertising

The Julia Robinson Mathematics Festival

which had this problem, proposed by Alan Frank:

How can you divide and distribute 5 muffins to 3 students so that every student gets $\frac{5}{3}$ where nobody gets a tiny sliver?
Five Muffins, Three Students, Proc by Picture

<table>
<thead>
<tr>
<th>Person</th>
<th>Color</th>
<th>What they Get</th>
</tr>
</thead>
<tbody>
<tr>
<td>Alice</td>
<td>RED</td>
<td>$1 + \frac{2}{3} = \frac{5}{3}$</td>
</tr>
<tr>
<td>Bob</td>
<td>BLUE</td>
<td>$1 + \frac{2}{3} = \frac{5}{3}$</td>
</tr>
<tr>
<td>Carol</td>
<td>GREEN</td>
<td>$1 + \frac{1}{3} + \frac{1}{3} = \frac{5}{3}$</td>
</tr>
</tbody>
</table>

Smallest Piece: $\frac{1}{3}$
Can We Do Better?

The smallest piece in the above solution is $\frac{1}{3}$.

Is there a procedure with a larger smallest piece?

VOTE
Can We Do Better?

The smallest piece in the above solution is $\frac{1}{3}$.

**Is there a procedure with a larger smallest piece?**

**VOTE**

- YES
- NO
Can We Do Better?

The smallest piece in the above solution is $\frac{1}{3}$.

Is there a procedure with a larger smallest piece?

VOTE

- YES
- NO

YES WE CAN!

We use ! since we are excited that we can!
## Five Muffins, Three People–Proc by Picture

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**Smallest Piece:** $\frac{5}{12}$
Can We Do Better?

The smallest piece in the above solution is $\frac{5}{12}$.

Is there a procedure with a larger smallest piece?

VOTE

▶ YES
▶ NO
Can We Do Better?

The smallest piece in the above solution is \( \frac{5}{12} \).

Is there a procedure with a larger smallest piece?

VOTE

- YES
- NO

NO WE CAN’T!

We use ! since we are excited to prove we can’t do better!
Assumption We Can Make

There is a procedure for 5 muffins, 3 students where each student gets $\frac{5}{3}$ muffins, smallest piece $N$. We want $N \leq \frac{5}{12}$.

We ASSUME each muffin cut into at least 2 pieces: If not then cut that muffin $(\frac{1}{2}, \frac{1}{2})$.

THIS TALK ALL proofs will be about opt being $\leq 1/2$. We assume each muffin is cut into at least 2 pieces.

PIECES VS SHARES: They are the same.

- PIECE is muffin-view,
- SHARE is student-view.
Muffin Principle

If a muffin is cut into $\geq u$ pieces then there is a piece $\leq \frac{1}{u}$.

Example: If a Muffin cut into 3 pieces:

\[
\text{some piece is } \leq \frac{1}{3}.
\]
Student Principle (not Principal)

If a student gets $\geq u$ shares then there is a share $\leq \frac{m}{s} \times \frac{1}{u}$

Example: 5 muffins, 3 students. All student gets $\frac{5}{3}$.
If some student gets $\geq 4$ shares:

Then one of these pieces is $\leq \frac{5}{3} \times \frac{1}{4}$
Pieces Principle

If there are $P$ pieces then:

Some student gets $\geq \left\lceil \frac{P}{s} \right\rceil$
Some student gets $\leq \left\lfloor \frac{P}{s} \right\rfloor$

Example: 5 muffins, 3 people. If there are 10 pieces:

Some student gets $\geq \left\lceil \frac{10}{3} \right\rceil = 4$
Some student gets $\leq \left\lfloor \frac{10}{3} \right\rfloor = 3$
There is a procedure for 5 muffins, 3 students where each student gets $\frac{5}{3}$ muffins, smallest piece $N$. We want $N \leq \frac{5}{12}$.

**Case 1:** Some muffin is cut into $\geq 3$ pieces. Then $N \leq \frac{1}{3} < \frac{5}{12}$. 
(Negation: All muffins are cut into 2 pieces.)

**Case 2:** All muffins are cut into 2 pieces. 10 pieces, 3 students: **Someone** gets $\geq 4$ pieces. He has some piece

$$\leq \frac{5}{3} \times \frac{1}{4} = \frac{5}{12}$$

Great to see $\frac{5}{12}$
Be Amazed Now! And Later!

1. Procedure for 5 muffins, 3 people, smallest piece $\frac{5}{12}$.
2. NO Procedure for 5 muffins, 3 people, smallest piece $\frac{5}{12}$.

Amazing That Have Exact Result!
Be Amazed Now! And Later!

1. Procedure for 5 muffins, 3 people, smallest piece \( \frac{5}{12} \).
2. NO Procedure for 5 muffins, 3 people, smallest piece \( > \frac{5}{12} \).

Amazing That Have Exact Result!

Prepare To Be More Amazed!
Be Amazed Now! And Later!

1. Procedure for 5 muffins, 3 people, smallest piece $\frac{5}{12}$.
2. NO Procedure for 5 muffins, 3 people, smallest piece $\geq \frac{5}{12}$.

Amazing That Have Exact Result!

Prepare To Be More Amazed!

We have many results like this:

\[
\begin{align*}
  f(47, 9) &= \frac{111}{234} \\
  f(52, 11) &= \frac{83}{176} \\
  f(35, 13) &= \frac{64}{143}
\end{align*}
\]
General Problem

How can you divide and distribute $m$ muffins to $s$ students so that each student gets $\frac{m}{s}$ AND the MIN piece is MAXIMIZED?

An $(m, s)$-procedure is a way to divide and distribute $m$ muffins to $s$ students so that each student gets $\frac{m}{s}$ muffins.

An $(m, s)$-procedure is optimal if it has the largest smallest piece of any procedure.

$f(m, s)$ be the smallest piece in an optimal $(m, s)$-procedure.

We have shown $f(5, 3) = \frac{5}{12}$.

Note: $f(m, s) \geq \frac{1}{s}$: divide each M into $s$ pieces of size $\frac{1}{s}$ and give each S $m$ of them.
Terminology Issue

Let $m, s \in \mathbb{N}$. 

$m$ is the number of muffins.  
$s$ is the number of students.

1. $f(m, s) \geq \alpha$ means that there is a procedure with smallest piece $\alpha$. We call this *A Procedure*.

2. $f(m, s) \leq \alpha$ means that there is NO procedure with smallest piece $> \alpha$. We call this *An Optimality Result* or *An Opt Result*.

**DO NOT** use terms *upper bound* and *lower bounds*:

1. Procedures are lower bounds, *opposite* of usual terminology.
2. Opt results are upper bounds, *opposite* of usual terminology.
Floor-Ceiling Theorem

\[ f(m, s) \leq \max \left\{ \frac{1}{3}, \min \left\{ \frac{m}{s \lfloor 2m/s \rfloor}, 1 - \frac{m}{s \lceil 2m/s \rceil} \right\} \right\}. \]

Proof:

**Case 1:** Some muffin is cut into \( \geq 3 \) pieces. Some piece \( \leq \frac{1}{3} \).

**Case 2:** Every muffin is cut into 2 pieces, so 2m pieces.

**Someone** gets \( \geq \left\lfloor \frac{2m}{s} \right\rfloor \) pieces. Some piece is \( \leq \frac{(m/s)}{\lfloor 2m/s \rfloor} = \frac{m}{s \lfloor 2m/s \rfloor} \).

**Someone** gets \( \leq \left\lceil \frac{2m}{s} \right\rceil \) pieces. Some piece is \( \geq \frac{(m/s)}{\lceil 2m/s \rceil} = \frac{m}{s \lceil 2m/s \rceil} \).

The other piece from that muffin is of size \( \leq 1 - \frac{m}{s \lceil 2m/s \rceil} \).
THREE Students

**CLEVERNESS, COMP PROGS** for the procedure.

**Floor-Ceiling Theorem** for optimality.

\[ f(1, 3) = \frac{1}{3} \]

\[ f(3k, 3) = 1. \]

\[ f(3k + 1, 3) = \frac{3k-1}{6k}, \quad k \geq 1. \]

\[ f(3k + 2, 3) = \frac{3k+2}{6k+6}. \]
FOUR Students

CLEVERNESS, COMP PROGS for procedures.

Floor-Ceiling Theorem for optimality.

\[ f(4k, 4) = 1 \text{ (easy)} \]

\[ f(1, 4) = \frac{1}{4} \text{ (easy)} \]

\[ f(4k + 1, 4) = \frac{4k-1}{8k}, \ k \geq 1. \]

\[ f(4k + 2, 4) = \frac{1}{2}. \]

\[ f(4k + 3, 4) = \frac{4k+1}{8k+4}. \]

Is FIVE student case a Mod 5 pattern?
VOTE YES or NO
FOUR Students

CLEVERNESS, COMP PROGS for procedures.

Floor-Ceiling Theorem for optimality.

\[ f(4k, 4) = 1 \text{ (easy)} \]

\[ f(1, 4) = \frac{1}{4} \text{ (easy)} \]

\[ f(4k + 1, 4) = \frac{4k-1}{8k}, \quad k \geq 1. \]

\[ f(4k + 2, 4) = \frac{1}{2}. \]

\[ f(4k + 3, 4) = \frac{4k+1}{8k+4}. \]

Is FIVE student case a Mod 5 pattern?

VOTE YES or NO

NO! (excited because YES would be boring)
FIVE Students, $m = 1, 2, 3, 4, 7, 11, 10k$

$f(1, 5) = \frac{1}{5}$ (easy)

$f(2, 5) = \frac{1}{5}$ (easy)

$f(3, 5) = \frac{1}{4}$ (Will discuss briefly later)

$f(4, 5) = \frac{3}{10}$ (Will not discuss later)

$f(7, 5) = \frac{1}{3}$ (Use Floor-Ceiling Thm)

$f(11, 5) = \text{(Will come back to this later)}$

$f(10k, 5) = 1$ (Trivial)
FIVE Students

Results on the next few slides:

**CLEVERNESS, COMP PROGS** for the procedure.

**Floor-Ceiling Theorem** for optimality.
FIVE Students $m = 10k + 1, 10k + 2, 10k + 3$

If $k$ not specified then $k \geq 0$.

$m = 10k + 1$:

$$f(30k + 1, 5) = \frac{30k+1}{60k+5}$$

$$f(30k + 11, 5) = \frac{30k+11}{60k+25} \quad (k \geq 1)$$

$$f(30k + 21, 5) = \frac{10k+7}{20k+15}$$

$f(10k + 2, 5) = \frac{10k-2}{20k} \quad (k \geq 1)$

$$f(10k + 3, 5) = \frac{10k+3}{20k+10} \quad (k \geq 1)$$
FIVE Students $m = 10k + 4, 10k + 5, 10k + 6$

$m = 10k + 4$

$$f(30k + 4, 5) = \frac{30k+1}{60k+5}$$

$$f(30k + 14, 5) = \frac{30k+11}{60k+25}$$

$$f(30k + 24, 5) = \frac{10k+7}{20k+15}$$

$f(10k + 5, 5) = 1$

$m = 10k + 6$: 

$$f(30k + 6, 5) = \frac{10k+2}{20k+5}$$

$$f(30k + 16, 5) = \frac{30k+16}{60k+35}$$

$$f(30k + 26, 5) = \frac{30k+26}{60k+55}$$
FIVE Students $m = 10k + 7, 10k + 8, 10k + 9$

\[
f(10k + 7, 5) = \frac{10k+3}{20k+10}
\]

\[
f(10k + 8, 5) = \frac{5k+4}{10k+10}
\]

\[
m = 10k + 9
\]

\[
f(30k + 9, 5) = \frac{10k+2}{20k+5}
\]

\[
f(30k + 19, 5) = \frac{30k+16}{60k+35}
\]

\[
f(30k + 29, 5) = \frac{30k+26}{60k+55}
\]
What About FIVE students, ELEVEN muffins?

Procedure:

Divide the Muffins in to Pieces:

1. Divide 6 muffins into \( \left( \frac{13}{30}, \frac{17}{30} \right) \).
2. Divide 4 muffins into \( \left( \frac{9}{20}, \frac{11}{20} \right) \).
3. Divide 1 muffin into \( \left( \frac{1}{2}, \frac{1}{2} \right) \).

Distribute the Shares to Students:

1. Give 2 students \( \left[ \frac{17}{30}, \frac{17}{30}, \frac{17}{30}, \frac{1}{2} \right] \).
2. Give 2 students \( \left[ \frac{13}{30}, \frac{13}{30}, \frac{13}{30}, \frac{9}{20}, \frac{9}{20} \right] \).
3. Give 1 student \( \left[ \frac{11}{20}, \frac{11}{20}, \frac{11}{20}, \frac{11}{20} \right] \).

So

\[
 f(11, 5) \geq \frac{13}{30}
\]
What About FIVE students, ELEVEN muffins? Opt

Recall: **Floor-Ceiling Theorem:**

\[
f(m, s) \leq \max \left\{ \frac{1}{3}, \min \left\{ \frac{m}{s \lceil 2m/s \rceil}, 1 - \frac{m}{s \lfloor 2m/s \rfloor} \right\} \right\}.
\]

\[
f(11, 5) \leq \max \left\{ \frac{1}{3}, \min \left\{ \frac{11}{5 \lceil 22/5 \rceil}, 1 - \frac{11}{5 \lfloor 22/5 \rfloor} \right\} \right\}.
\]

\[
f(11, 5) \leq \max \left\{ \frac{1}{3}, \min \left\{ \frac{11}{5 \times 5}, 1 - \frac{11}{5 \times 4} \right\} \right\}.
\]

\[
f(11, 5) \leq \max \left\{ \frac{1}{3}, \min \left\{ \frac{11}{25}, \frac{9}{20} \right\} \right\}.
\]

\[
f(11, 5) \leq \max \left\{ \frac{1}{3}, \frac{11}{25} \right\} = \frac{11}{25}.
\]
Where Are We On FIVE students, ELEVEN muffins?

- By **Procedure** \( \frac{13}{30} \leq f(11, 5) \).
- By **Floor-Ceiling** \( f(11, 5) \leq \frac{11}{25} \).

So

\[
\frac{13}{30} \leq f(11, 5) \leq \frac{11}{25}
\]

\( \text{Diff=} \ 0.006666 \ldots \)
Where Are We On FIVE students, ELEVEN muffins?

- By **Procedure** \( \frac{13}{30} \leq f(11, 5) \).
- By **Floor-Ceiling** \( f(11, 5) \leq \frac{11}{25} \).

So

\[
\frac{13}{30} \leq f(11, 5) \leq \frac{11}{25} \quad \text{Diff} = 0.006666 \ldots
\]

**VOTE:**

1. **KNOWN:** \( f(11, 5) = \frac{13}{30} \): New opt technique.
2. **KNOWN:** \( f(11, 5) = \frac{11}{25} \): New procedure.
3. **KNOWN:** \( \frac{13}{30} < f(11, 5) < \frac{11}{25} \): New opt and new proc.
4. **UNKNOWN TO SCIENCE!**
5. **HARAMBE THE GORILLA!**
   (In Poll of Discrete Math Students for Presidential Election 3 wrote in Harambe.)
Where Are We On FIVE students, ELEVEN muffins?

- By **Procedure** $\frac{13}{30} \leq f(11, 5)$.
- By **Floor-Ceiling** $f(11, 5) \leq \frac{11}{25}$.

So

$$\frac{13}{30} \leq f(11, 5) \leq \frac{11}{25} \quad \text{Diff=} \ 0.006666 \ldots$$

**VOTE:**

1. **KNOWN:** $f(11, 5) = \frac{13}{30}$: New opt technique.
2. **KNOWN:** $f(11, 5) = \frac{11}{25}$: New procedure.
3. **KNOWN:** $\frac{13}{30} < f(11, 5) < \frac{11}{25}$: New opt and new proc.
4. **UNKNOWN TO SCIENCE!**
5. **HARAMBE THE GORILLA!**
   (In Poll of Discrete Math Students for Presidential Election 3 wrote in Harambe.)

**KNOWN:** $f(11, 5) = \frac{13}{30}$

**HAPPY:** New opt tech more interesting than new proc.
$f(11, 5) = \frac{13}{30}$, Easy Case Based on Muffins

$N$ is smallest piece.

**Case 1:** Some muffin is cut into $\geq 3$ pieces. $N \leq \frac{1}{3} < \frac{13}{30}$.

(Negation: All muffins cut into 2 pieces.)
$f(11, 5) = \frac{13}{30}$, Easy Case Based on Students

**Case 2:** Some student gets $\geq 6$ pieces.

$$N \leq \frac{11}{5} \times \frac{1}{6} = \frac{11}{30} < \frac{13}{30}.$$  

**Case 3:** Some student gets $\leq 3$ pieces.

One of the shares is

$$\geq \frac{11}{5} \times \frac{1}{3} = \frac{11}{15}.$$  

Look at the muffin it came from to find a piece that is

$$\leq 1 - \frac{11}{15} = \frac{4}{15} < \frac{13}{30}.$$  

*(Negation of Cases 2 and 3: Every student gets 4 or 5 shares.)*
Case 4: Every muffin is cut in 2 pieces, every student gets 4 or 5 pieces. Number of pieces: 22. Note $\leq 11$ pieces are $> \frac{1}{2}$.

$\triangleright$ $s_4$ is number of students who get 4 shares

$\triangleright$ $s_5$ is number of students who get 5 shares

\[
4s_4 + 5s_5 = 22
\]
\[
s_4 + s_5 = 5
\]

$s_4 = 3$: There are 3 students who have 4 shares.
$s_5 = 2$: There are 2 students who have 5 shares.
\( f(11, 5) = \frac{13}{30}, \text{ Fun Cases} \)

\( \Diamond \) and \( \circ \) are shares.

\[ \begin{align*}
\Diamond & \quad \Diamond \quad \Diamond \quad \Diamond \quad \Diamond \quad \Diamond \quad \text{(Sums to 11/5)} \\
\Diamond & \quad \Diamond \quad \Diamond \quad \Diamond \quad \Diamond \quad \Diamond \quad \text{(Sums to 11/5)}
\end{align*} \]

\[ \begin{align*}
\circ & \quad \circ \quad \circ \quad \circ \quad \circ \quad \circ \quad \text{(Sums to 11/5)} \\
\circ & \quad \circ \quad \circ \quad \circ \quad \circ \quad \circ \quad \text{(Sums to 11/5)} \\
\circ & \quad \circ \quad \circ \quad \circ \quad \circ \quad \circ \quad \text{(Sums to 11/5)}
\end{align*} \]

**Case 3.1:** One of (say)

\[ \begin{align*}
\circ & \quad \circ \quad \circ \quad \circ \quad \circ \quad \circ \quad \text{(Sums to 11/5)}
\end{align*} \]

is \( \leq \frac{1}{2} \). Then there is a share

\[
\geq \frac{(11/5) - (1/2)}{3} = \frac{17}{30}.
\]

The other piece from the muffin is

\[
\leq 1 - \frac{17}{30} = \frac{13}{30} \quad \text{Great to see } \frac{13}{30}.
\]
\( f(11, 5) = \frac{13}{30}, \text{ Fun Cases} \)

**Case 3.2: All**

\[ \begin{align*}
\circ & \circ & \circ & \circ & \circ \quad (\text{Sums to } 11/5) \\
\circ & \circ & \circ & \circ & \circ \quad (\text{Sums to } 11/5) \\
\circ & \circ & \circ & \circ & \circ \quad (\text{Sums to } 11/5)
\end{align*} \]

are \( > \frac{1}{2} \).

There are \( \geq 12 \) shares \( > \frac{1}{2} \). Can’t occur.
The Techniques Generalizes!

**Good News!**
The technique used to get \( f(11, 5) \leq \frac{13}{30} \) lead to a theorem that apply to other cases! We call it *The Interval Theorem*

**Bad News!**
*Interval Theorem* is hard to state, so you don’t get to see it.

**Good News!**
*Interval Theorem* is hard to state, so you don’t have to see it.
Does $f(m, s)$ Always Exist? Rational? Computable?

Let $x_{ij}$ be the fraction of Muffin $i$ that Student $j$ gets. Each Muffin adds to 1:

$$(\forall i)[\sum_{j=1}^{s} x_{ij} = 1].$$

Each Student gets $\frac{m}{s}$:

$$(\forall j)[\sum_{i=1}^{m} x_{ij} = \frac{m}{s}].$$

Each Piece is of size between 0 and 1:

$$(\forall i, j)[0 \leq x_{ij} \leq 1].$$

Maximize $\min_{1\leq i\leq m, 1\leq j\leq s} x_{ij}$

relative to the constraints above.
Rephrase the Problem

Maximize $z$
Relative to constraints:

\[(\forall i)\left[\sum_{j=1}^{s} x_{ij} = 1\right]\]

\[(\forall j)\left[\sum_{i=1}^{m} x_{ij} = \frac{m}{s}\right]\]

\[(\forall i, j)\left[z \leq x_{ij} \leq 1\right]\]

This is a standard **Linear Programming Problem**!
There are very fast **packages** for it!
And Linear Programming is in P.

Does not work. Could have some $x_{ij} = 0$.
If NONE of Muffin 1's goes to Student 3, so $x_{13} = 0$.
Get $z = 0$. Not what we want.
Rephrase the Problem

Maximize $z$
Relative to constraints:

$$(\forall i)[\sum_{j=1}^{s} x_{ij} = 1]$$

$$(\forall j)[\sum_{i=1}^{m} x_{ij} = \frac{m}{s}]$$

$$(\forall i, j)[z \leq x_{ij} \leq 1]$$

This is a standard **Linear Programming Problem**!
There are very fast **packages** for it!
And Linear Programming is in P.

Does not work. Could have some $x_{ij} = 0$.
If NONE of Muffin 1’s goes to Student 3, so $x_{13} = 0$.
Get $z = 0$. Not what we want.
For each $i, j$ introduce variable $y_{ij} \in \{0, 1\}$ (0 OR 1).

**Plan:**

1. Will ensure that $x_{ij} = 0 \implies y_{ij} = 1$
2. Will ensure that $x_{ij} > 0 \implies y_{ij} = 0$
3. Will constrain $z$ by $z \leq x_{ij} + y_{ij}$
   
   3.1 If $x_{ij} = 0$ then constraint is $z \leq 1$, NO EFFECT.
   
   3.2 If $x_{ij} > 0$ then constraint is $z \leq x_{ij}$. WHAT WE WANT.
Correct Version of the Problem

Add to the constraints:

1. Add variable $y_{ij}$ which is in $\{0, 1\}$.
2. Add the constraint $x_{ij} + y_{ij} \leq 1$. Note that
   - $x_{ij} = 0 \implies x_{ij} + y_{ij} \leq 1$ (no constraint on $y_{ij}$)
   - $x_{ij} > 0 \implies y_{ij} < 1 \implies y_{ij} = 0$
3. Add the constraint $x_{ij} + y_{ij} \geq \frac{1}{s}$. Note that
   - $x_{ij} = 0 \implies y_{ij} \geq \frac{1}{s} \implies y_{ij} = 1 \implies x_{ij} + y_{ij} = 1$
   - $x_{ij} > 0 \implies x_{ij} \geq \frac{1}{s} \implies x_{ij} + y_{ij} \geq \frac{1}{s}$ (no constraint on $y_{ij}$)
4. Replace the constraint $z \leq x_{ij}$ with $z \leq x_{ij} + y_{ij}$.
**Definition:** A **Mixed Integer Problem** is defined by

1. linear constraints on the variables,
2. want to maximize (or minimize) a linear function,
3. some of the variables are constrainted to be integers, the rest reals.
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1. linear constraints on the variables,
2. want to maximize (or minimize) a linear function,
3. some of the variables are constrained to be integers, the rest reals.

**Known:**
1. All MIP’s with integer coefficients have rational solutions.
2. There is an algorithm to FIND the solutions to an MIP.
3. The problem is NP-complete (so thought to be hard to compute).

We have an MIP for $f(m, s)$ hence $f(m, s)$ is **rational!**
**Computable!**
Not Just Theoretical

**Good News:** $f(m,s)$ is rational and computable!
Not Just Theoretical

**Good News:** \( f(m, s) \) is rational and computable!

**Bad News:** Proof uses MIP’s where are NP-complete.
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**Good News:** There are packages for MIP’s that are . . . okay.
Not Just Theoretical

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**Bad News:** There is no more bad news which breaks the symmetry of good/bad/good/bad.

We HAVE coded it up and we HAVE gotten some results this way.
Not Just Theoretical

**Good News:** $f(m, s)$ is rational and computable!

**Bad News:** Proof uses MIP’s where are NP-complete

**Good News:** There are packages for MIP’s that are \ldots okay.

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The Synergy Between Fields

One often hears:

**Pure Math done without an application in mind often ends up being Applied!**

(Number theory and Cryptography is a great example.)
The Synergy Between Fields

One often hears: 
**Pure Math done without an application in mind often ends up being Applied!**
(Number theory and Cryptography is a **great** example.)

One seldom hears (though its true):
**Applied Math done for a real world applications often ends up being used for Pure Math!**
(MIP and Muffins is a ‘**great**’ example.)
The Synergy Between Fields

One often hears:
**Pure Math done without an application in mind often ends up being Applied!**
(Number theory and Cryptography is a great example.)

One seldom hears (though its true):
**Applied Math done for a real world applications often ends up being used for Pure Math!**
(MIP and Muffins is a ‘great’ example.)

Pure Math, Applied Math, Computer Science, Physics, all play off each other!
How Research Works

History:

1. Obtain particular results.
2. Prove a general theorem based on those results.
3. Run into a case we cannot solve (e.g., (11,5) and (35,13)).
4. Lather, Rinse, Repeat.
1. A formula for $f(s + 1, s)$.
2. A computer program that helps us get procedures—used MIP
3. For $1 \leq s \leq 15$, for all $m$, know $f(m, s)$.
4. Convinced 4 High School students, 1 college student, and one professor that the most important field of Mathematics is **Muffinry**.
Open Questions

1. For all $s$ there is a pattern for $f(m, s)$ that depends on $m \mod T$ where $s$ divides $T$.
2. $f(m, s) = \frac{a}{b}$ (lowest terms) where $s$ divides $b$.
3. For all $m \geq s$, $f(m, s)$ is always determined by either
   - Floor Ceiling Theorem
   - Interval Theorem
   - $f(s + 1, s)$ Theorem.