The Muffin Problem

Guangi Cui - Montgomery Blair HS
John Dickerson- University of MD
Naveen Durvasula - Montgomery Blair HS
William Gasarch - University of MD
Erik Metz - University of MD
Jacob Prinz-University of MD
Naveen Raman - Richard Montgomery HS
Daniel Smolyak- University of MD
Sung Hyun Yoo - Bergen County Academies (in NJ)
Five Muffins, Three Students

At

A Recreational Math Conference
(Gathering for Gardner)
May 2016

I found a pamphlet:

The Julia Robinson Mathematics Festival:
A Sample of Mathematical Puzzles
Compiled by Nancy Blachman

which had this problem, proposed by Alan Frank:

How can you divide and distribute 5 muffins to 3 students so that every student gets $\frac{5}{3}$ where nobody gets a tiny sliver?
Five Muffins, Three Students, Proc by Picture

<table>
<thead>
<tr>
<th>Person</th>
<th>Color</th>
<th>What they Get</th>
</tr>
</thead>
<tbody>
<tr>
<td>Alice</td>
<td>RED</td>
<td>$1 + \frac{2}{3} = \frac{5}{3}$</td>
</tr>
<tr>
<td>Bob</td>
<td>BLUE</td>
<td>$1 + \frac{2}{3} = \frac{5}{3}$</td>
</tr>
<tr>
<td>Carol</td>
<td>GREEN</td>
<td>$1 + \frac{1}{3} + \frac{1}{3} = \frac{5}{3}$</td>
</tr>
</tbody>
</table>

Smallest Piece: $\frac{1}{3}$
Can We Do Better?

The smallest piece in the above solution is $\frac{1}{3}$.

Is there a procedure with a larger smallest piece?

VOTE
Can We Do Better?

The smallest piece in the above solution is $\frac{1}{3}$.

**Is there a procedure with a larger smallest piece?**

**VOTE**

- **YES**
- **NO**
Can We Do Better?

The smallest piece in the above solution is $\frac{1}{3}$.

Is there a procedure with a larger smallest piece?

VOTE

- YES
- NO

YES WE CAN!

We use ! since we are excited that we can!
Five Muffins, Three People–Proc by Picture

<table>
<thead>
<tr>
<th>Person</th>
<th>Color</th>
<th>What they Get</th>
</tr>
</thead>
<tbody>
<tr>
<td>Alice</td>
<td>RED</td>
<td>$\frac{6}{12} + \frac{7}{12} + \frac{7}{12}$</td>
</tr>
<tr>
<td>Bob</td>
<td>BLUE</td>
<td>$\frac{6}{12} + \frac{7}{12} + \frac{7}{12}$</td>
</tr>
<tr>
<td>Carol</td>
<td>GREEN</td>
<td>$\frac{5}{12} + \frac{5}{12} + \frac{5}{12} + \frac{5}{12}$</td>
</tr>
</tbody>
</table>

Smallest Piece: $\frac{5}{12}$
Can We Do Better?

The smallest piece in the above solution is \( \frac{5}{12} \).

Is there a procedure with a larger smallest piece?

VOTE

- YES
- NO
Can We Do Better?

The smallest piece in the above solution is \( \frac{5}{12} \).

**Is there a procedure with a larger smallest piece?**

VOTE

- YES
- NO

NO WE CAN’T!

We use ! since we are excited to prove we can’t do better!
There is a procedure for 5 muffins, 3 students where each student gets $\frac{5}{3}$ muffins, smallest piece $N$. We want $N \leq \frac{5}{12}$.

**Case 0:** Some muffin is uncut. Cut it $\left(\frac{1}{2}, \frac{1}{2}\right)$ and give both $\frac{1}{2}$-sized pieces to whoever got the uncut muffin. (Note $\frac{1}{2} > \frac{5}{12}$.) Reduces to other cases. 

*(Henceforth: All muffins are cut into $\geq 2$ pieces.)*

**Case 1:** Some muffin is cut into $\geq 3$ pieces. Then $N \leq \frac{1}{3} < \frac{5}{12}$.

*(Henceforth: All muffins are cut into 2 pieces.)*

**Case 2:** All muffins are cut into 2 pieces. 10 pieces, 3 students: Someone gets $\geq 4$ pieces. He has some piece

$$\leq \frac{5}{3} \times \frac{1}{4} = \frac{5}{12}$$

Great to see $\frac{5}{12}$
Be Amazed Now! And Later!

1. Procedure for 5 muffins, 3 people, smallest piece \( \frac{5}{12} \).
2. NO Procedure for 5 muffins, 3 people, smallest piece > \( \frac{5}{12} \).

Amazing That Have Exact Result!
1. Procedure for 5 muffins, 3 people, smallest piece $\frac{5}{12}$.
2. NO Procedure for 5 muffins, 3 people, smallest piece $> \frac{5}{12}$.

**Amazing That Have Exact Result!**

Prepare To Be More Amazed! On Next Page!
Amazing Results!

1. Procedure for 47 muffins, 9 people, smallest piece $\frac{111}{234}$.
2. NO Procedure for 47 muffins, 9 people, smallest piece $\frac{111}{234}$.
Amazing Results!

1. Procedure for 47 muffins, 9 people, smallest piece $\frac{111}{234}$.
2. NO Procedure for 47 muffins, 9 people, smallest piece $\frac{111}{234}$.

1. Procedure for 52 muffins, 11 people, smallest piece $\frac{83}{176}$.
2. NO Procedure for 52 muffins, 11 people, smallest piece $\frac{83}{176}$.

All done by hand, no use of a computer.

Co-author Erik Metz is a muffin savant.
Amazing Results!

1. Procedure for 47 muffins, 9 people, smallest piece $\frac{111}{234}$.
2. NO Procedure for 47 muffins, 9 people, smallest piece $\geq \frac{111}{234}$.

1. Procedure for 52 muffins, 11 people, smallest piece $\frac{83}{176}$.
2. NO Procedure for 52 muffins, 11 people, smallest piece $\geq \frac{83}{176}$.

1. Procedure for 35 muffins, 13 people, smallest piece $\frac{64}{143}$.
2. NO Procedure for 35 muffins, 13 people, smallest piece $\geq \frac{64}{143}$.
Amazing Results!

1. Procedure for 47 muffins, 9 people, smallest piece $\frac{111}{234}$.
2. NO Procedure for 47 muffins, 9 people, smallest piece $\geq \frac{111}{234}$.

1. Procedure for 52 muffins, 11 people, smallest piece $\frac{83}{176}$.
2. NO Procedure for 52 muffins, 11 people, smallest piece $\geq \frac{83}{176}$.

1. Procedure for 35 muffins, 13 people, smallest piece $\frac{64}{143}$.
2. NO Procedure for 35 muffins, 13 people, smallest piece $\geq \frac{64}{143}$.

All done by hand, no use of a computer.
Amazing Results!

1. Procedure for 47 muffins, 9 people, smallest piece $\frac{111}{234}$.
2. NO Procedure for 47 muffins, 9 people, smallest piece $\geq \frac{111}{234}$.

1. Procedure for 52 muffins, 11 people, smallest piece $\frac{83}{176}$.
2. NO Procedure for 52 muffins, 11 people, smallest piece $\geq \frac{83}{176}$.

1. Procedure for 35 muffins, 13 people, smallest piece $\frac{64}{143}$.
2. NO Procedure for 35 muffins, 13 people, smallest piece $\geq \frac{64}{143}$.

All done by hand, no use of a computer

Co-author Erik Metz is a muffin savant
General Problem

How can you divide and distribute $m$ muffins to $s$ students so that each student gets $\frac{m}{s}$ AND the MIN piece is MAXIMIZED?

An $(m, s)$-procedure is a way to divide and distribute $m$ muffins to $s$ students so that each student gets $\frac{m}{s}$ muffins.

An $(m, s)$-procedure is optimal if it has the largest smallest piece of any procedure.

$f(m, s)$ be the smallest piece in an optimal $(m, s)$-procedure.

We have shown $f(5, 3) = \frac{5}{12}$.

Note: $f(m, s) \geq \frac{1}{s}$: divide each muffin into $s$ pieces of size $\frac{1}{s}$ and give each student $m$ of them.
$f(3, 5) \geq \text{?}$

Clearly $f(3, 5) \geq \frac{1}{5}$. Can we get $f(3, 5) > \frac{1}{5}$?
Think about it at your desk.
Clearly $f(3, 5) \geq \frac{1}{5}$. Can we get $f(3, 5) > \frac{1}{5}$?

Think about it at your desk.

$f(3, 5) \geq \frac{1}{4}$

1. Divide 2 muffin $[\frac{6}{20}, \frac{7}{20}, \frac{7}{20}]$
2. Divide 1 muffin $[\frac{5}{20}, \frac{5}{20}, \frac{5}{20}, \frac{5}{20}]$
3. Give 4 students $(\frac{5}{20}, \frac{7}{20})$
4. Give 1 students $(\frac{6}{20}, \frac{6}{20})$
$f(3, 5) \geq ?$

Clearly $f(3, 5) \geq \frac{1}{5}$. Can we get $f(3, 5) > \frac{1}{5}$?

Think about it at your desk.

$f(3, 5) \geq \frac{1}{4}$

1. Divide 2 muffin $[\frac{6}{20}, \frac{7}{20}, \frac{7}{20}]$
2. Divide 1 muffin $[\frac{5}{20}, \frac{5}{20}, \frac{5}{20}, \frac{5}{20}]$
3. Give 4 students $[\frac{5}{20}, \frac{7}{20} ]$
4. Give 1 students $[\frac{6}{20}, \frac{6}{20} ]$

Can we do better? Vote!

YES
NO
UNKNOWN TO SCIENCE
Clearly \( f(3, 5) \geq \frac{1}{5} \). Can we get \( f(3, 5) > \frac{1}{5} \)?

Think about it at your desk.

\[
f(3, 5) \geq \frac{1}{4}
\]

1. Divide 2 muffin \( \left[ \frac{6}{20}, \frac{7}{20}, \frac{7}{20} \right] \)
2. Divide 1 muffin \( \left[ \frac{5}{20}, \frac{5}{20}, \frac{5}{20}, \frac{5}{20} \right] \)
3. Give 4 students \( \left( \frac{5}{20}, \frac{7}{20} \right) \)
4. Give 1 students \( \left( \frac{6}{20}, \frac{6}{20} \right) \)

Can we do better? Vote!

**YES**

**NO**

**UNKNOWN TO SCIENCE**

**NO** Proof on next slide.
\[ f(3, 5) \leq \frac{1}{4} \]

There is a procedure for 3 muffins, 5 students where each student gets \( \frac{3}{5} \) muffins, smallest piece \( N \). We want \( N \leq \frac{1}{4} \).

**Case 0:** Some student gets 1 piece, so size \( \frac{3}{5} \). Cut that piece in half and give both \( \frac{3}{10} \)-sized pieces to that student. (Note \( \frac{3}{10} > \frac{1}{4} \).) Reduces to other cases. (*Henceforth:* All students get \( \geq 2 \) pieces.)

**Case 1:** Some student gets \( \geq 3 \) pieces. Then \( N \leq \frac{3}{5} \times \frac{3}{3} = \frac{1}{5} < \frac{1}{4} \). (*Henceforth:* All students get 2 pieces.)

**Case 2:** All students get 2 pieces. 5 students, so 10 pieces. **Some muffin** gets cut into \( \geq 4 \) pieces. Some piece \( \leq \frac{1}{4} \).
3 People, 5 Muffins VS 5 People, 3 Muffins

\[ f(5, 3) \geq \frac{5}{12} \]

1. Divide 4 muffins \([\frac{5}{12}, \frac{7}{12}]\)
2. Divide 1 muffin \([\frac{6}{12}, \frac{6}{12}]\)
3. Give 2 students \((\frac{6}{12}, \frac{7}{12}, \frac{7}{12})\)
4. Give 1 students \((\frac{5}{12}, \frac{5}{12}, \frac{5}{12}, \frac{5}{12})\)
3 People, 5 Muffins VS 5 People, 3 Muffins

\[ f(5, 3) \geq \frac{5}{12} \]

1. Divide 4 muffins \([\frac{5}{12}, \frac{7}{12}]\)
2. Divide 1 muffin \([\frac{6}{12}, \frac{6}{12}]\)
3. Give 2 students \((\frac{6}{12}, \frac{7}{12}, \frac{7}{12})\)
4. Give 1 student \((\frac{5}{12}, \frac{5}{12}, \frac{5}{12}, \frac{5}{12})\)

\[ f(3, 5) \geq \frac{1}{4} \]

1. Divide 2 muffin \([\frac{6}{20}, \frac{7}{20}, \frac{7}{20}]\)
2. Divide 1 muffin \([\frac{5}{20}, \frac{5}{20}, \frac{5}{20}, \frac{5}{20}]\)
3. Give 4 students \((\frac{5}{20}, \frac{7}{20})\)
4. Give 1 students \((\frac{6}{20}, \frac{6}{20})\)
3 People, 5 Muffins VS 5 People, 3 Muffins

\[ f(5, 3) \geq \frac{5}{12} \]

1. Divide 4 muffins \([\frac{5}{12}, \frac{7}{12}]\)
2. Divide 1 muffin \([\frac{6}{12}, \frac{6}{12}]\)
3. Give 2 students \((\frac{6}{12}, \frac{7}{12}, \frac{7}{12})\)
4. Give 1 student \((\frac{5}{12}, \frac{5}{12}, \frac{5}{12}, \frac{5}{12})\)

\[ f(3, 5) \geq \frac{1}{4} \]

1. Divide 2 muffins \([\frac{6}{20}, \frac{7}{20}, \frac{7}{20}]\)
2. Divide 1 muffin \([\frac{5}{20}, \frac{5}{20}, \frac{5}{20}, \frac{5}{20}]\)
3. Give 4 students \((\frac{5}{20}, \frac{7}{20})\)
4. Give 1 student \((\frac{6}{20}, \frac{6}{20})\)

\[ f(3, 5) \text{ proc is } f(5, 3) \text{ proc but swap Divide/Give and mult by } 3/5. \]
3 People, 5 Muffins VS 5 People, 3 Muffins

\[ f(5, 3) \geq \frac{5}{12} \]

1. Divide 4 muffins \([\frac{5}{12}, \frac{7}{12}]\)
2. Divide 1 muffin \([\frac{6}{12}, \frac{6}{12}]\)
3. Give 2 students \((\frac{6}{12}, \frac{7}{12}, \frac{7}{12})\)
4. Give 1 students \((\frac{5}{12}, \frac{5}{12}, \frac{5}{12}, \frac{5}{12})\)

\[ f(3, 5) \geq \frac{1}{4} \]

1. Divide 2 muffin \([\frac{6}{20}, \frac{7}{20}, \frac{7}{20}]\)
2. Divide 1 muffin \([\frac{5}{20}, \frac{5}{20}, \frac{5}{20}, \frac{5}{20}]\)
3. Give 4 students \((\frac{5}{20}, \frac{7}{20})\)
4. Give 1 students \((\frac{6}{20}, \frac{6}{20})\)

\(f(3, 5)\) proc is \(f(5, 3)\) proc but swap Divide/Give and mult by 3/5.

**Theorem:** \(f(m, s) = \frac{m}{s} f(s, m)\).
Floor-Ceiling Theorem (Generalize $f(5, 3) \leq \frac{5}{12}$)

$$f(m, s) \leq \max\left\{ \frac{1}{3}, \min\left\{ \frac{m}{s \lceil 2m/s \rceil}, 1 - \frac{m}{s \lfloor 2m/s \rfloor} \right\} \right\}.$$ 

**Case 0:** Some muffin is uncut. Cut it ($\frac{1}{2}, \frac{1}{2}$) and give both halves to whoever got the uncut muffin, so reduces to other cases.

**Case 1:** Some muffin is cut into $\geq 3$ pieces. Some piece $\leq \frac{1}{3}$.

**Case 2:** Every muffin is cut into 2 pieces, so $2m$ pieces.

**Someone** gets $\geq \left\lceil \frac{2m}{s} \right\rceil$ pieces. $\exists$ piece $\leq \frac{m}{s} \times \frac{1}{\lceil 2m/s \rceil} = \frac{m}{s \lceil 2m/s \rceil}$.

**Someone** gets $\leq \left\lfloor \frac{2m}{s} \right\rfloor$ pieces. $\exists$ piece $\geq \frac{m}{s} \frac{1}{\lfloor 2m/s \rfloor} = \frac{m}{s \lfloor 2m/s \rfloor}$.

The other piece from that muffin is of size $\leq 1 - \frac{m}{s \lfloor 2m/s \rfloor}$.
CLEVERNESS, COMP PROGS for the procedure.

Floor-Ceiling Theorem for optimality.

\[ f(1, 3) = \frac{1}{3} \]

\[ f(3k, 3) = 1. \]

\[ f(3k + 1, 3) = \frac{3k-1}{6k}, \quad k \geq 1. \]

\[ f(3k + 2, 3) = \frac{3k+2}{6k+6}. \]
FOUR Students

CLEVERNESS, COMP PROGS for procedures.

Floor-Ceiling Theorem for optimality.

\[ f(4k, 4) = 1 \text{ (easy)} \]

\[ f(1, 4) = \frac{1}{4} \text{ (easy)} \]

\[ f(4k + 1, 4) = \frac{4k-1}{8k}, \quad k \geq 1. \]

\[ f(4k + 2, 4) = \frac{1}{2}. \]

\[ f(4k + 3, 4) = \frac{4k+1}{8k+4}. \]

Is FIVE student case a Mod 5 pattern?

VOTE YES or NO
FOUR Students

CLEVERNESS, COMP PROGS for procedures.

Floor-Ceiling Theorem for optimality.

\[
f(4k, 4) = 1 \text{ (easy)}
\]

\[
f(1, 4) = \frac{1}{4} \text{ (easy)}
\]

\[
f(4k + 1, 4) = \frac{4k-1}{8k}, \quad k \geq 1.
\]

\[
f(4k + 2, 4) = \frac{1}{2}.
\]

\[
f(4k + 3, 4) = \frac{4k+1}{8k+4}.
\]

Is FIVE student case a Mod 5 pattern?

VOTE YES or NO

YES but with some exceptions
FIVE Students, \( m = 1, \ldots, 11 \)

\[
f(1, 5) = \frac{1}{5} \text{ (easy or use } f(1, 5) = \frac{5}{1} f(5, 1).)\]

\[
f(2, 5) = \frac{1}{5} \text{ (easy or use } f(2, 5) = \frac{5}{2} f(5, 2).)\]

\[
f(3, 5) = \frac{1}{4} \text{ (use } f(3, 5) = \frac{3}{5} f(5, 3).)\]

\[
f(4, 5) = \frac{3}{10} \text{ (use } f(4, 5) = \frac{4}{5} f(5, 4).)\]

\[
f(5, 5) = 1 \text{ (Easy and fits pattern)}\]

\[
f(6, 5) = \frac{2}{5} \text{ (Use Floor-Ceiling Thm, fits pattern)}\]

\[
f(7, 5) = \frac{1}{3} \text{ (Use Floor-Ceiling Thm, NOT pattern)}\]

\[
f(8, 5) = \frac{2}{5} \text{ (Use Floor-Ceiling Thm, fits pattern)}\]

\[
f(9, 5) = \frac{2}{5} \text{ (Use Floor-Ceiling Thm, fits pattern )}\]

\[
f(10, 5) = 1 \text{ (Easy and fits pattern)}\]

\[
f(11, 5) = \text{(Will come back to this later)}\]
FIVE Students

**CLEVERNESS, COMP PROGS** for procedures.

**Floor-Ceiling Theorem** for optimality.

For $k \geq 1$, $f(5k, 5) = 1$.

For $k = 1$ and $k \geq 3$, $f(5k + 1, 5) = \frac{5k+1}{10k+5}$

For $k \geq 2$, $f(5k + 2, 5) = \frac{5k-2}{10k}$

For $k \geq 1$, $f(5k + 3, 5) = \frac{5k+3}{10k+10}$

For $k \geq 1$, $f(5k + 4, 5) = \frac{5k+1}{10k+5}$
What About FIVE students, ELEVEN muffins?

Procedure:

Divide the Muffins in to Pieces:

1. Divide 6 muffins into \( \left( \frac{13}{30}, \frac{17}{30} \right) \).
2. Divide 4 muffins into \( \left( \frac{9}{20}, \frac{11}{20} \right) \).
3. Divide 1 muffin into \( \left( \frac{1}{2}, \frac{1}{2} \right) \).

Distribute the Shares to Students:

1. Give 2 students \( \left[ \frac{17}{30}, \frac{17}{30}, \frac{17}{30}, \frac{1}{2} \right] \).
2. Give 2 students \( \left[ \frac{13}{30}, \frac{13}{30}, \frac{13}{30}, \frac{9}{20}, \frac{9}{20} \right] \).
3. Give 1 student \( \left[ \frac{11}{20}, \frac{11}{20}, \frac{11}{20}, \frac{11}{20} \right] \).

So

\[
f(11, 5) \geq \frac{13}{30} \sim 0.43333.
\]
What About FIVE students, ELEVEN muffins? Opt

Recall: **Floor-Ceiling Theorem:**

\[
f(m, s) \leq \max \left\{ \frac{1}{3}, \min \left\{ \frac{m}{s \lceil 2m/s \rceil}, 1 - \frac{m}{s \lfloor 2m/s \rfloor} \right\} \right\}.
\]

\[
f(11, 5) \leq \max \left\{ \frac{1}{3}, \min \left\{ \frac{11}{5 \lceil 22/5 \rceil}, 1 - \frac{11}{5 \lfloor 22/5 \rfloor} \right\} \right\}.
\]

\[
f(11, 5) \leq \max \left\{ \frac{1}{3}, \min \left\{ \frac{11}{5 \times 5}, 1 - \frac{11}{5 \times 4} \right\} \right\}.
\]

\[
f(11, 5) \leq \max \left\{ \frac{1}{3}, \min \left\{ \frac{11}{25}, \frac{9}{20} \right\} \right\}.
\]

\[
f(11, 5) \leq \max \left\{ \frac{1}{3}, \frac{11}{25} \right\} = \frac{11}{25} = 0.44.
\]
Where Are We On FIVE students, ELEVEN muffins?

- By **Procedure** $\frac{13}{30} \sim 0.43333 \leq f(11, 5)$
- By **Floor-Ceiling** $f(11, 5) \leq \frac{11}{25} \sim .44$

So

$$\frac{13}{30} \leq f(11, 5) \leq \frac{11}{25} \quad \text{Diff} = 0.0066666\ldots$$
Where Are We On FIVE students, ELEVEN muffins?

- By **Procedure** \( \frac{13}{30} \sim 0.43333 \leq f(11, 5) \)
- By **Floor-Ceiling** \( f(11, 5) \leq \frac{11}{25} \sim .44 \)

So

\[
\frac{13}{30} \leq f(11, 5) \leq \frac{11}{25} \quad \text{Diff} = 0.0066666\ldots
\]

**Darling:** 0.0066666 close enough?
Where Are We On FIVE students, ELEVEN muffins?

- By **Procedure** $\frac{13}{30} \sim 0.43333 \leq f(11, 5)$
- By **Floor-Ceiling** $f(11, 5) \leq \frac{11}{25} \sim .44$

So

$$\frac{13}{30} \leq f(11, 5) \leq \frac{11}{25} \quad \text{Diff} = 0.006666\ldots$$

**Darling:** 0.0066666 close enough?

**VOTE:**

1. $f(11, 5) = \frac{13}{30}$: Needs NEW technique to show limits on procedures.
2. $f(11, 5) = \frac{11}{25}$: Needs NEW better procedure.
3. $f(11, 5) = \alpha$ where $\frac{13}{30} < \alpha < \frac{11}{25}$. Needs both:
4. **UNKNOWN TO SCIENCE!**
Where Are We On FIVE students, ELEVEN muffins?

- By **Procedure** \(\frac{13}{30} \sim 0.43333 \leq f(11, 5)\)
- By **Floor-Ceiling** \(f(11, 5) \leq \frac{11}{25} \sim .44\)

So
\[
\frac{13}{30} \leq f(11, 5) \leq \frac{11}{25} \quad \text{Diff} = 0.0066666\ldots
\]

**Darling:** 0.0066666 close enough?

**VOTE:**

1. \(f(11, 5) = \frac{13}{30}\): Needs NEW technique to show limits on procedures.
2. \(f(11, 5) = \frac{11}{25}\): Needs NEW better procedure.
3. \(f(11, 5) = \alpha\) where \(\frac{13}{30} < \alpha < \frac{11}{25}\). Needs both:
4. **UNKNOWN TO SCIENCE!**

**KNOWN:** \(f(11, 5) = \frac{13}{30}\)

**HAPPY:** New opt tech more interesting than new proc.
There is a procedure for 11 muffins, 5 students where each student gets \( \frac{11}{5} \) muffins, smallest piece \( N \). We want \( N \leq \frac{13}{30} \).

**Case 0:** Some muffin is uncut. Cut it \( \left( \frac{1}{2}, \frac{1}{2} \right) \) and give both halves to whoever got the uncut muffin. Reduces to other cases.

**Case 1:** Some muffin is cut into \( \geq 3 \) pieces. \( N \leq \frac{1}{3} < \frac{13}{30} \).

(**Negation of Case 0 and Case 1:** All muffins cut into 2 pieces.)
$f(11, 5) = \frac{13}{30}$, Easy Case Based on Students

**Case 2:** Some student gets $\geq 6$ pieces.

\[ N \leq \frac{11}{5} \times \frac{1}{6} = \frac{11}{30} < \frac{13}{30}. \]

**Case 3:** Some student gets $\leq 3$ pieces.

One of the pieces is

\[ \geq \frac{11}{5} \times \frac{1}{3} = \frac{11}{15}. \]

Look at the muffin it came from to find a piece that is

\[ \leq 1 - \frac{11}{15} = \frac{4}{15} < \frac{13}{30}. \]

(Negation of Cases 2 and 3: Every student gets 4 or 5 pieces.)
Case 4: Every muffin is cut in 2 pieces, every student gets 4 or 5 pieces. Number of pieces: 22. Note \( \leq 11 \) pieces are \( > \frac{1}{2} \).

▶ \( s_4 \) is number of students who get 4 pieces
▶ \( s_5 \) is number of students who get 5 pieces

\[
4s_4 + 5s_5 = 22 \\
\text{ } \\
\quad s_4 + s_5 = 5
\]

\( s_4 = 3 \): There are 3 students who have 4 pieces.
\( s_5 = 2 \): There are 2 students who have 5 pieces.
\( f(11, 5) = \frac{13}{30}, \) Fun Cases

\[ \begin{array}{ccccccc}
\Diamond & \Diamond & \Diamond & \Diamond & \Diamond & \Diamond & \Diamond \\
\Diamond & \Diamond & \Diamond & \Diamond & \Diamond & \Diamond & \Diamond \\
\end{array} \]  \hspace{1cm} (Sums to \( 11/5 \))

\[ \begin{array}{ccccccc}
\bigcirc & \bigcirc & \bigcirc & \bigcirc & \bigcirc & \bigcirc & \bigcirc \\
\bigcirc & \bigcirc & \bigcirc & \bigcirc & \bigcirc & \bigcirc & \bigcirc \\
\end{array} \]  \hspace{1cm} (Sums to \( 11/5 \))

\[ \begin{array}{ccccccc}
\bigcirc & \bigcirc & \bigcirc & \bigcirc & \bigcirc & \bigcirc & \bigcirc \\
\bigcirc & \bigcirc & \bigcirc & \bigcirc & \bigcirc & \bigcirc & \bigcirc \\
\end{array} \]  \hspace{1cm} (Sums to \( 11/5 \))

\[ \begin{array}{ccccccc}
\bigcirc & \bigcirc & \bigcirc & \bigcirc & \bigcirc & \bigcirc & \bigcirc \\
\bigcirc & \bigcirc & \bigcirc & \bigcirc & \bigcirc & \bigcirc & \bigcirc \\
\end{array} \]  \hspace{1cm} (Sums to \( 11/5 \))

**Case 4.1:** One of (say)

\[ \begin{array}{ccccccc}
\bigcirc & \bigcirc & \bigcirc & \bigcirc & \bigcirc & \bigcirc & \bigcirc \\
\bigcirc & \bigcirc & \bigcirc & \bigcirc & \bigcirc & \bigcirc & \bigcirc \\
\end{array} \]  \hspace{1cm} (Sums to \( 11/5 \))

is \( \leq \frac{1}{2} \). Then there is a piece

\[ \geq \frac{(11/5) - (1/2)}{3} = \frac{17}{30}. \]

The other piece from the muffin is

\[ \leq 1 - \frac{17}{30} = \frac{13}{30} \]  \hspace{1cm} Great to see \( \frac{13}{30} \).
$f(11, 5) = \frac{13}{30}$, Fun Cases

Case 4.2: All

○ ○ ○ ○ ○ (Sums to 11/5)
○ ○ ○ ○ ○ (Sums to 11/5)
○ ○ ○ ○ ○ (Sums to 11/5)

are $> \frac{1}{2}$.
There are $\geq 12$ pieces $> \frac{1}{2}$. Can't occur.
The Techniques Generalizes!

Good News!
The technique used to get $f(11, 5) \leq \frac{13}{30}$ lead to a theorem that apply to other cases! We call it The Interval Theorem

Bad News!
Interval Theorem is hard to state, so you don’t get to see it.

Good News!
Interval Theorem is hard to state, so you don’t have to see it.
Notation

$FC(m, s)$ is the upper bound provided by Floor-Ceiling Thm.

$IN(m, s)$ is the upper bound provided by Interval Thm.

$SP(s + 1, s) = f(s + 1, s)$. We have a theorem that tells us this exactly.
How Good Is the FC Bound? Mod Pattern?

1. For all $s$ for all $m \geq \frac{s^3 + 2s^2 + s}{2}$, $f(m, s) = FC(m, s)$. (Empirical evidence $O(s^2)$).

2. For all $s$ there is a mod-$s$-formula $FORM(m, s)$ such that for all $m \geq \frac{s^2 + s}{4}$, $f(m, s) = FORM(m, s)$.

3. Hence: For all $s$ there is a mod-$s$-formula $FORM(m, s)$ such that for all $m \geq \frac{s^3 + 2s^2 + s}{2}$, $f(m, s) = FORM(m, s)$.

4. For $1 \leq s \leq 6$ we have the $FORM(m, s)$.

5. For $7 \leq s \leq 60$ have conjectures for $FORM(m, s)$ that are surely true.
The Exceptions

For all $s$ there is a mod-$s$-formula $FORM(m, s)$ such that for all $m \geq \frac{s^2 + s}{4}$, $f(m, s) = FORM(m, s)$.

What happens when $FORM(m, s) \neq f(m, s)$.

1. $f(s + 1, s)$. Have Sep theorem for that case, known exactly.
2. $f(m, s) = \frac{1}{3}$.
3. $f(m, s)$ used Interval Theorem.

So far these are the only exceptions.
Does $f(m, s)$ Exist? Rational? Debatable?

**Plausible:**

1. There is a protocol showing $f(m, s) \geq \frac{1}{5}$
2. There is a protocol showing $f(m, s) \geq \frac{1}{5} + \frac{1}{5^2}$
3. There is a protocol showing $f(m, s) \geq \frac{1}{5} + \frac{1}{5^2} + \frac{1}{5^3}$

4. 

But NO protocol shows $f(m, s) \geq \frac{1}{5} + \frac{1}{5^2} + \cdots = \frac{1}{4}$. 
Plausible:

1. There is a protocol showing \( f(m, s) \geq \frac{1}{5} \)
2. There is a protocol showing \( f(m, s) \geq \frac{1}{5} + \frac{1}{5^2} \)
3. There is a protocol showing \( f(m, s) \geq \frac{1}{5} + \frac{1}{5^2} + \frac{1}{5^3} \)
4. ...

But NO protocol shows \( f(m, s) \geq \frac{1}{5} + \frac{1}{5^2} + \cdots = \frac{1}{4} \).
Does $f(m,s)$ Exist? Rational? Debatable?

**Plausible:**

1. There is a protocol showing $f(m,s) \geq \frac{1}{5}$
2. There is a protocol showing $f(m,s) \geq \frac{1}{5} + \frac{1}{5^2}$
3. There is a protocol showing $f(m,s) \geq \frac{1}{5} + \frac{1}{5^2} + \frac{1}{5^3}$
4. ...

But NO protocol shows $f(m,s) \geq \frac{1}{5} + \frac{1}{5^2} + \cdots = \frac{1}{4}$.

**Plausible:** $f(m,s) = \frac{1}{\pi}$ (so $\pi$ is key to muffins!)
Plausible:

1. There is a protocol showing \( f(m, s) \geq \frac{1}{5} \)
2. There is a protocol showing \( f(m, s) \geq \frac{1}{5} + \frac{1}{5^2} \)
3. There is a protocol showing \( f(m, s) \geq \frac{1}{5} + \frac{1}{5^2} + \frac{1}{5^3} \)
4. :

But NO protocol shows \( f(m, s) \geq \frac{1}{5} + \frac{1}{5^2} + \cdots = \frac{1}{4} \).

Plausible: \( f(m, s) = \frac{1}{\pi} \) (so \( \pi \) is key to muffins!)

Plausible: \( f(m, s) \) is not computable.
**Theorem**

1. *There is a mixed integer program with \( O(ms) \) binary variables, \( O(ms) \) real variables, \( O(ms) \) constraints, and all coefficients integers of absolute value \( \leq \max\{m, s\} \) such that, from the solution, one can extract \( f(m, s) \) and a protocol that achieves this bound. This MIP can easily be obtained given \( m, s \).*

2. *\( f(m, s) \) is always rational. This follows from part 1.*

3. *The problem of, given \( m, s \), determine \( f(m, s) \), is decidable. This follows from part 1.*
Good News: $f(m, s)$ exists, is rational and computable!
Not Just Theoretical

**Good News:** $f(m,s)$ exists, is rational and computable!

**Bad News:** Proof uses MIP’s which are NP-complete
**Good News:** $f(m, s)$ exists, is rational and computable!

**Bad News:** Proof uses MIP’s which are NP-complete

**Good News:** There are packages for MIP’s that are . . . okay.
Not Just Theoretical

**Good News:** $f(m, s)$ exists, is rational and computable!

**Bad News:** Proof uses MIP’s which are NP-complete

**Good News:** There are packages for MIP’s that are . . . okay.

**Bad News:** There is no more bad news which breaks the symmetry of good/bad/good/bad.

We HAVE coded it up and we HAVE gotten some results this way.
Not Just Theoretical

**Good News:** $f(m, s)$ exists, is rational and computable!

**Bad News:** Proof uses MIP’s which are NP-complete

**Good News:** There are packages for MIP’s that are . . . okay.

**Bad News:** There is no more bad news which breaks the symmetry of good/bad/good/bad.

**Good News:** We HAVE coded it up and we HAVE gotten some results this way.
The Synergy Between Fields

One often hears:
Pure Math done without an application in mind often ends up being Applied!
(Number theory and Cryptography is a great example.)
One often hears:

**Pure Math done without an application in mind often ends up being Applied!**
(Number theory and Cryptography is a great example.)

One seldom hears (though its true):

**Applied Math done for a real world applications often ends up being used for Pure Math!**
(MIP and Muffins is a ‘great’ example.)
The Synergy Between Fields

One often hears:

**Pure Math done without an application in mind often ends up being Applied!**
(Number theory and Cryptography is a great example.)

One seldom hears (though its true):

**Applied Math done for a real world applications often ends up being used for Pure Math!**
(MIP and Muffins is a ‘great’ example.)

**Pure Math, Applied Math, Computer Science, Physics,** all play off each other! None of the four has moral superiority!
How Research Works

1. Obtain particular results.
2. Prove a general theorem based on those results.
3. Run into a case we cannot solve (e.g., (11,5) and (35,13)).
4. Lather, Rinse, Repeat.
Conjectures

**Conjecture:** The following program computes $f(m, s)$ for $m > s$.

- If $d = \gcd(m, s) \neq 1$ then call $f(m/d, s/d)$.
- If $m = s + 1$ output $SP(s + 1, s)$.
- If $s = 1$ then output 1.
- Otherwise output the MIN of $FC(m, s)$ and $INT(m, s)$

Empirically true for $1 \leq s \leq 15$, $1 \leq m \leq 100$.

**If True:**

1. $f(m, s)$ can be computed with a constant number of arith operations on numbers $\leq O(s + m)$.
2. $f(m, s)$ can be computed in time $O(M(s + m))$, where $M$ is speed of multiplication.
3. $f(m, s)$ is in P.
Accomplishment I Am Most Proud of:

Accomplishment I Am Most Proud of:

I convinced 4 high school students (Guang, Naveen, Naveen, Sunny), 1 college student (Erik), and 1 professor (John D.) that the most important field of Mathematics is Muffinry.
Accomplishment I Am Most Proud of:

**Convinced**

- 4 High School students (Guang, Naveen, Naveen, Sunny)
- 1 college student (Erik)
- 1 professor (John D.)

that the most important field of Mathematics is **Muffinry**.