The Muffin Problem

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How it Began

A Recreational Math Conference
(Gathering for Gardner)
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I found a pamphlet:

The Julia Robinson Mathematics Festival:
A Sample of Mathematical Puzzles
Compiled by Nancy Blachman

which had this problem, proposed by Alan Frank:

How can you divide and distribute 5 muffins to 3 students so that every student gets $\frac{5}{3}$ where nobody gets a tiny sliver?
### Five Muffins, Three Students, Proc by Picture

<table>
<thead>
<tr>
<th>Person</th>
<th>Color</th>
<th>What they Get</th>
</tr>
</thead>
<tbody>
<tr>
<td>Alice</td>
<td>RED</td>
<td>$1 + \frac{2}{3} = \frac{5}{3}$</td>
</tr>
<tr>
<td>Bob</td>
<td>BLUE</td>
<td>$1 + \frac{2}{3} = \frac{5}{3}$</td>
</tr>
<tr>
<td>Carol</td>
<td>GREEN</td>
<td>$1 + \frac{1}{3} + \frac{1}{3} = \frac{5}{3}$</td>
</tr>
</tbody>
</table>

Smallest Piece: $\frac{1}{3}$
Can We Do Better?

The smallest piece in the above solution is $\frac{1}{3}$.

Is there a procedure with a larger smallest piece?
Can We Do Better?

The smallest piece in the above solution is $\frac{1}{3}$.

*Is there a procedure with a larger smallest piece?*

*YES WE CAN!*
## Five Muffins, Three People–Proc by Picture

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**Smallest Piece:** $\frac{5}{12}$
Can We Do Better?

The smallest piece in the above solution is $\frac{5}{12}$.

*Is there a procedure with a larger smallest piece?*
Can We Do Better?

The smallest piece in the above solution is \(\frac{5}{12}\).

Is there a procedure with a larger smallest piece? NO WE CAN’T!
There is a procedure for 5 muffins, 3 students where each student gets $\frac{5}{3}$ muffins, smallest piece $N$. We want $N \leq \frac{5}{12}$.

**Case 0:** Some muffin is uncut. Cut it $(\frac{1}{2}, \frac{1}{2})$ and give both $\frac{1}{2}$-sized pieces to whoever got the uncut muffin. (Note $\frac{1}{2} > \frac{5}{12}$.) Reduces to other cases. 

(Henceforth: All muffins are cut into $\geq 2$ pieces.)

**Case 1:** Some muffin is cut into $\geq 3$ pieces. Then $N \leq \frac{1}{3} < \frac{5}{12}$. 

(Henceforth: All muffins are cut into 2 pieces.)

**Case 2:** All muffins are cut into 2 pieces. 10 pieces, 3 students: Someone gets $\geq 4$ pieces. He has some piece 

$$\leq \frac{5}{3} \times \frac{1}{4} = \frac{5}{12}$$

Great to see $\frac{5}{12}$
General Problem

\( f(m, s) \) be the smallest piece in the best procedure (best in that the smallest piece is maximized) to divide \( m \) muffins among \( s \) students so that everyone gets \( \frac{m}{s} \).

We have shown \( f(5, 3) = \frac{5}{12} \) here.

We have shown \( f(m, s) \) exists, is rational, and is computable using a Mixed Int Program (in paper).
1. \( f(43, 33) = \frac{91}{264} \).
2. \( f(52, 11) = \frac{83}{176} \).
3. \( f(35, 13) = \frac{64}{143} \).

All done by hand, no use of a computer.
by Co-author Erik Metz is a muffin savant!

Have General Theorems from which upper bounds follow.
Have General Procedures from which lower bounds follow.
Clearly $f(3, 5) \geq \frac{1}{5}$. Can we get $f(3, 5) > \frac{1}{5}$?
Clearly $f(3, 5) \geq \frac{1}{5}$. Can we get $f(3, 5) > \frac{1}{5}$?

$f(3, 5) \geq \frac{1}{4}$

1. Divide 2 muffin $[\frac{6}{20}, \frac{7}{20}, \frac{7}{20}]$
2. Divide 1 muffin $[\frac{5}{20}, \frac{5}{20}, \frac{5}{20}, \frac{5}{20}]$
3. Give 4 students $(\frac{5}{20}, \frac{7}{20})$
4. Give 1 students $(\frac{6}{20}, \frac{6}{20})$
$f(3, 5) \geq \frac{1}{5}$. Can we get $f(3, 5) > \frac{1}{5}$?

$f(3, 5) \geq \frac{1}{4}$

1. Divide 2 muffin $[\frac{6}{20}, \frac{7}{20}, \frac{7}{20}]
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3. Give 4 students $(\frac{5}{20}, \frac{7}{20})$
4. Give 1 student $(\frac{6}{20}, \frac{6}{20})$

Can we do better?
$f(3, 5) \geq?$

Clearly $f(3, 5) \geq \frac{1}{5}$. Can we get $f(3, 5) > \frac{1}{5}$?

$f(3, 5) \geq \frac{1}{4}$

1. Divide 2 muffin $[\frac{6}{20}, \frac{7}{20}, \frac{7}{20}]$
2. Divide 1 muffin $[\frac{5}{20}, \frac{5}{20}, \frac{5}{20}, \frac{5}{20}]$
3. Give 4 students $(\frac{5}{20}, \frac{7}{20})$
4. Give 1 students $(\frac{6}{20}, \frac{6}{20})$

Can we do better?

NO
3 People, 5 Muffins VS 5 People, 3 Muffins

\[ f(5, 3) \geq \frac{5}{12} \]

1. Divide 4 muffins \([\frac{5}{12}, \frac{7}{12}]\)
2. Divide 1 muffin \([\frac{6}{12}, \frac{6}{12}]\)
3. Give 2 students \((\frac{6}{12}, \frac{7}{12}, \frac{7}{12})\)
4. Give 1 students \((\frac{5}{12}, \frac{5}{12}, \frac{5}{12}, \frac{5}{12})\)
3 People, 5 Muffins VS 5 People, 3 Muffins

\[ f(5, 3) \geq \frac{5}{12} \]

1. Divide 4 muffins \([\frac{5}{12}, \frac{7}{12}]\)
2. Divide 1 muffin \([\frac{6}{12}, \frac{6}{12}]\)
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4. Give 1 students \((\frac{5}{12}, \frac{5}{12}, \frac{5}{12}, \frac{5}{12})\)

\[ f(3, 5) \geq \frac{1}{4} \]

1. Divide 2 muffin \([\frac{6}{20}, \frac{7}{20}, \frac{7}{20}]\)
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3 People, 5 Muffins VS 5 People, 3 Muffins

\[ f(5, 3) \geq \frac{5}{12} \]

1. Divide 4 muffins \([\frac{5}{12}, \frac{7}{12}]\)
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3. Give 2 students \((\frac{6}{12}, \frac{7}{12}, \frac{7}{12})\)
4. Give 1 students \((\frac{5}{12}, \frac{5}{12}, \frac{5}{12}, \frac{5}{12})\)

\[ f(3, 5) \geq \frac{1}{4} \]

1. Divide 2 muffin \([\frac{6}{20}, \frac{7}{20}, \frac{7}{20}]\)
2. Divide 1 muffin \([\frac{5}{20}, \frac{5}{20}, \frac{5}{20}, \frac{5}{20}]\)
3. Give 4 students \((\frac{5}{20}, \frac{7}{20})\)
4. Give 1 students \((\frac{6}{20}, \frac{6}{20})\)

\[ f(3, 5) \text{ proc is } f(5, 3) \text{ proc but swap Divide/Give and mult by } 3/5. \]
3 People, 5 Muffins VS 5 People, 3 Muffins

\[ f(5, 3) \geq \frac{5}{12} \]

1. Divide 4 muffins \( \left[ \frac{5}{12}, \frac{7}{12} \right] \)
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4. Give 1 students \( \left( \frac{5}{12}, \frac{5}{12}, \frac{5}{12}, \frac{5}{12} \right) \)

\[ f(3, 5) \geq \frac{1}{4} \]

1. Divide 2 muffin \( \left[ \frac{6}{20}, \frac{7}{20}, \frac{7}{20} \right] \)
2. Divide 1 muffin \( \left[ \frac{5}{20}, \frac{5}{20}, \frac{5}{20}, \frac{5}{20} \right] \)
3. Give 4 students \( \left( \frac{5}{20}, \frac{7}{20} \right) \)
4. Give 1 students \( \left( \frac{6}{20}, \frac{6}{20} \right) \)

\( f(3, 5) \) proc is \( f(5, 3) \) proc but swap Divide/Give and mult by 3/5.

**Theorem:** \( f(m, s) = \frac{m}{s} f(s, m) \).
Floor-Ceiling Thm (FC Thm) Generalizes $f(5, 3) \leq \frac{5}{12}$

$$f(m, s) \leq FC(m, s) = \max\left\{\frac{1}{3}, \min\left\{\frac{m}{s \lceil 2m/s \rceil}, 1 - \frac{m}{s \lceil 2m/s \rceil}\right\}\right\}.$$ 

**Case 0:** Some muffin is uncut. Cut it $(\frac{1}{2}, \frac{1}{2})$ and give both halves to whoever got the uncut muffin, so reduces to other cases.

**Case 1:** Some muffin is cut into $\geq 3$ pieces. Some piece $\leq \frac{1}{3}$.

**Case 2:** Every muffin is cut into 2 pieces, so $2m$ pieces.

**Someone** gets $\geq \left\lceil \frac{2m}{s} \right\rceil$ pieces. $\exists$ piece $\leq \frac{m}{s \lceil 2m/s \rceil} = \frac{m}{s \lceil 2m/s \rceil}$.

**Someone** gets $\leq \left\lfloor \frac{2m}{s} \right\rfloor$ pieces. $\exists$ piece $\geq \frac{m}{s \lfloor 2m/s \rfloor} = \frac{m}{s \lfloor 2m/s \rfloor}$.

The other piece from that muffin is of size $\leq 1 - \frac{m}{s \lceil 2m/s \rceil}$. 

THREE Students

**CLEVERNESS, COMP PROGS** for the procedure.

**FC Theorem** for optimality.

\[ f(1, 3) = \frac{1}{3} \]

\[ f(3k, 3) = 1. \]

\[ f(3k + 1, 3) = \frac{3k-1}{6k}, \quad k \geq 1. \]

\[ f(3k + 2, 3) = \frac{3k+2}{6k+6}. \]

**Note:** A Mod 3 Pattern.

**Theorem:** For all \( m \geq 3, \ f(m, 3) = FC(m, 3). \)
FOUR Students

CLEVERNESS, COMP PROGS for procedures.

FC Theorem for optimality.

\[ f(4k, 4) = 1 \text{ (easy)} \]
\[ f(1, 4) = \frac{1}{4} \text{ (easy)} \]
\[ f(4k + 1, 4) = \frac{4k-1}{8k}, \quad k \geq 1. \]
\[ f(4k + 2, 4) = \frac{1}{2}. \]
\[ f(4k + 3, 4) = \frac{4k+1}{8k+4}. \]

Note: A Mod 4 Pattern.

Theorem: For all \( m \geq 4 \), \( f(m, 4) = FC(m, 4) \).

FC-Conjecture: For all \( m, s \) with \( m \geq s \), \( f(m, s) = FC(m, s) \).
CLEVERNESS, COMP PROGS for procedures.

FC Theorem for optimality.

For $k \geq 1$, $f(5k, 5) = 1$.

For $k = 1$ and $k \geq 3$, $f(5k + 1, 5) = \frac{5k+1}{10k+5}$. $f(11, 5)$?

For $k \geq 2$, $f(5k + 2, 5) = \frac{5k-2}{10k}$. $f(7, 5) = FC(7, 5) = \frac{1}{3}$

For $k \geq 1$, $f(5k + 3, 5) = \frac{5k+3}{10k+10}$

For $k \geq 1$, $f(5k + 4, 5) = \frac{5k+1}{10k+5}$

Note: A Mod 5 Pattern.

Theorem: For all $m \geq 5$ except $m=11$, $f(m, 5) = FC(m, 5)$. 
What About FIVE students, ELEVEN muffins?

1. We have a procedure which shows \( f(11, 5) \geq \frac{13}{30} \).
2. \( f(11, 5) \leq \max\{\frac{1}{3}, \min\{\frac{11}{5\lceil 22/5 \rceil}, 1 - \frac{11}{5\lfloor 22/5 \rfloor}\}\} = \frac{11}{25} \).

So
\[
\frac{13}{30} \leq f(11, 5) \leq \frac{11}{25} \quad \text{Diff= 0.006666...}
\]

If \( f(5, 11) < \frac{11}{25} \) then FC-conjecture is false!
What About FIVE students, ELEVEN muffins?

1. We have a procedure which shows $f(11, 5) \geq \frac{13}{30}$.
2. $f(11, 5) \leq \max\{\frac{1}{3}, \min\{\frac{11}{5\lceil 22/5 \rceil}, 1 - \frac{11}{5\lfloor 22/5 \rfloor}\}\} = \frac{11}{25}$.

So

$$\frac{13}{30} \leq f(11, 5) \leq \frac{11}{25} \quad \text{Diff= 0.006666 \ldots}$$

If $f(5, 11) < \frac{11}{25}$ then FC-conjecture is false!

WE SHOW: $f(11, 5) = \frac{13}{30}$
There is a procedure for 11 muffins, 5 students where each student gets \( \frac{11}{5} \) muffins, smallest piece \( N \). We want \( N \leq \frac{13}{30} \).

**Case 0:** Some muffin is uncut. Cut it \( \left( \frac{1}{2}, \frac{1}{2} \right) \) and give both halves to whoever got the uncut muffin. Reduces to other cases.

**Case 1:** Some muffin is cut into \( \geq 3 \) pieces. \( N \leq \frac{1}{3} < \frac{13}{30} \).

(Negation of Case 0 and Case 1: All muffins cut into 2 pieces.)
$f(11, 5) = \frac{13}{30}$, Easy Case Based on Students

**Case 2:** Some student gets $\geq 6$ pieces.

$$N \leq \frac{11}{5} \times \frac{1}{6} = \frac{11}{30} < \frac{13}{30}.$$

**Case 3:** Some student gets $\leq 3$ pieces.

One of the pieces is

$$\geq \frac{11}{5} \times \frac{1}{3} = \frac{11}{15}.$$

Look at the muffin it came from to find a piece that is

$$\leq 1 - \frac{11}{15} = \frac{4}{15} < \frac{13}{30}.$$

*(Negation of Cases 2 and 3: Every student gets 4 or 5 pieces.)*
$f(11, 5) = \frac{13}{30}$, Fun Cases

**Case 4:** Every muffin is cut in 2 pieces, every student gets 4 or 5 pieces. Number of pieces: 22. Note $\leq 11$ pieces are $> \frac{1}{2}$.

- $s_4$ is number of students who get 4 pieces
- $s_5$ is number of students who get 5 pieces

\[ 4s_4 + 5s_5 = 22 \]
\[ s_4 + s_5 = 5 \]

$s_4 = 3$: There are 3 students who have 4 shares.
$s_5 = 2$: There are 2 students who have 5 shares.

We call a share that goes to a person who gets 4 shares a **4-share**. We call a share that goes to a person who gets 5 shares a **5-share**.
\( f(11, 5) = \frac{13}{30}, \) Fun Cases

**Case 4.1:** is \( \leq \frac{1}{2} \). Then there is a piece

\[
\geq \frac{(11/5) - (1/2)}{3} = \frac{17}{30}.
\]

The other piece from the muffin is

\[
\leq 1 - \frac{17}{30} = \frac{13}{30} \quad \text{Great to see } \frac{13}{30}.
\]

**Case 4.2:** All 4-shares are \( > \frac{1}{2} \). So there are \( 4s_4 = 12 \) 4-shares. There are \( \geq 12 \) pieces \( > \frac{1}{2} \). Can’t occur.
Essence of the Interval Method

1. Every muffin cut into two pieces.
2. Find $L$ such that some students get either $L$ or $L + 1$ pieces.
3. Find how many students get $L (L + 1)$ pieces.
4. Find intervals that these pieces must be in.
5. Find how many pieces are in an interval.
6. Get a contradiction out of this.

Note: Can turn Interval Theorem into a function $INT$ such that $f(m, s) \leq INT(m, s)$. 
FC CONJECTURE STILL SORT OF TRUE

**FC Conj:** For all $m \geq s$, $f(m, s) = FC(m, s)$. FALSE

**Theorem:** For fixed $s$, for $m \geq \frac{s^3 + 2s^2 + s}{2}$, $f(m, s) = FC(m, s)$.

**Statistics:** For $3 \leq s \leq 50$, $s + 1 \leq m \leq 59$:

- $f(m, s) = FC(m, s)$ in 683 cases
- $f(m, s) = INT(m, s)$ in 194 cases

Still 108 cases left. Need new technique!
The Buddy-Match Method! (BM)

Can FC and INT do everything?
No.
They are very good when \( \frac{2m}{s} > 3 \) but NOT so good otherwise.
We do a concrete example of **The Buddy-Match Method**

\[
\begin{align*}
f(43, 39) & \leq \frac{53}{156} \\
\end{align*}
\]

(We have matching lower bound also)

**Definition:** Assume we have a protocol where all students get 2 or 3 shares. If \( x \) is a 2-share then the other share that student has is the shares **match**. Note that \( M(x) = \frac{m}{s} - x \).

**Warning:** We will apply \( M \) to intervals. These intervals have to have only 2-shares in them! But they will!
Theorem \( f(43, 39) \leq \frac{53}{156} \) (\( \geq \) also known).
Assume there is an \((43, 39)\)-procedure with smallest piece \( > \frac{53}{156} \).
Can assume all muffins cut in 2 pieces, all students get \( \geq 2 \) shares.

**Case 1:** A student gets \( \geq 4 \) shares. Some share \( \leq \frac{43}{39 \times 4} < \frac{53}{156} \).

**Case 2:** A student gets \( \leq 1 \) shares. Can’t occur.

**Case 3:** Every muffin is cut in 2 pieces and every student gets either 2 or 3 shares. The total number of shares is 86.
How Many Students get Two Shares? Three Shares?

Let \( s_2 \) (\( s_3 \)) be the number of 2-students (3-students).

\[
2s_2 + 3s_3 = 86
\]
\[
s_2 + s_3 = 39 \text{ Get } s_2 = 31 \text{ and } s_3 = 8
\]

**Case 3.1, 3.2, 3.3, 3.4:**

(\( \exists \)) 3-share \( \geq \frac{66}{156} \). Rm. Now 2-shares \( \geq \frac{43}{39} - \frac{66}{156} = \frac{53}{78} \).

So some share \( \leq \frac{53}{156} \).

By similar reasoning (Case 3.2, 3.3, 3.4) we have:

\[
\begin{pmatrix}
\frac{53}{156} & 24 & 3\text{-shs} & \frac{66}{156} & 0 & \text{shs} & \frac{69}{156} & 62 & 2\text{-shs} & \frac{103}{156}
\end{pmatrix}
\]
The Buddy-Match Method

\[
\left( \begin{array}{c}
24 \\ 3-\text{shs}
\end{array} \right) \ \begin{array}{c}
0 \\ \text{shs}
\end{array} \ \begin{array}{c}
62 \\ 2-\text{shs}
\end{array}
\]

\[
\begin{array}{c}
\frac{53}{156} \\ \frac{66}{156} \\ \frac{69}{156} \\ \frac{103}{156}
\end{array}
\]

\[
\left| \left( \begin{array}{c}
\frac{53}{156} \\ \frac{69}{156}
\end{array} \right) \right| = 24
\]

\[
|B\left( \begin{array}{c}
\frac{53}{156} \\ \frac{69}{156}
\end{array} \right)| = \left| \begin{array}{c}
\frac{87}{156} \\ \frac{103}{156}
\end{array} \right| = 24
\]

\[
|M\left( \begin{array}{c}
\frac{87}{156} \\ \frac{103}{156}
\end{array} \right)| = \left| \begin{array}{c}
\frac{69}{156} \\ \frac{85}{156}
\end{array} \right| = 24
\]

\[
\left| \left( \begin{array}{c}
\frac{53}{156} \\ \frac{69}{156}
\end{array} \right) \cup \left( \begin{array}{c}
\frac{69}{156} \\ \frac{85}{156}
\end{array} \right) \cup \left( \begin{array}{c}
\frac{87}{156} \\ \frac{103}{156}
\end{array} \right) \right| = 24 \times 3 = 72
\]

\[
\left| \left( \begin{array}{c}
\frac{85}{156} \\ \frac{87}{156}
\end{array} \right) \right| = 86 - 72 = 14.
\]
More Buddy-Match Method

\( |(\frac{85}{156}, \frac{87}{156})| = 14 \). Buddy-Match yields \( |(\frac{53}{156}, \frac{55}{156})| = 14 \)

\( |[\frac{66}{156}, \frac{69}{156}]| = 0 \). Buddy-Match yields \( |[\frac{55}{156}, \frac{58}{156}]| = 0 \).

The following picture captures what we know so far about 3-shares.

\[
\begin{pmatrix}
\frac{53}{156} & 14 & \frac{55}{156} \\
0 & \frac{58}{156} & \frac{66}{156}
\end{pmatrix}
\]
Big Shares and Small Shares

\[
\begin{pmatrix}
\frac{53}{156} & 14 & 0 & \frac{58}{156} & 10 \\
\frac{55}{156} & 0 & & \frac{66}{156}
\end{pmatrix}
\]

- Shares in \(\left(\frac{53}{156}, \frac{55}{156}\right)\) are small shares;
- Shares in \(\left(\frac{58}{156}, \frac{66}{156}\right)\) are large shares;

**Notation** \(d_i\) is numb of students who have \(i\) small shares (\(3 - i\) large shares).

\[d_0 = 0 \text{ since } 3 \times \frac{58}{156} = \frac{174}{156} > \frac{172}{156} = \frac{43}{39} .\]

\[d_3 = 0 \text{ since } 3 \times \frac{55}{156} = \frac{165}{156} < \frac{172}{156} = \frac{43}{39} .\]

SO there are NO \(d_0\)-students or \(d_3\)-students.
$d_1$ and $d_2$ Students Cause a Gap!

\[
\begin{pmatrix}
\frac{53}{156} & 14 \\
\frac{55}{156} & 0 \\
\frac{58}{156} & 10 \\
\frac{66}{156}
\end{pmatrix}
\]

$d_1$: If a $d_1$-student has a large shares $\geq \frac{61}{156}$ then he will have

\[
\frac{53}{156} + \frac{58}{156} + \frac{61}{156} = \frac{172}{156} = \frac{43}{39}.
\]

**Upshot:** Large shares of a $d_1$-student are in ($\frac{58}{156}, \frac{61}{156}$).

$d_2$: If a $d_2$-student has a large shares $\leq \frac{62}{156}$ then he will have

\[
\frac{55}{156} + \frac{55}{156} + \frac{62}{156} = \frac{172}{156} = \frac{43}{39}.
\]

**Upshot:** Large shares of a $d_2$-student are in ($\frac{62}{156}, \frac{66}{156}$).

**Upshot Upshot:** There are NO shares in $[\frac{61}{156}, \frac{62}{156}]$. 
Even More Buddy Match

The following picture captures what we know so far about 3-shares.

\[
\begin{pmatrix}
14 & 0 \\
53/156 & 55/156
\end{pmatrix}
\begin{pmatrix}
x \\
58/156
\end{pmatrix}
\begin{pmatrix}
0 \\
61/156
\end{pmatrix}
\begin{pmatrix}
y \\
62/156
\end{pmatrix}
\begin{pmatrix}
58 \\
66/156
\end{pmatrix}
\]

Use Buddy-Match to show that \(|\left(\frac{61}{156}, \frac{62}{156}\right)| = |\left(\frac{62}{156}, \frac{63}{156}\right)|\). So:

\[
\begin{pmatrix}
14 & 0 \\
53/156 & 55/156
\end{pmatrix}
\begin{pmatrix}
x \\
58/156
\end{pmatrix}
\begin{pmatrix}
0 \\
61/156
\end{pmatrix}
\begin{pmatrix}
y \\
63/156
\end{pmatrix}
\begin{pmatrix}
58 \\
66/156
\end{pmatrix}
\]

\[x + y = 10.\]

Use Buddy-Match to show that \(|\left(\frac{58}{156}, \frac{61}{156}\right)| = |\left(\frac{63}{156}, \frac{66}{156}\right)|\) so they are both 5.

\[
\begin{pmatrix}
14 & 0 \\
53/156 & 55/156
\end{pmatrix}
\begin{pmatrix}
5 \\
58/156
\end{pmatrix}
\begin{pmatrix}
0 \\
61/156
\end{pmatrix}
\begin{pmatrix}
5 \\
63/156
\end{pmatrix}
\begin{pmatrix}
5 \\
66/156
\end{pmatrix}
\]

Equations

\[
\begin{pmatrix}
\frac{53}{156} & 14 \\
\frac{55}{156} & \frac{5}{0}
\end{pmatrix}
\begin{pmatrix}
\frac{58}{156} \\
\frac{61}{156}
\end{pmatrix}
\begin{pmatrix}
0 \\
0
\end{pmatrix}
\begin{pmatrix}
\frac{63}{156} \\
\frac{66}{156}
\end{pmatrix}
\]

Only the \(d_2\)-students use \((\frac{63}{156}, \frac{66}{156})\). Every \(d_2\) student uses one share from that interval:

\[d_2 = 5.\]

Each \(d_i\) student uses \(i\) shares from \((\frac{53}{156}, \frac{55}{156})\):

\[1 \times d_1 + 2 \times d_2 = 14 :\ So \ d_1 = 4\]

There are 8 3-students:

\[d_1 + d_2 = 8 :\ So 5 + 4 = 8. \text{CONTRADICTION!}\]
The Essence of The Buddy-Match Method

1. Works when $\left\lceil \frac{2m}{s} \right\rceil = 3$: Just 2-shares and 3-shares.
2. $2m$ pieces, $s_2$ students get 2 shares, $s_3$ students get 3 shares.
3. Find a GAP
4. Using BM Sequence on 3Shares-interval find intervals that cover almost the entire interval. Missing an interval $(a, b)$.
5. Use BM on $(a, b)$ to get info on an initial interval of 3-shares.
6. Use BM on GAP to get GAPs within the 3-shares.
7. Set up linear equations relating intervals and types of students.
8. Show that system has no solution in $\mathbb{N}$.

**Note:** Can turn BM technique into a function $BM(m, s)$ such that $f(m, s) \leq BM(m, s)$. 

Statistics

For $3 \leq s \leq 50$, $s + 1 \leq m \leq 59$:

\[ f(m, s) = FC(m, s) \text{ in 683 cases} \]
\[ f(m, s) = INT(m, s) \text{ in 193 cases} \]
\[ f(m, s) = BM(m, s) \text{ in 85 cases.} \]
\[ f(m, s) = ERIK(m, s) \text{ in 5 cases.} \]

\[ f(m, s) \text{ is OPEN in 18 cases.} \]

\[ \sim 98\% \text{ of the cases solved!} \]
A Guess that Works. But Why?

1) We suspected there was a constant $X$ such that:

$$(\forall k \geq 1) \left[ f(21k + 11, 21k + 4) \leq \frac{7k + X}{21k + 4} \right]$$

2) We knew that $f(11, 4) = \frac{9}{20}$ so we conjectured $X = \frac{9}{5}$.

3) We prove the result with $X = \frac{9}{5}$ and $k \geq 1$ using BM. We prove matching lower bound for several $k$.

4) But the proof for $f(11, 4) \ (k = 0)$ cannot use BM and is totally unrelated to the proof for $k \geq 1$.

Note: This technique always worked!
Want to know $f(41, 19)$. Can’t use BM. 
$41 - 19 = 22$. So try to prove, diff $d$ is always Mod $3d$ pattern. 
Need $X$:

$$(\forall k \geq 1) \left[ f(66k + 41, 66k + 19) \leq \frac{22k + X}{66k + 19} \right]$$

Find $X$ using BM and linear algebra (have program for that). 
Get conj: $f(41, 19) = \frac{X}{19}$.

Note: This seems to always work but have not been able to use to get new results yet.
Open Problems: $1 \leq s \leq 40, s + 1 \leq 59$

<table>
<thead>
<tr>
<th>M</th>
<th>S</th>
<th>LB</th>
<th>UB</th>
<th>Method for UB</th>
</tr>
</thead>
<tbody>
<tr>
<td>41</td>
<td>19</td>
<td>980/2280</td>
<td>983/2280</td>
<td>ERIK</td>
</tr>
<tr>
<td>41</td>
<td>23</td>
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<td>196/4839</td>
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</tr>
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<td>54</td>
<td>25</td>
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<td>216/500</td>
<td>FC</td>
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<tr>
<td>48</td>
<td>37</td>
<td>512/1480</td>
<td>515/1480</td>
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</tr>
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</table>
Open Problems: $41 \leq s \leq 50$, $s + 1 \leq 59$

<table>
<thead>
<tr>
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<tr>
<td>59</td>
<td>45</td>
<td>92/270</td>
<td>93/270</td>
<td>FC</td>
</tr>
</tbody>
</table>
We have a program that on input \((m, s)\):

1. We used FC, INT, BM to get upper bounds.
2. BM method is a theorem generator.
3. Use linear algebra to try to find a lower bound (a procedure).
Results

1. FC, INT, and BM upper bounds on $f(m,s)$
2. For fixed $s$, for $m \gtrsim s^3$, $f(m,s) = FC(m,s)$.
3. For all $m \geq s$ $f(m,s) \geq \frac{1}{3}$.
4. For $1 \leq s \leq 7$ have proven formulas for $f(m,s)$. Mod $s$ pattern
5. For $s = 8, \ldots, 100$ conjectures for $f(m,s)$. $f(m,s)$ seems to be a mod $s$ pattern.
6. For $1 \leq d \leq 7$ have proven formulas for $f(s+d,s)$. Mod $3d$ pattern.
7. For all $d$ conjecture that our Theorem Generator gives $f(s+d,s)$.
8. Conjecture that for all $a, d$ there exists $X$ such that

$$(\forall k \geq 0) \left[ f(3dk + a + d, 3dk + a) \leq \frac{dk + X}{3dk + a} \right]$$
Open Problems-Complexity

Consider:
Given \( m, s \) in binary, compute \( f(m, s) \).

1. Is the problem in P? We keep on finding techniques that we think cover all cases (so it would be in P) but then finding a case not covered.

2. Is it in NP? The procedure might be very large compared to the input.

3. Is it NP-complete or NP-hard?

4. The problem IS in FPT: \( (\forall m \geq s^3)[f(m, s) = FC(m, s)] \).
Conjectures That are Surely True

1. $f(m, s)$ has mod $s$ pattern with a few exceptions (known for large $m$).
2. $f(s + d, s)$ has mod $3d$ pattern with no exceptions.
3. $f(m, s)$ only depends on $m/s$.
4. If $f(m, s) = \frac{a}{b}$ then $s$ divides $b$. 
Accomplishment I Am Most Proud of

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