The Muffin Problem

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How it Began

A Recreational Math Conference
(Gathering for Gardner)
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I found a pamphlet:
The Julia Robinson Mathematics Festival:
A Sample of Mathematical Puzzles
Compiled by Nancy Blachman

which had this problem, proposed by Alan Frank:

How can you divide and distribute 5 muffins to 3 students so that every student gets $\frac{5}{3}$ where nobody gets a tiny sliver?
## Five Muffins, Three Students, Proc by Picture

<table>
<thead>
<tr>
<th>Person</th>
<th>Color</th>
<th>What they Get</th>
</tr>
</thead>
<tbody>
<tr>
<td>Alice</td>
<td>RED</td>
<td>$1 + \frac{2}{3} = \frac{5}{3}$</td>
</tr>
<tr>
<td>Bob</td>
<td>BLUE</td>
<td>$1 + \frac{2}{3} = \frac{5}{3}$</td>
</tr>
<tr>
<td>Carol</td>
<td>GREEN</td>
<td>$1 + \frac{1}{3} + \frac{1}{3} = \frac{5}{3}$</td>
</tr>
</tbody>
</table>

**Smallest Piece:** $\frac{1}{3}$
Can We Do Better?

The smallest piece in the above solution is $\frac{1}{3}$.

**Is there a procedure with a larger smallest piece?**
Can We Do Better?

The smallest piece in the above solution is \( \frac{1}{3} \).

**Is there a procedure with a larger smallest piece?**

**YES WE CAN!**
### Five Muffins, Three People–Proc by Picture

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**Smallest Piece:** $\frac{5}{12}$
Can We Do Better?

The smallest piece in the above solution is $\frac{5}{12}$. Is there a procedure with a larger smallest piece?
Can We Do Better?

The smallest piece in the above solution is $\frac{5}{12}$.

Is there a procedure with a larger smallest piece?

NO WE CAN’T!
There is a procedure for 5 muffins, 3 students where each student gets $\frac{5}{3}$ muffins, smallest piece $N$. We want $N \leq \frac{5}{12}$.

**Case 0:** Some muffin is uncut. Cut it $(\frac{1}{2}, \frac{1}{2})$ and give both $\frac{1}{2}$-sized pieces to whoever got the uncut muffin. (Note $\frac{1}{2} > \frac{5}{12}$.) Reduces to other cases.

(***Henceforth:** All muffins are cut into $\geq 2$ pieces.)

**Case 1:** Some muffin is cut into $\geq 3$ pieces. Then $N \leq \frac{1}{3} < \frac{5}{12}$. (**Henceforth:** All muffins are cut into 2 pieces.)

**Case 2:** All muffins are cut into 2 pieces. 10 pieces, 3 students: **Someone** gets $\geq 4$ pieces. He has some piece

$$\leq \frac{5}{3} \times \frac{1}{4} = \frac{5}{12}$$

Great to see $\frac{5}{12}$
General Problem

\( f(m, s) \) be the smallest piece in the best procedure (best in that the smallest piece is maximized) to divide \( m \) muffins among \( s \) students so that everyone gets \( \frac{m}{s} \).

We have shown \( f(5, 3) = \frac{5}{12} \) here.

We have shown \( f(m, s) \) exists, is rational, and is computable using a Mixed Int Program (in paper).
Amazing Results! / Amazing Theorems!

1. \( f(43, 33) = \frac{91}{264} \).
2. \( f(52, 11) = \frac{83}{176} \).
3. \( f(35, 13) = \frac{64}{143} \).

All done by hand, no use of a computer by Co-author Erik Metz is a **muffin savant**!

Have **General Theorems** from which **upper bounds** follow. Have **General Procedures** from which **lower bounds** follow.
Clearly $f(3, 5) \geq \frac{1}{5}$. Can we get $f(3, 5) > \frac{1}{5}$?
Clearly \( f(3,5) \geq \frac{1}{5} \). Can we get \( f(3,5) > \frac{1}{5} \)?

\[
\begin{align*}
\text{1.} & \quad \text{Divide 2 muffins } \left[ \frac{6}{20}, \frac{7}{20}, \frac{7}{20} \right] \\
\text{2.} & \quad \text{Divide 1 muffin } \left[ \frac{5}{20}, \frac{5}{20}, \frac{5}{20}, \frac{5}{20} \right] \\
\text{3.} & \quad \text{Give 4 students } \left( \frac{5}{20}, \frac{7}{20} \right) \\
\text{4.} & \quad \text{Give 1 students } \left( \frac{6}{20}, \frac{6}{20} \right)
\end{align*}
\]
$f(3, 5) \geq \frac{1}{5}$.

Clearly $f(3, 5) \geq \frac{1}{5}$. Can we get $f(3, 5) > \frac{1}{5}$?

$f(3, 5) \geq \frac{1}{4}$

1. Divide 2 muffin $[\frac{6}{20}, \frac{7}{20}, \frac{7}{20}]$
2. Divide 1 muffin $[\frac{5}{20}, \frac{5}{20}, \frac{5}{20}, \frac{5}{20}]$
3. Give 4 students $(\frac{5}{20}, \frac{7}{20})$
4. Give 1 students $(\frac{6}{20}, \frac{6}{20})$

Can we do better?
Clearly $f(3, 5) \geq \frac{1}{5}$. Can we get $f(3, 5) > \frac{1}{5}$?

$f(3, 5) \geq \frac{1}{4}$

1. Divide 2 muffin $\left[ \frac{6}{20}, \frac{7}{20}, \frac{7}{20} \right]$
2. Divide 1 muffin $\left[ \frac{5}{20}, \frac{5}{20}, \frac{5}{20}, \frac{5}{20} \right]$
3. Give 4 students $\left( \frac{5}{20}, \frac{7}{20} \right)$
4. Give 1 students $\left( \frac{6}{20}, \frac{6}{20} \right)$

Can we do better?

NO
3 People, 5 Muffins VS 5 People, 3 Muffins

\[ f(5, 3) \geq \frac{5}{12} \]

1. Divide 4 muffins \([\frac{5}{12}, \frac{7}{12}]\)
2. Divide 1 muffin \([\frac{6}{12}, \frac{6}{12}]\)
3. Give 2 students \((\frac{6}{12}, \frac{7}{12}, \frac{7}{12})\)
4. Give 1 students \((\frac{5}{12}, \frac{5}{12}, \frac{5}{12}, \frac{5}{12})\)
3 People, 5 Muffins VS 5 People, 3 Muffins

\[ f(5, 3) \geq \frac{5}{12} \]

1. Divide 4 muffins \([\frac{5}{12}, \frac{7}{12}]\)
2. Divide 1 muffin \([\frac{6}{12}, \frac{6}{12}]\)
3. Give 2 students \((\frac{6}{12}, \frac{7}{12}, \frac{7}{12})\)
4. Give 1 students \((\frac{5}{12}, \frac{5}{12}, \frac{5}{12}, \frac{5}{12})\)

\[ f(3, 5) \geq \frac{1}{4} \]

1. Divide 2 muffin \([\frac{6}{20}, \frac{7}{20}, \frac{7}{20}]\)
2. Divide 1 muffin \([\frac{5}{20}, \frac{5}{20}, \frac{5}{20}, \frac{5}{20}]\)
3. Give 4 students \((\frac{5}{20}, \frac{7}{20})\)
4. Give 1 students \((\frac{6}{20}, \frac{6}{20})\)
3 People, 5 Muffins VS 5 People, 3 Muffins

\[ f(5, 3) \geq \frac{5}{12} \]

1. Divide 4 muffins \([\frac{5}{12}, \frac{7}{12}]\)
2. Divide 1 muffin \([\frac{6}{12}, \frac{6}{12}]\)
3. Give 2 students \((\frac{6}{12}, \frac{7}{12}, \frac{7}{12})\)
4. Give 1 students \((\frac{5}{12}, \frac{5}{12}, \frac{5}{12}, \frac{5}{12})\)

\[ f(3, 5) \geq \frac{1}{4} \]

1. Divide 2 muffin \([\frac{6}{20}, \frac{7}{20}, \frac{7}{20}]\)
2. Divide 1 muffin \([\frac{5}{20}, \frac{5}{20}, \frac{5}{20}, \frac{5}{20}]\)
3. Give 4 students \((\frac{5}{20}, \frac{7}{20})\)
4. Give 1 students \((\frac{6}{20}, \frac{6}{20})\)

\(f(3, 5)\) proc is \(f(5, 3)\) proc but swap Divide/Give and mult by 3/5.
3 People, 5 Muffins VS 5 People, 3 Muffins

\[ f(5, 3) \geq \frac{5}{12} \]

1. Divide 4 muffins \([\frac{5}{12}, \frac{7}{12}]\)
2. Divide 1 muffin \([\frac{6}{12}, \frac{6}{12}]\)
3. Give 2 students \((\frac{6}{12}, \frac{7}{12}, \frac{7}{12})\)
4. Give 1 students \((\frac{5}{12}, \frac{5}{12}, \frac{5}{12}, \frac{5}{12})\)

\[ f(3, 5) \geq \frac{1}{4} \]

1. Divide 2 muffin \([\frac{6}{20}, \frac{7}{20}, \frac{7}{20}]\)
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3. Give 4 students \((\frac{5}{20}, \frac{7}{20})\)
4. Give 1 students \((\frac{6}{20}, \frac{6}{20})\)

\( f(3, 5) \) proc is \( f(5, 3) \) proc but swap Divide/Give and mult by \( \frac{3}{5} \). 

**Theorem:** \( f(m, s) = \frac{m}{s} f(s, m) \).
Floor-Ceiling Thm (FC Thm) Generalizes $f(5, 3) \leq \frac{5}{12}$

$$f(m, s) \leq FC(m, s) = \max\left\{\frac{1}{3}, \min\left\{\frac{m}{s\lfloor2m/s\rfloor}, 1 - \frac{m}{s\lfloor2m/s\rfloor}\right\}\right\}. $$

Case 0: Some muffin is uncut. Cut it ($\frac{1}{2}, \frac{1}{2}$) and give both halves to whoever got the uncut muffin, so reduces to other cases.

Case 1: Some muffin is cut into $\geq 3$ pieces. Some piece $\leq \frac{1}{3}$.

Case 2: Every muffin is cut into 2 pieces, so $2m$ pieces.

Someone gets $\geq \left\lceil \frac{2m}{s} \right\rceil$ pieces. $\exists$ piece $\leq \frac{m}{s} \times \frac{1}{\lfloor2m/s\rfloor} = \frac{m}{s\lfloor2m/s\rfloor}$.

Someone gets $\leq \left\lfloor \frac{2m}{s} \right\rfloor$ pieces. $\exists$ piece $\geq \frac{m}{s} \frac{1}{\lceil2m/s\rceil} = \frac{m}{s\lceil2m/s\rceil}$.

The other piece from that muffin is of size $\leq 1 - \frac{m}{s\lceil2m/s\rceil}$. 
THREE Students

**CLEVERNESS, COMP PROGS** for the procedure.

**FC Theorem** for optimality.

\[ f(1, 3) = \frac{1}{3} \]

\[ f(3k, 3) = 1. \]

\[ f(3k + 1, 3) = \frac{3k-1}{6k}, \quad k \geq 1. \]

\[ f(3k + 2, 3) = \frac{3k+2}{6k+6}. \]

**Note:** A Mod 3 Pattern.

**Theorem:** For all \( m \geq 3 \), \( f(m, 3) = FC(m, 3) \).
CLEVERNESS, COMP PROGS for procedures.

FC Theorem for optimality.

\[ f(4k, 4) = 1 \text{ (easy)} \]

\[ f(1, 4) = \frac{1}{4} \text{ (easy)} \]

\[ f(4k + 1, 4) = \frac{4k-1}{8k}, \quad k \geq 1. \]

\[ f(4k + 2, 4) = \frac{1}{2}. \]

\[ f(4k + 3, 4) = \frac{4k+1}{8k+4}. \]

**Note:** A Mod 4 Pattern.

**Theorem:** For all \( m \geq 4 \), \( f(m, 4) = FC(m, 4) \).

**FC-Conjecture:** For all \( m, s \) with \( m \geq s \), \( f(m, s) = FC(m, s) \).
CLEVERNESS, COMP PROGS for procedures.

FC Theorem for optimality.

For \( k \geq 1 \), \( f(5k, 5) = 1 \).

For \( k = 1 \) and \( k \geq 3 \), \( f(5k + 1, 5) = \frac{5k+1}{10k+5} \). \( f(11, 5) \)?

For \( k \geq 2 \), \( f(5k + 2, 5) = \frac{5k-2}{10k} \). \( f(7, 5) = FC(7, 5) = \frac{1}{3} \)

For \( k \geq 1 \), \( f(5k + 3, 5) = \frac{5k+3}{10k+10} \)

For \( k \geq 1 \), \( f(5k + 4, 5) = \frac{5k+1}{10k+5} \)

Note: A Mod 5 Pattern.
Theorem: For all \( m \geq 5 \) except \( m=11 \), \( f(m, 5) = FC(m, 5) \).
What About FIVE students, ELEVEN muffins?

1. We have a procedure which shows $f(11, 5) \geq \frac{13}{30}$.

2. $f(11, 5) \leq \max\{\frac{1}{3}, \min\{\frac{11}{5\lceil 22/5 \rceil}, 1 - \frac{11}{5\lfloor 22/5 \rfloor}\}\} = \frac{11}{25}$.

So

$$\frac{13}{30} \leq f(11, 5) \leq \frac{11}{25} \quad \text{Diff= 0.006666...}$$

If $f(5, 11) < \frac{11}{25}$ then FC-conjecture is false!
What About FIVE students, ELEVEN muffins?

1. We have a procedure which shows $f(11, 5) \geq \frac{13}{30}$.
2. $f(11, 5) \leq \max\{\frac{1}{3}, \min\{\frac{11}{5\lceil 22/5 \rceil}, 1 - \frac{11}{5\lfloor 22/5 \rfloor}\}\} = \frac{11}{25}$.

So
\[
\frac{13}{30} \leq f(11, 5) \leq \frac{11}{25} \quad \text{Diff} = 0.0066666...\]

If $f(5, 11) < \frac{11}{25}$ then FC-conjecture is false!

WE SHOW: $f(11, 5) = \frac{13}{30}$
There is a procedure for 11 muffins, 5 students where each student gets $\frac{11}{5}$ muffins, smallest piece $N$. We want $N \leq \frac{13}{30}$.

**Case 0:** Some muffin is uncut. Cut it ($\frac{1}{2}$, $\frac{1}{2}$) and give both halves to whoever got the uncut muffin. Reduces to other cases.

**Case 1:** Some muffin is cut into $\geq 3$ pieces. $N \leq \frac{1}{3} < \frac{13}{30}$.

*(Negation of Case 0 and Case 1: All muffins cut into 2 pieces.)*
\(f(11, 5) = \frac{13}{30}\), Easy Case Based on Students

**Case 2:** Some student gets \(\geq 6\) pieces.

\[
N \leq \frac{11}{5} \times \frac{1}{6} = \frac{11}{30} < \frac{13}{30}.
\]

**Case 3:** Some student gets \(\leq 3\) pieces.

One of the pieces is

\[
\geq \frac{11}{5} \times \frac{1}{3} = \frac{11}{15}.
\]

Look at the muffin it came from to find a piece that is

\[
\leq 1 - \frac{11}{15} = \frac{4}{15} < \frac{13}{30}.
\]

*(Negation of Cases 2 and 3: Every student gets 4 or 5 pieces.)*
\( f(11, 5) = \frac{13}{30} \), Fun Cases

**Case 4:** Every muffin is cut in 2 pieces, every student gets 4 or 5 pieces. Number of pieces: 22. Note \( \leq 11 \) pieces are \( > \frac{1}{2} \).

- \( s_4 \) is number of students who get 4 pieces
- \( s_5 \) is number of students who get 5 pieces

\[
4s_4 + 5s_5 = 22 \\
\quad s_4 + s_5 = 5
\]

\( s_4 = 3 \): There are 3 students who have 4 shares. \\
\( s_5 = 2 \): There are 2 students who have 5 shares.

We call a share that goes to a person who gets 4 shares a **4-share**. 
We call a share that goes to a person who gets 5 shares a **5-share**.
$f(11, 5) = \frac{13}{30}$, Fun Cases

**Case 4.1:** is $\leq \frac{1}{2}$. Then there is a piece

$$\geq \frac{(11/5) - (1/2)}{3} = \frac{17}{30}.$$ 

The other piece from the muffin is

$$\leq 1 - \frac{17}{30} = \frac{13}{30} \quad \text{Great to see } \frac{13}{30}.$$ 

**Case 4.2:** All 4-shares are $> \frac{1}{2}$. So there are $4s_4 = 12$ 4-shares. There are $\geq 12$ pieces $> \frac{1}{2}$. Can’t occur.
Essence of the Interval Method

1. Every muffin cut into two pieces.
2. Find $L$ such that some students get either $L$ or $L + 1$ pieces.
3. Find how many students get $L \cdot (L + 1)$ pieces.
4. Find intervals that these pieces must be in.
5. Find how many pieces are in an interval
6. Get a contradiction out of this.

**Note:** Can turn Interval Theorem into a function $\text{INT}$ such that $f(m, s) \leq \text{INT}(m, s)$. 
FC CONJECTURE STILL SORT OF TRUE

**FC Conj:** For all $m \geq s$, $f(m, s) = FC(m, s)$. FALSE

**Theorem:** For fixed $s$, for $m \geq \frac{s^3+2s^2+s}{2} f(m, s) = FC(m, s)$.

**Statistics:** For $3 \leq s \leq 50$, $s + 1 \leq m \leq 59$:

$f(m, s) = FC(m, s)$ in 683 cases
$f(m, s) = INT(m, s)$ in 194 cases

Still 108 cases left. Need new technique!
The Buddy-Match Method! (BM)

Can FC and INT do everything?
No.
They are very good when $\frac{2m}{s} > 3$ but NOT so good otherwise.
We do a concrete example of The Buddy-Match Method

$$f(43, 39) \leq \frac{53}{156}$$

(We have matching lower bound also)

Definition: Assume we have a protocol where all students get 2 or 3 shares. If $x$ is a 2-share then the other share that student has is the shares match. Note that $M(x) = \frac{m}{s} - x$.

Warning: We will apply $M$ to intervals. These intervals have to have only 2-shares in them! But they will!
\( f(43, 39) \leq \frac{53}{156} \)

**Theorem** \( f(43, 39) \leq \frac{53}{156} \) (≥ also known).

Assume there is an \((43, 39)\)-procedure with smallest piece \(>\frac{53}{156}\).

Can assume all muffins cut in 2 pieces, all students get \(\geq 2\) shares.

**Case 1:** A student gets \(\geq 4\) shares. Some share \(\leq \frac{43}{39 \times 4} < \frac{53}{156}\).

**Case 2:** A student gets \(\leq 1\) shares. Can’t occur.

**Case 3:** Every muffin is cut in 2 pieces and every student gets either 2 or 3 shares. The total number of shares is 86.
Let $s_2$ ($s_3$) be the number of 2-students (3-students).

\[ 2s_2 + 3s_3 = 86 \]
\[ s_2 + s_3 = 39 \]

Get $s_2 = 31$ and $s_3 = 8$

**Case 3.1, 3.2, 3.3, 3.4:**

$(\exists)$ 3-share $\geq \frac{66}{156}$. Rm. Now 2-shares $\geq \frac{43}{39} - \frac{66}{156} = \frac{53}{78}$.

So some share $\leq \frac{53}{156}$.

By similar reasoning (Case 3.2, 3.3, 3.4) we have:

\[
\begin{pmatrix}
\frac{53}{156} & \frac{24}{156} & 0 \times 156 \\
\frac{66}{156} & 0 \times 156 & \frac{69}{156} \\
\frac{103}{156} & \frac{62}{156} & 1 \times 156
\end{pmatrix}
\]
The Buddy-Match Method

\[
\begin{align*}
( &\quad 24 \text{ 3-shs} \quad )[ &\quad 0 \text{ shs} \quad ]( &\quad 62 \text{ 2-shs} \quad ) \\
\frac{53}{156} &\quad &\frac{66}{156} &\quad &\frac{69}{156} &\quad &\frac{103}{156}
\end{align*}
\]

\[
\left| \left( \frac{53}{156}, \frac{69}{156} \right) \right| = 24
\]

\[
\left| B \left( \frac{53}{156}, \frac{69}{156} \right) \right| = \left| \frac{87}{156}, \frac{103}{156} \right| = 24
\]

\[
\left| M \left( \frac{87}{156}, \frac{103}{156} \right) \right| = \left| \frac{69}{156}, \frac{85}{156} \right| = 24
\]

\[
\left| \left( \frac{53}{156}, \frac{69}{156} \right) \cup \left( \frac{69}{156}, \frac{85}{156} \right) \cup \left( \frac{87}{156}, \frac{103}{156} \right) \right| = 24 \times 3 = 72
\]

\[
\left| \left( \frac{85}{156}, \frac{87}{156} \right) \right| = 86 - 72 = 14.
\]
More Buddy-Match Method

\[|\left( \frac{85}{156}, \frac{87}{156} \right)| = 14. \text{ Buddy-Match yields } |\left( \frac{53}{156}, \frac{55}{156} \right)| = 14\]

\[|\left[ \frac{66}{156}, \frac{69}{156} \right]| = 0. \text{ Buddy-Match yields } |\left[ \frac{55}{156}, \frac{58}{156} \right]| = 0.\]

The following picture captures what we know so far about 3-shares.

\[
\begin{pmatrix}
53 & 14 & 0 \\
\frac{53}{156} & \frac{55}{156} & \frac{58}{156} \\
10 & 66 & 0 \\
\frac{58}{156} & \frac{66}{156} & \frac{66}{156}
\end{pmatrix}
\]
Big Shares and Small Shares

\[
\begin{pmatrix}
53 \\
55 \\
58 \\
156 \\
156 \\
156 \\
156
\end{pmatrix} \begin{bmatrix}
14 \\
0 \\
10 \\
156 \\
156 \\
156 \\
156
\end{bmatrix}
\]

- Shares in \( \left( \frac{53}{156}, \frac{55}{156} \right) \) are *small shares*;
- Shares in \( \left( \frac{58}{156}, \frac{66}{156} \right) \) are *large shares*;

**Notation** \( d_i \) is numb of students who have \( i \) small shares (\( 3 - i \) large shares).

\[
d_0 = 0 \text{ since } 3 \times \frac{58}{156} = \frac{174}{156} > \frac{172}{156} = \frac{43}{39}.
\]

\[
d_3 = 0 \text{ since } 3 \times \frac{55}{156} = \frac{165}{156} < \frac{172}{156} = \frac{43}{39}.
\]

SO there are NO \( d_0 \)-students or \( d_3 \)-students.
$d_1$ and $d_2$ Students Cause a Gap!

$\begin{pmatrix} 14 \\ \frac{53}{156} \end{pmatrix} \left[ \begin{array}{c} 0 \\ \frac{55}{156} \end{array} \right] \left( \begin{array}{c} 10 \\ \frac{58}{156} \end{array} \right)$

$d_1$: If a $d_1$-student has a large shares $\geq \frac{61}{156}$ then he will have

$$> \frac{53}{156} + \frac{58}{156} + \frac{61}{156} = \frac{172}{156} = \frac{43}{39}.$$

**Upshot:** Large shares of a $d_1$-student are in $(\frac{58}{156}, \frac{61}{156})$.

$d_2$: If a $d_2$-student has a large shares $\leq \frac{62}{156}$ then he will have

$$< \frac{55}{156} + \frac{55}{156} + \frac{62}{156} = \frac{172}{156} = \frac{43}{39}.$$

**Upshot:** Large shares of a $d_2$-student are in $(\frac{62}{156}, \frac{66}{156})$.

**Upshot Upshot:** There are NO shares in $[\frac{61}{156}, \frac{62}{156}]$.
Even More Buddy Match

The following picture captures what we know so far about 3-shares.

\[
\begin{bmatrix}
\frac{53}{156} & 14 & 0 \\
\frac{55}{156} & 0 & \times \\
\frac{61}{156} & 0 & 5 \\
\end{bmatrix}
\begin{bmatrix}
\frac{58}{156} \\
\frac{62}{156} \\
\frac{66}{156} \\
\end{bmatrix}
\]

Use Buddy-Match to show that \(|(\frac{61}{156}, \frac{62}{156})| = |(\frac{62}{156}, \frac{63}{156})|\). So:

\[
\begin{bmatrix}
\frac{53}{156} & 14 & 0 \\
\frac{55}{156} & 0 & \times \\
\frac{61}{156} & 0 & 5 \\
\end{bmatrix}
\begin{bmatrix}
\frac{58}{156} \\
\frac{63}{156} \\
\frac{66}{156} \\
\end{bmatrix}
\]

\[x + y = 10.\]

Use Buddy-Match to show that \(|(\frac{58}{156}, \frac{61}{156})| = |(\frac{63}{156}, \frac{66}{156})|\) so they are both 5.
Only the $d_2$-students use $(\frac{63}{156}, \frac{66}{156})$. Every $d_2$ student uses one share from that interval:

$$d_2 = 5.$$ 

Each $d_i$ student uses $i$ shares from $(\frac{53}{156}, \frac{55}{156})$:

$$1 \times d_1 + 2 \times d_2 = 14 : \text{ So } d_1 = 4$$

There are 8 3-students:

$$d_1 + d_2 = 8 : \text{ So } 5 + 4 = 8.\text{CONTRADICTION!}$$
The Essence of The Buddy-Match Method

1. Works when $\lceil \frac{2m}{s} \rceil = 3$: Just 2-shares and 3-shares.
2. $2m$ pieces, $s_2$ students get 2 shares, $s_3$ students get 3 shares.
3. Find a GAP
4. Using BM Sequence on 3-shares-interval find intervals that cover almost the entire interval. Missing an interval $(a, b)$.
5. Use BM on $(a, b)$ to get info on an initial interval of 3-shares.
6. Use BM on GAP to get GAPs within the 3-shares.
7. Set up linear equations relating intervals and types of students.
8. Show that system has no solution in $\mathbb{N}$.

Note: Can turn BM technique into a function $BM(m, s)$ such that $f(m, s) \leq BM(m, s)$. 
Statistics

For $3 \leq s \leq 60$, $s + 1 \leq m \leq 70$, $m$, $s$ rel prime:

$f(m, s) = FC(m, s)$ in 927 cases. $\sim 68\%$

$f(m, s) = INT(m, s)$ in 268 cases. $\sim 20\%$

$f(m, s) = BM(m, s)$ in 85 cases. $\sim 6\%$

$f(m, s) = ERIK(m, s)$ in 80 cases. $\sim 6\%$

All cases solved!
A Guess that Works. But Why?

1) We suspected there was a constant $X$ such that:

$$(\forall k \geq 1) \left[ f(21k + 11, 21k + 4) \leq \frac{7k + X}{21k + 4} \right]$$

2) We knew that $f(11, 4) = \frac{9}{20}$ so we conjectured $X = \frac{9}{5}$.

3) We prove the result with $X = \frac{9}{5}$ and $k \geq 1$ using BM. We prove matching lower bound for several $k$.

4) But the proof for $f(11, 4) (k = 0)$ cannot use BM and is totally unrelated to the proof for $k \geq 1$.

**Note:** This technique always worked!
Want to know \( f(41, 19) \). Can’t use BM.

\[ 41 - 19 = 22. \] So try to prove, diff \( d \) is always Mod 3\( d \) pattern.

Need \( X \):

\[
(\forall k \geq 1) \left[ f(66k + 41, 66k + 19) \leq \frac{22k + X}{66k + 19} \right]
\]

Find \( X \) using BM and linear algebra (have program for that).

Get conj: \( f(41, 19) = \frac{X}{19} \).

**Note:** This seems to always work but have not been able to use to get new results yet.
Programs

We have a program that on input \((m, s)\):

1. We used FC, INT, BM to get upper bounds.
2. BM method is a theorem generator.
3. Use linear algebra to try to find a lower bound (a procedure).
Results

1. FC, INT, and BM upper bounds on $f(m, s)$
2. For fixed $s$, for $m \geq s^3$, $f(m, s) = FC(m, s)$.
3. For all $m \geq s$ $f(m, s) \geq \frac{1}{3}$.
4. For $1 \leq s \leq 7$ have proven formulas for $f(m, s)$. Mod $s$ pattern
5. For $s = 8, \ldots, 100$ conjectures for $f(m, s)$. $f(m, s)$ seems to be a mod $s$ pattern.
6. For $1 \leq d \leq 7$ have proven formulas for $f(s + d, s)$. Mod $3d$ pattern.
7. For all $d$ conjecture that our Theorem Generator gives $f(s + d, s)$.
8. Conjecture that for all $a, d$ there exists $X$ such that

\[
(\forall k \geq 0) \left[f(3dk + a + d, 3dk + a) \leq \frac{dk + X}{3dk + a}\right]
\]
Open Problems-Complexity

Consider:
Given $m, s$ in binary, compute $f(m, s)$.

1. Is the problem in P? We keep on finding techniques that we think cover all cases (so it would be in P) but then finding a case not covered.
2. Is it in NP? The procedure might be very large compared to the input.
3. Is it NP-complete or NP-hard?
4. The problem IS in FPT: $(\forall m \geq s^3)[f(m, s) = FC(m, s)]$. 
1. $f(m, s)$ has mod $s$ pattern with a few exceptions (known for large $m$).
2. $f(s + d, s)$ has mod $3d$ pattern with no exceptions.
3. $f(m, s)$ only depends on $m/s$.
4. If $f(m, s) = \frac{a}{b}$ then $s$ divides $b$. 
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