The Muffin Problem

Guangi Cui - Montgomery Blair HS
John Dickerson- University of MD
Naveen Durvasula - Montgomery Blair HS
William Gasarch - University of MD
Erik Metz - University of MD
Jacob Prinz-University of MD
Naveen Raman - Richard Montgomery HS
Daniel Smolyak- University of MD
Sung Hyun Yoo - Bergen County Academies (in NJ)
How it Began

A Recreational Math Conference
(Gathering for Gardner)
May 2016

I found a pamphlet:

The Julia Robinson Mathematics Festival:
A Sample of Mathematical Puzzles
Compiled by Nancy Blachman

which had this problem, proposed by Alan Frank:

How can you divide and distribute 5 muffins to 3 students so that every student gets \( \frac{5}{3} \) where nobody gets a tiny sliver?
### Five Muffins, Three Students, Proc by Picture

<table>
<thead>
<tr>
<th>Person</th>
<th>Color</th>
<th>What they Get</th>
</tr>
</thead>
<tbody>
<tr>
<td>Alice</td>
<td>RED</td>
<td>$1 + \frac{2}{3} = \frac{5}{3}$</td>
</tr>
<tr>
<td>Bob</td>
<td>BLUE</td>
<td>$1 + \frac{2}{3} = \frac{5}{3}$</td>
</tr>
<tr>
<td>Carol</td>
<td>GREEN</td>
<td>$1 + \frac{1}{3} + \frac{1}{3} = \frac{5}{3}$</td>
</tr>
</tbody>
</table>

**Smallest Piece:** $\frac{1}{3}$
Can We Do Better?

The smallest piece in the above solution is $\frac{1}{3}$. Is there a procedure with a larger smallest piece? Work on it with your neighbor.
YES WE CAN!

<table>
<thead>
<tr>
<th>Person</th>
<th>Color</th>
<th>What they Get</th>
</tr>
</thead>
<tbody>
<tr>
<td>Alice</td>
<td>RED</td>
<td>$\frac{6}{12} + \frac{7}{12} + \frac{7}{12}$</td>
</tr>
<tr>
<td>Bob</td>
<td>BLUE</td>
<td>$\frac{6}{12} + \frac{7}{12} + \frac{7}{12}$</td>
</tr>
<tr>
<td>Carol</td>
<td>GREEN</td>
<td>$\frac{5}{12} + \frac{5}{12} + \frac{5}{12} + \frac{5}{12}$</td>
</tr>
</tbody>
</table>

Smallest Piece: $\frac{5}{12}$
Can We Do Better?

The smallest piece in the above solution is \( \frac{5}{12} \).

*Is there a procedure with a larger smallest piece?*

*Work on it with your neighbor*
NO WE CAN’T!

There is a procedure for 5 muffins, 3 students where each student gets \( \frac{5}{3} \) muffins, smallest piece \( N \). We want \( N \leq \frac{5}{12} \).

**Case 0:** Some muffin is uncut. Cut it \( \left( \frac{1}{2}, \frac{1}{2} \right) \) and give both \( \frac{1}{2} \)-sized pieces to whoever got the uncut muffin. (Note \( \frac{1}{2} > \frac{5}{12} \).) Reduces to other cases.

(*Henceforth:* All muffins are cut into \( \geq 2 \) pieces.)

**Case 1:** Some muffin is cut into \( \geq 3 \) pieces. Then \( N \leq \frac{1}{3} < \frac{5}{12} \).

(*Henceforth:* All muffins are cut into 2 pieces.)

**Case 2:** All muffins are cut into 2 pieces. 10 pieces, 3 students: **Someone** gets \( \geq 4 \) pieces. He has some piece

\[
\leq \frac{5}{3} \times \frac{1}{4} = \frac{5}{12}
\]

Great to see \( \frac{5}{12} \)
**General Problem**

\[ f(m, s) \] be the smallest piece in the best procedure (best in that the smallest piece is maximized) to divide \( m \) muffins among \( s \) students so that everyone gets \( \frac{m}{s} \).

We have shown \( f(5, 3) = \frac{5}{12} \) here.

We have shown \( f(m, s) \) exists, is rational, and is computable using a Mixed Int Program (in paper).
Amazing Results! / Amazing Theorems!

1. \( f(43, 33) = \frac{91}{264} \).
2. \( f(52, 11) = \frac{83}{176} \).
3. \( f(35, 13) = \frac{64}{143} \).

All done by hand, no use of a computer by Co-author Erik Metz is a muffin savant!

Have General Theorems from which upper bounds follow. Have General Procedures from which lower bounds follow.
Clearly $f(3, 5) \geq \frac{1}{5}$.

Can we get $f(3, 5) > \frac{1}{5}$?

Work on it with your neighbor.
\( f(3, 5) \geq \frac{1}{4} \)

1. Divide 2 muffin \([\frac{6}{20}, \frac{7}{20}, \frac{7}{20}]\)
2. Divide 1 muffin \([\frac{5}{20}, \frac{5}{20}, \frac{5}{20}, \frac{5}{20}]\)
3. Give 4 students \((\frac{5}{20}, \frac{7}{20})\)
4. Give 1 students \((\frac{6}{20}, \frac{6}{20})\)
\[ f(3, 5) \geq \frac{1}{4} \]

1. Divide 2 muffin \( \left[ \frac{6}{20}, \frac{7}{20}, \frac{7}{20} \right] \)
2. Divide 1 muffin \( \left[ \frac{5}{20}, \frac{5}{20}, \frac{5}{20}, \frac{5}{20} \right] \)
3. Give 4 students \( \left( \frac{5}{20}, \frac{7}{20} \right) \)
4. Give 1 student \( \left( \frac{6}{20}, \frac{6}{20} \right) \)

**Can we do better?**

**Work on it with your neighbor**
NO WE CAN’T!
There is a procedure for 3 muffins, 5 students where each student gets $\frac{3}{5}$ muffins, smallest piece $N$. We want $N \leq \frac{1}{4}$.

**Case 0:** Alice gets 1 piece of size $\frac{3}{5}$. Look at the rest of that muffin which totals to $\frac{2}{5}$. (1) That piece is cut. Have piece $\leq \frac{2}{5} \times \frac{1}{2} = \frac{1}{5}$, OR (2) That piece uncut. So someone gets a $\frac{2}{5}$-piece. Must also get a $\frac{1}{5}$ piece.

**Henceforth:** All people get $\geq 2$ pieces.

**Case 1:** Alice gets $\geq 3$ pieces. Then $N \leq \frac{3}{5} \times \frac{1}{3} = \frac{1}{5}$.

**Henceforth:** Everyone gets 2 pieces.

**Case 2:** Everyone gets 2 pieces. 10 pieces, 3 muffins: Some muffin gets $\geq 4$ pieces. So some piece is $\leq \frac{1}{4}$. 
$f(3, 5)$ and $f(5, 3)$

1. Divide 4 muffins $\left[ \frac{5}{12}, \frac{7}{12} \right]$
2. Divide 1 muffin $\left[ \frac{6}{12}, \frac{6}{12} \right]$
3. Give 2 students $\left( \frac{6}{12}, \frac{7}{12}, \frac{7}{12} \right)$
4. Give 1 students $\left( \frac{5}{12}, \frac{5}{12}, \frac{5}{12}, \frac{5}{12} \right)$
$f(3, 5)$ and $f(5, 3)$

1. Divide 4 muffins $\left[ \frac{5}{12}, \frac{7}{12} \right]$
2. Divide 1 muffin $\left[ \frac{6}{12}, \frac{6}{12} \right]$
3. Give 2 students $\left( \frac{6}{12}, \frac{7}{12}, \frac{7}{12} \right)$
4. Give 1 student $\left( \frac{5}{12}, \frac{5}{12}, \frac{5}{12}, \frac{5}{12} \right)$

$f(3, 5) \geq \frac{1}{4}$

1. Divide 2 muffin $\left[ \frac{6}{20}, \frac{7}{20}, \frac{7}{20} \right]$
2. Divide 1 muffin $\left[ \frac{5}{20}, \frac{5}{20}, \frac{5}{20}, \frac{5}{20} \right]$
3. Give 4 students $\left( \frac{5}{20}, \frac{7}{20} \right)$
4. Give 1 student $\left( \frac{6}{20}, \frac{6}{20} \right)$
$f(3, 5)$ and $f(5, 3)$

1. Divide 4 muffins $\left[\frac{5}{12}, \frac{7}{12}\right]$
2. Divide 1 muffin $\left[\frac{6}{12}, \frac{6}{12}\right]$
3. Give 2 students $\left(\frac{6}{12}, \frac{7}{12}, \frac{7}{12}\right)$
4. Give 1 students $\left(\frac{5}{12}, \frac{5}{12}, \frac{5}{12}, \frac{5}{12}\right)$

$f(3, 5) \geq \frac{1}{4}$

1. Divide 2 muffin $\left[\frac{6}{20}, \frac{7}{20}, \frac{7}{20}\right]$
2. Divide 1 muffin $\left[\frac{5}{20}, \frac{5}{20}, \frac{5}{20}, \frac{5}{20}\right]$
3. Give 4 students $\left(\frac{5}{20}, \frac{7}{20}\right)$
4. Give 1 students $\left(\frac{6}{20}, \frac{6}{20}\right)$

$f(3, 5)$ proc is $f(5, 3)$ proc but swap Divide/Give and mult by $3/5$. 

Theorem:

\[ f(m, s) = m \cdot s = f(s, m) \]
**f(3, 5) and f(5, 3)**

1. Divide 4 muffins \([\frac{5}{12}, \frac{7}{12}]\)
2. Divide 1 muffin \([\frac{6}{12}, \frac{6}{12}]\)
3. Give 2 students \((\frac{6}{12}, \frac{7}{12}, \frac{7}{12})\)
4. Give 1 students \((\frac{5}{12}, \frac{5}{12}, \frac{5}{12}, \frac{5}{12})\)

\[f(3, 5) \geq \frac{1}{4}\]

1. Divide 2 muffin \([\frac{6}{20}, \frac{7}{20}, \frac{7}{20}]\)
2. Divide 1 muffin \([\frac{5}{20}, \frac{5}{20}, \frac{5}{20}, \frac{5}{20}]\)
3. Give 4 students \((\frac{5}{20}, \frac{7}{20})\)
4. Give 1 students \((\frac{6}{20}, \frac{6}{20})\)

\[f(3, 5)\] proc is \(f(5, 3)\) proc but swap Divide/Give and mult by \(3/5\).

**Theorem:** \(f(m, s) = \frac{m}{s} f(s, m)\).
Floor-Ceiling Thm (FC Thm) Generalizes $f(5, 3) \leq \frac{5}{12}$

$$f(m, s) \leq FC(m, s) = \max\left\{\frac{1}{3}, \min\left\{\frac{m}{s\lfloor 2m/s\rfloor}, 1 - \frac{m}{s\lfloor 2m/s\rfloor}\right\}\right\}.$$ 

**Case 0:** Some muffin is uncut. Cut it $(\frac{1}{2}, \frac{1}{2})$ and give both halves to whoever got the uncut muffin, so reduces to other cases.

**Case 1:** Some muffin is cut into $\geq 3$ pieces. Some piece $\leq \frac{1}{3}$.

**Case 2:** Every muffin is cut into 2 pieces, so $2m$ pieces.

Someone gets $\geq \left\lfloor \frac{2m}{s}\right\rfloor$ pieces. $\exists$ piece $\leq \frac{m}{s} \times \frac{1}{\lfloor 2m/s\rfloor} = \frac{m}{s\lfloor 2m/s\rfloor}$.

Someone gets $\leq \left\lfloor \frac{2m}{s}\right\rfloor$ pieces. $\exists$ piece $\geq \frac{m}{s} \left\lfloor \frac{1}{2m/s}\right\rfloor = \frac{m}{s\lfloor 2m/s\rfloor}$.

The other piece from that muffin is of size $\leq 1 - \frac{m}{s\lfloor 2m/s\rfloor}$. 
THREE Students

**CLEVERNESS, COMP PROGS** for the procedure.

**FC Theorem** for optimality.

\[ f(1, 3) = \frac{1}{3} \]

\[ f(3k, 3) = 1. \]

\[ f(3k + 1, 3) = \frac{3k-1}{6k}, \quad k \geq 1. \]

\[ f(3k + 2, 3) = \frac{3k+2}{6k+6}. \]

**Note:** A Mod 3 Pattern.

**Theorem:** For all \( m \geq 3 \), \( f(m, 3) = FC(m, 3) \).
FOUR Students

**CLEVERNESS, COMP PROGS** for procedures.

**FC Theorem** for optimality.

\[ f(4k, 4) = 1 \text{ (easy)} \]

\[ f(1, 4) = \frac{1}{4} \text{ (easy)} \]

\[ f(4k + 1, 4) = \frac{4k - 1}{8k}, \quad k \geq 1. \]

\[ f(4k + 2, 4) = \frac{1}{2}. \]

\[ f(4k + 3, 4) = \frac{4k + 1}{8k + 4}. \]

**Note:** A Mod 4 Pattern.

**Theorem:** For all \( m \geq 4 \), \( f(m, 4) = FC(m, 4). \)

**FC-Conjecture:** For all \( m, s \) with \( m \geq s \), \( f(m, s) = FC(m, s). \)
CLEVERNESS, COMP PROGS for procedures.

FC Theorem for optimality.

For $k \geq 1$, $f(5k, 5) = 1$.

For $k = 1$ and $k \geq 3$, $f(5k + 1, 5) = \frac{5k+1}{10k+5}$. $f(11, 5)$?

For $k \geq 2$, $f(5k + 2, 5) = \frac{5k-2}{10k}$. $f(7, 5) = FC(7, 5) = \frac{1}{3}$

For $k \geq 1$, $f(5k + 3, 5) = \frac{5k+3}{10k+10}$

For $k \geq 1$, $f(5k + 4, 5) = \frac{5k+1}{10k+5}$

Note: A Mod 5 Pattern.

Theorem: For all $m \geq 5$ except $m=11$, $f(m, 5) = FC(m, 5)$. 
What About FIVE students, ELEVEN muffins?

1. We have a procedure which shows \( f(11, 5) \geq \frac{13}{30} \).
2. \( f(11, 5) \leq \max\left\{ \frac{1}{3}, \min\left\{ \frac{11}{5\lceil 22/5 \rceil}, 1 - \frac{11}{5\lfloor 22/5 \rfloor} \right\} \right\} = \frac{11}{25} \).

So
\[
\frac{13}{30} \leq f(11, 5) \leq \frac{11}{25} \quad \text{Diff} = 0.0066666\ldots
\]

Options:

1. \( f(11, 5) = \frac{11}{25} \). Need to find procedure.
2. \( f(11, 5) = \frac{13}{30} \). Need to find new technique for upper bounds.
3. \( f(11, 5) \) in between. Need to find both.
4. \( f(11, 5) \) unknown to science!

Vote
What About FIVE students, ELEVEN muffins?

1. We have a procedure which shows $f(11, 5) \geq \frac{13}{30}$.
2. $f(11, 5) \leq \max\{\frac{1}{3}, \min\{\frac{11}{5\lfloor22/5\rfloor}, 1 - \frac{11}{5\lceil22/5\rceil}\}\} = \frac{11}{25}$.

So

$$\frac{13}{30} \leq f(11, 5) \leq \frac{11}{25} \quad \text{Diff} = 0.006666\ldots$$

Options:

1. $f(11, 5) = \frac{11}{25}$. Need to find procedure.
2. $f(11, 5) = \frac{13}{30}$. Need to find new technique for upper bounds.
3. $f(11, 5)$ in between. Need to find both.
4. $f(11, 5)$ unknown to science!

**Vote** WE SHOW: $f(11, 5) = \frac{13}{30}$. **Exciting** new technique!
Terminology: Buddy

Assume that in some protocol every muffin is cut into two pieces.

Let $x$ be a piece from muffin $M$. The other piece from muffin $M$ is the buddy of $x$.

Note that the buddy of $x$ is of size $1 - x$. 
There is a procedure for 11 muffins, 5 students where each student gets \( \frac{11}{5} \) muffins, smallest piece \( N \). We want \( N \leq \frac{13}{30} \).

**Case 0:** Some muffin is uncut. Cut it \((\frac{1}{2}, \frac{1}{2})\) and give both halves to whoever got the uncut muffin. Reduces to other cases.

**Case 1:** Some muffin is cut into \( \geq 3 \) pieces. \( N \leq \frac{1}{3} < \frac{13}{30} \).

*(Negation of Case 0 and Case 1: All muffins cut into 2 pieces.)*
Case 2: Some student gets $\geq 6$ pieces.

$$N \leq \frac{11}{5} \times \frac{1}{6} = \frac{11}{30} < \frac{13}{30}.$$ 

Case 3: Some student gets $\leq 3$ pieces.
One of the pieces is

$$\geq \frac{11}{5} \times \frac{1}{3} = \frac{11}{15}.$$ 

Look at the muffin it came from to find a piece that is

$$\leq 1 - \frac{11}{15} = \frac{4}{15} < \frac{13}{30}.$$ 

(Negation of Cases 2 and 3: Every student gets 4 or 5 pieces.)
Case 4: Every muffin is cut in 2 pieces, every student gets 4 or 5 pieces. Number of pieces: 22. Note $\leq 11$ pieces are $> \frac{1}{2}$.

- $s_4$ is number of students who get 4 pieces
- $s_5$ is number of students who get 5 pieces

\[
4s_4 + 5s_5 = 22
\]
\[
s_4 + s_5 = 5
\]

$s_4 = 3$: There are 3 students who have 4 shares.
$s_5 = 2$: There are 2 students who have 5 shares.

We call a share that goes to a person who gets 4 shares a 4-share. We call a share that goes to a person who gets 5 shares a 5-share.
Case 4.1: Some 4-share is \( \leq \frac{1}{2} \).
Alice gets \( w, x, y, z \) and \( w \leq \frac{1}{2} \).
Since \( w + x + y + z = \frac{11}{5} \) and \( w \leq \frac{1}{2} \)

\[
x + y + z \geq \frac{11}{5} - \frac{1}{2} = \frac{17}{10}
\]

Let \( x \) be the largest of \( x, y, z \)

\[
x \geq \frac{17}{10} \times \frac{1}{3} = \frac{17}{30}
\]

Look at buddy of \( x \).

\[
B(x) \leq 1 - x = 1 - \frac{17}{30} = \frac{13}{30}
\]

GREAT! This is where \( \frac{13}{30} \) comes from!
\[ f(11, 5) = \frac{13}{30}, \text{ Fun Cases} \]

**Case 4.2:** All 4-shares are \( > \frac{1}{2} \). There are \( 4s_4 = 12 \) 4-shares. There are \( \geq 12 \) pieces \( > \frac{1}{2} \). Can’t occur.
Proof that $f(11, 5) \leq \frac{13}{30}$ was an example of the INT method. We give a more sophisticated example.
Assume \((24, 11)\)-procedure with smallest piece \(> \frac{19}{44}\).
Can assume all muffin cut in two and all student gets \(\geq 2\) shares.
We show that there is a piece \(\leq \frac{19}{44}\).

**Case 1:** A student gets \(\geq 6\) shares. Some piece \(\leq \frac{24}{11 \times 6} < \frac{19}{44}\).

**Case 2:** A student gets \(\leq 3\) shares. Some piece \(\geq \frac{24}{11 \times 3} = \frac{8}{11}\).
Buddy of that piece \(\leq 1 - \frac{8}{11} \leq \frac{3}{11} < \frac{19}{44}\).

**Case 3:** Every muffin is cut in 2 pieces and every student gets either 4 or 5 shares. Total number of shares is 48.
How many students get 4? 5? Where are the Shares?

4-students: a student who gets 4 shares. \( s_4 \) is the number of them.
5-students: a student who gets 5 shares. \( s_5 \) is the number of them.

4-share: a share that a 4-student who gets.
5-share: a share that a 5-student who gets.

\[
4s_4 + 5s_5 \quad = \quad 48 \\
\quad s_4 + s_5 \quad = \quad 11 \\
\]

\( s_4 = 7 \). Hence there are \( 4s_4 = 4 \times 7 = 28 \) 4-shares.
\( s_5 = 4 \). Hence there are \( 5s_5 = 5 \times 4 = 20 \) 5-shares.
Case 3.1: There is a share $\geq \frac{25}{44}$. Then its buddy is

$$\leq 1 - \frac{25}{44} = \frac{19}{44}$$

Case 3.2: There is a share $\leq \frac{19}{44}$. Duh.

Henceforth assume that all shares are in

$$\left( \frac{19}{44}, \frac{25}{44} \right)$$
Case 3.3: Some 5-shares $\geq \frac{20}{44}$

5-share: a share that a 5-student who gets.

Claim: If some 5-shares is $\geq \frac{20}{44}$ then some share $\leq \frac{19}{44}$.

Proof: Assume that Alice 5 pieces $A, B, C, D, E$ and $E \geq \frac{20}{44}$.

Since $A + B + C + D + E = \frac{24}{11}$ and $E > \frac{20}{44}$

\[
A + B + C + D < \frac{24}{11} - \frac{20}{44} = \frac{76}{44}
\]

Assume $A$ is the smallest of $A, B, C, D$.

\[
A \leq \frac{76}{44} \times \frac{1}{4} = \frac{19}{44}
\]

Henceforth we assume all 5-shares are in

\[
\left( \frac{19}{44}, \frac{20}{44} \right).
\]
Case 3.4: Some 4-shares $\leq \frac{21}{44}$

4-share: a share that a 4-student who gets.

Claim: If some 4-shares is $\leq \frac{21}{44}$ then some share $\leq \frac{19}{44}$.

Proof: Assume that Alice 4 pieces $A, B, C, D$ and $D \leq \frac{21}{44}$.

Since $A + B + C + D = \frac{24}{11}$ and $D \leq \frac{21}{44}$

$$A + B + C > \frac{24}{11} - \frac{21}{44} = \frac{75}{44}$$

Assume $A$ is the largest of $A, B, C$.

$$A \geq \frac{75}{44} \times \frac{1}{3} = \frac{25}{44}$$

The buddy of $A$ is of size

$$\leq 1 - \frac{25}{44} = \frac{19}{44}$$

Henceforth we assume all 4-shares are in

$$\left( \frac{21}{44}, \frac{25}{44} \right).$$
Case 3.5: All Shares in Their Proper Intervals

**Case 3.5:** 4-shares in \( \left( \frac{21}{44}, \frac{25}{44} \right) \), 5-shares in \( \left( \frac{19}{44}, \frac{20}{44} \right) \).

\[
\begin{align*}
\left( \begin{array}{c}
?\text{ 5-shs} \\
\frac{19}{44}
\end{array} \right) & \left[ \begin{array}{c} 
0 \text { shs} \\
\frac{20}{44}
\end{array} \right]
\left( \begin{array}{c} 
?\text{ 4-shs} \\
\frac{21}{44}
\end{array} \right) \left( \begin{array}{c} 
\frac{25}{44}
\end{array} \right)
\end{align*}
\]
**Case 3.5: All Shares in Their Proper Intervals**

**Case 3.5:** 4-shares in \((\frac{21}{44}, \frac{25}{44})\), 5-shares in \((\frac{19}{44}, \frac{20}{44})\).

\[
\left( \begin{array}{ccc}
\frac{19}{44} & ?? & \frac{25}{44} \\
\frac{20}{44} & 0 & \frac{21}{44} \\
\end{array} \right)
\]

**Recall:** there are \(4s_4 = 4 \times 7 = 28\) 4-shares.

**Recall:** there are \(5s_5 = 5 \times 4 = 20\) 5-shares.
Case 3.5: All Shares in Their Proper Intervals

**Case 3.5:** 4-shares in \((\frac{21}{44}, \frac{25}{44})\), 5-shares in \((\frac{19}{44}, \frac{20}{44})\).

\[
(?? \text{ 5-shs})[0 \text{ shs}](?? \text{ 4-shs})
\]
\[
\frac{19}{44} \quad \frac{20}{44} \quad \frac{21}{44} \quad \frac{25}{44}
\]

**Recall:** there are \(4s_4 = 4 \times 7 = 28\) 4-shares.

**Recall:** there are \(5s_5 = 5 \times 4 = 20\) 5-shares.

\[
(20 \text{ 5-shs})[0 \text{ shs}](28 \text{ 4-shs})
\]
\[
\frac{19}{44} \quad \frac{20}{44} \quad \frac{21}{44} \quad \frac{25}{44}
\]
More Refined Picture of What is Going On

\[
\begin{pmatrix}
20 
\end{pmatrix}[
0 
\begin{pmatrix}
28 
\end{pmatrix}
\]

\[
20 \quad 44
\]

\[
21 \quad 44
\]

\[
25 \quad 44
\]

Claim 1: There are no shares \( x \in \left[ \frac{23}{44}, \frac{24}{44} \right] \).

If there was such a share then buddy is in \( \left[ \frac{20}{44}, \frac{21}{44} \right] \).
Claim 1: There are no shares \( x \in \left[ \frac{23}{44}, \frac{24}{44} \right] \).

If there was such a share then buddy is in \( \left[ \frac{20}{44}, \frac{21}{44} \right] \).

The following picture captures what we know so far.

\[
\begin{pmatrix}
\frac{19}{44} & 20 & 5\text{-shs} & \frac{20}{44} & 0 & \text{shs} & \frac{21}{44} & 28 & 4\text{-shs} & \frac{25}{44}
\end{pmatrix}
\]

S4 = Small 4-shares
L4 = Large 4-shares. L4 shares, 5-share: buddies, so \(|L4| = 20\).
Claim 2: Every 4-student has at least 3 L4 shares.

If a 4-student had \( \leq 2 \) L4 shares then he has

\[
\begin{align*}
< 2 \times \left( \frac{23}{44} \right) + 2 \times \left( \frac{25}{44} \right) &= \frac{24}{11}.
\end{align*}
\]
Claim 2: Every 4-student has at least 3 L4 shares.

If a 4-student had $\leq 2$ L4 shares then he has

$$< 2 \times \left( \frac{23}{44} \right) + 2 \times \left( \frac{25}{44} \right) = \frac{24}{11}.$$  

Contradiction: Each 4-student gets $\geq 3$ L4 shares. There are $s_4 = 7$ 4-students. Hence there are $\geq 21$ L4-shares. But there are only 20.
INT Technique

INT is generalization of $f(24, 11) \leq \frac{19}{44}$ proof.

**Definition:** Let INT($m, s$) be the bound obtained.

1. INT proofs can get more complicated than this one.
2. INT($m, s$) can be computed in $O\left(\frac{2m \log m}{s}\right)$. Note: do not need to know the answer ahead of time.
3. For $1 \leq s \leq 60$, $s < m \leq 70$, $m, s$ rel prime:
   3.1 There are 1360 cases total.
   3.2 For 927 of the $(m, s)$, $f(m, s) = FC(m, s)$. $\sim 68\%$
   3.3 For 268 of the $(m, s)$, $f(m, s) = INT(m, s)$. $\sim 20\%$
   3.4 The cases not covered use **interesting** new techniques!
Example of GAPS Technique: \( f(31, 19) \leq \frac{54}{133} \)

We show \( f(31, 19) \leq \frac{54}{133} \).

Assume \((31, 19)\)-procedure with smallest piece \(> \frac{54}{133}\).

By INT-technique methods obtain:

\[ s_3 = 14, \ s_4 = 5. \]

\[
\begin{pmatrix}
\frac{54}{133} \\
\frac{55}{133} \\
\frac{59}{133} \\
\frac{74}{133} \\
\frac{78}{133} \\
\frac{79}{133}
\end{pmatrix}
\begin{pmatrix}
20 \\
4 \text{-shs}
\end{pmatrix}
\begin{pmatrix}
0
\end{pmatrix}
\begin{pmatrix}
\text{S3 shs}
\end{pmatrix}
\begin{pmatrix}
0
\end{pmatrix}
\begin{pmatrix}
20 \text{ L3-shs}
\end{pmatrix}
\]

We just look at the 3-shares:

\[
\begin{pmatrix}
\frac{59}{133} \\
\frac{74}{133} \\
\frac{78}{133} \\
\frac{79}{133}
\end{pmatrix}
\begin{pmatrix}
\text{S3 shs}
\end{pmatrix}
\begin{pmatrix}
0
\end{pmatrix}
\begin{pmatrix}
20 \text{ L3-shs}
\end{pmatrix}
\]
GAPS Technique: \( f(31, 19) \leq \frac{54}{133} \)

\[
\begin{pmatrix}
\frac{59}{133} & 0 & \frac{78}{133} \\
\frac{74}{133} & \frac{79}{133}
\end{pmatrix}
\]

1. \( J_1 = (\frac{59}{133}, \frac{66.5}{133}) \)
2. \( J_2 = (\frac{66.5}{133}, \frac{74}{133}) \) \( (|J_1| = |J_2|) \)
3. \( J_3 = (\frac{78}{133}, \frac{79}{133}) \) \( (|J_3| = 20) \)

**Note:** Split the shares of size 66.5 between \( J_1 \) and \( J_2 \).

**Notation:** An \( e(1, 1, 3) \) students is a student who has a \( J_1 \)-share, a \( J_1 \)-share, and a \( J_3 \)-share.

Generalize to \( e(i, j, k) \) easily.
GAPS Technique: $f(31, 19) \leq \frac{54}{133}$

1. $J_1 = \left( \frac{59}{133} , \frac{66.5}{133} \right)$

2. $J_2 = \left( \frac{66.5}{133} , \frac{74}{133} \right) \ (|J_1| = |J_2|)$

3. $J_3 = \left( \frac{78}{133} , \frac{79}{133} \right) \ (|J_3| = 20)$

1) Only students allowed: $e(1, 2, 3), e(1, 3, 3), e(2, 2, 2), e(2, 2, 3)$. All others have either $< \frac{31}{19}$ or $> \frac{31}{19}$.

2) No shares in $[\frac{61}{133}, \frac{64}{133}]$. Look at $J_1$-shares:
An $e(1, 2, 3)$-student has $J_1$-share $> \frac{31}{19} - \frac{74}{133} - \frac{79}{133} = \frac{64}{133}$.
An $e(1, 3, 3)$-student has $J_1$-share $< \frac{31}{19} - 2 \times \frac{78}{133} = \frac{61}{133}$.

3) No shares in $[\frac{69}{133}, \frac{72}{133}]$: $x \in [\frac{69}{133}, \frac{72}{133}] \implies 1 - x \in [\frac{61}{133}, \frac{64}{133}]$. 
GAPS Technique: $f(31, 19) \leq \frac{54}{133}$

1. $J_1 = (\frac{59}{133}, \frac{61}{133})$
2. $J_2 = (\frac{64}{133}, \frac{66.5}{133})$
3. $J_3 = (\frac{66.5}{133}, \frac{69}{133})$ ($|J_2| = |J_3|$)
4. $J_4 = (\frac{72}{133}, \frac{74}{133})$ ($|J_1| = |J_4|$)
5. $J_5 = (\frac{78}{133}, \frac{79}{133})$ ($|J_5| = 20$)

The following are the only students who are allowed.

$e(1, 5, 5)$.
$e(2, 4, 5)$.
$e(3, 4, 5)$.
$e(4, 4, 4)$. 
GAPS Technique: \( f(31, 19) \leq \frac{54}{133} \)

\( e(1, 5, 5) \). Let the number of such students be \( x \)
\( e(2, 4, 5) \). Let the number of such students be \( y_1 \)
\( e(3, 4, 5) \). Let the number of such students be \( y_2 \).
\( e(4, 4, 4) \). Let the number of such students be \( z \).

1) \( |J_2| = |J_3| \),
only students using \( J_2 \) are \( e(2, 4, 5) \) – they use one share each,
only students using \( J_3 \) are \( e(3, 4, 5) \) – they use one share each.
Hence \( y_1 = y_2 \). We call them both \( y \).

2) Since \( |J_1| = |J_4| \), \( x = 2y + 3z \).

3) Since \( s_3 = 14 \), \( x + 2y + z = 14 \).

\[
(2y + 3z) + 2y + z = 14 \implies 4(y + z) = 14 \implies y + z = \frac{7}{2}.
\]
Contradiction.
MATRIX Technique: \( f(5, 3) \geq \frac{5}{12} \)

Want proc for \( f(5, 3) \geq \frac{5}{12} \).

1) **Guess** that the only piece sizes are \( \frac{5}{12}, \frac{6}{12}, \frac{7}{12} \)

2) **Muffin**=pieces add to 1: \( \{ \frac{6}{12}, \frac{6}{12} \}, \{ \frac{5}{12}, \frac{7}{12} \} \). Vectors \( \{ \frac{6}{12}, \frac{6}{12} \} \) is \( (0, 2, 0) \), \( m_1 \) muffins of this type. 
   \( \{ \frac{5}{12}, \frac{7}{12} \} \) is \( (1, 0, 1) \), \( m_2 \) muffins of this type.

3) **Student**=pieces add to \( \frac{5}{3} \) 
   \( \{ \frac{6}{12}, \frac{7}{12}, \frac{7}{12} \} \) is \( (0, 1, 2) \), \( s_1 \) students of this type. 
   \( \{ \frac{5}{12}, \frac{5}{12}, \frac{5}{12}, \frac{5}{12} \} \) is \( (4, 0, 0) \), \( s_2 \) students of this type.

4) **Set up equations:**
   \( m_1(0, 2, 0) + m_2(1, 0, 1) = s_1(0, 1, 2) + s_2(4, 0, 0) \)
   \( m_1 + m_2 = 5 \)
   \( s_1 + s_2 = 3 \)

**Natural Number Solution:** \( m_1 = 1, m_2 = 4, s_1 = 2, s_2 = 1 \)
MATRIX Technique

Want proc for \( f(m, s) \geq \frac{a}{b} \).

1) **Guess** that the only piece sizes are \( \frac{a}{b}, \ldots, \frac{b-a}{b} \)

2) **Muffin**=pieces add to 1: Vectors \( \vec{v}_i \). \( x \) types. 
\( m_i \) muffins of type \( \vec{v}_i \)

3) **Student**=pieces add to \( \frac{m}{s} \): Vectors \( \vec{u}_j \). \( y \) types. 
\( s_j \) students of type \( \vec{u}_j \)

4) **Set up equations:**
\[ m_1 \vec{v}_1 + \cdots + m_x \vec{v}_x = s_1 \vec{u}_1 + \cdots + s_y \vec{u}_y \]
\[ m_1 + \cdots + m_x = m \]
\[ s_1 + \cdots + s_y = s \]

5) **Look for Nat Numb sol.** If find can translate into procedure.
Scott Huddleston has an algorithm that is REALLY FAST and seems to ALWAYS WORK. We are still trying to figure that out.
More Is Known

1) For a fixed $s$, $m \geq \frac{s^3 + 2s^2 + s}{2} \implies f(m, s) = FC(m, s)$.
1-E) Empirical: $m \geq 0.63s^2 \implies f(m, s) = FC(m, s)$.

2) Have formulas for $f(m, s)$ with $1 \leq s \leq 12$
2-F) Future: Working on using ML to derive procedures.

3) Have formulas for $f(3ad + a + d, 3ad + a)$ for $1 \leq d \leq 8$
3-F) FUN with Algorithms Paper has bizarre conjecture about fml.

4) Have program that finds procedures if there is one.
4-E) Empirical: Works very well in practice – most of the time.

5) There is far more I could talk about.
5-E) Empirical: Far more than you want to know.
1) Is computing $f(m, s)$ poly in $\lg m, \lg s$? NP? (No, No)
2) Is computing $f(m, s)$ poly in $m, s$? NP? (Yes, Yes)
3) Is our algorithm efficient? (Yes)
4) Does our algorithm always work? (Yes)
5) Are there other Upper Bound Techniques? (Define “other”)
6) Are there other Lower Bound Techniques? (Define “other”)
7) Does $f(m, s)$ only depend on $m/s$? (Yes)
7-F) Scott Huddleston claims proof. We are in contact.