Open Problems Column
Edited by William Gasarch

This Issue’s Column!

This issue’s Open Problem Column is by William Gasarch and Erik Metz. It is on Generalizing the 3SUM Problem.

Request for Columns!

I invite any reader who has knowledge of some area to contact me and arrange to write a column about open problems in that area. That area can be (1) broad or narrow or anywhere inbetween, and (2) really important or really unimportant or anywhere inbetween.

Generalizing the 3SUM Problem
By William Gasarch\(^1\) and Erik Metz\(^2\)

1 3SUM-Hardness and Completeness

Def 1.1 3SUM is the following problem:

1. Input: A set \(A\) of \(n\) integers.

2. Output: YES if there is \(x, y, z \in A\) such that \(x + y + z = 0\), NO otherwise.

3. Caveat: The complexity of an algorithm is the number of operations. Hence we count one multiplication, even if the numbers involved are huge, to be one step.

Def 1.2 An algorithm is subquadratic if there exists an \(\epsilon > 0\) such that the algorithm runs in time \(O(n^{2-\epsilon})\).

There is an \(O(n^2)\) algorithm for 3SUM. Is there a subquadratic algorithm? The consensus is that there is not.

Imagine if we did not have the Cook-Levin theorem, but the consensus was that SAT was hard. We could still define notions of hardness and even completeness. This is what Gajentaan and Overmars [GO12] did in the context of 3SUM where there is no analog to the Cook-Levin Theorem.

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**Def 1.3** Let $A$ and $B$ be problems.

1. $A \leq B$ if, given an oracle for $B$, one can solve $A$ in subquadratic time.
2. A problem $B$ is 3SUM-hard if $3\text{SUM} \leq B$.
3. A problem $B$ is 3SUM-complete if $3\text{SUM} \leq B$ and $B \leq 3\text{SUM}$.


2 Generalizing 3SUM

**Def 2.1** Let $a, b, c \in \mathbb{Z}$. The abcSUM problem is as follows: Given a set of $n$ integers, $A$, is there $x, y, z \in A$ such that $ax + by + cz = 0$.

If $a = 0$ or $b = 0$ or $c = 0$ then abcSUM is 2SUM which is in $O(n)$ time. If $a, b, c$ share a factor than you can just divide by it. What about the other cases? Henceforth we will assume $a \neq 0$, $b \neq 0$, and $c \neq 0$. We will also assume that $a, b, c$ have no common factors.

3 abcSUM is $\leq$ 3SUM

**Theorem 3.1** Let $a, b, c \in \mathbb{Z}$. Then abcSUM $\leq$ 3SUM.

**Proof:** Assume there is an $O(n^{2-\epsilon})$ algorithm for 3SUM. Here is an $O(n^{2-\epsilon})$ algorithm for abcSUM.

1. Input $A$.

2. Let

$$A' = \{7ar + 1 : r \in A\} \cup \{7br + 2 : r \in A\} \cup \{7cr - 3 : r \in A\}.$$ 

3. Run the 3SUM algorithm on $A'$. If it says YES, output YES. If it says NO, then output NO.
This algorithm is clearly in \( O(n^{2-\epsilon}) \) time. We show that it is correct.

**If alg says YES then there is** \( x, y, z \in A \) **with** \( ax + by + cz = 0 \).

Assume the algorithm says YES. Then there is an \( x, y, z \in A' \) such that \( x + y + z = 0 \). Let \( x = 7r_1 + d_1 \), \( y = 7br_2 + d_2 \), \( z = 7cr_3 + d_3 \) where \( r_1, r_2, r_3 \in A \) and \( d_1, d_2, d_3 \in \{1, 2, -3\} \).

\[
(7r_1 + d_1) + (7br_2 + d_2) + (7cr_3 + d_3) = 0
\]

\[
d_1 + d_2 + d_3 \equiv 0 \pmod{7}.
\]

By cases one can see that you must have \( \{d_1, d_2, d_3\} = \{1, 2, -3\} \).

Since \( x + y + z = 0 \) we have \( ar_1 + br_2 + cr_3 = 0 \).

**If there is** \( x, y, z \in A \) **with** \( ax + by + cz = 0 \) **then alg says YES.**

Assume that there is an \( x, y, z \in A \) such that \( ax + by + cz = 0 \). Then

\[
(7ax + 1) + (7by + 2) + (7cz - 3) = 7(ax + by + cz) = 0
\]

Hence the algorithm will find this triple and say YES. \( \blacksquare \)

**4 For Many** \( a, b, c \): 3SUM \( \leq \) abcSUM

**Def 4.1** Let \( a, b, c \in \mathbb{Z} \) with \( a, b, c \neq 0 \) and \( a, b, c \) have no common factor. \((a, b, c)\) are cool if there exists \( D \in \mathbb{N} \) and \( k_1, k_2, k_3 \in \mathbb{Z} \) (all distinct) such that the following hold:

- \( ak_1 + bk_2 + ck_3 = 0 \).
- The only solution to

\[
ak' + bk'' + ck''' \equiv 0 \pmod{D}
\]

with \( k', k'', k''' \in \{k_1, k_2, k_3\} \) (repeats allowed) is \( k_1, k_2, k_3 \).

**Theorem 4.2** Let \( a, b, c \in \mathbb{Z} \) be cool. Then 3SUM \( \leq \) abcSUM.
Proof: Assume there is an $O(n^{2-\epsilon})$ algorithm for abcSUM. Here is an $O(n^{2-\epsilon})$ algorithm for 3SUM.

Let $D, k_1, k_2, k_3$ be from $(a, b, c)$ being cool.

1. Input $A$.
2. Let
   
   $A' = \{Dbr + k_1 : r \in A\} \cup \{Dac + k_2 : r \in A\} \cup \{Dab + k_3 : r \in A\}$.

3. Run the abcSUM algorithm on $A'$. If it says YES, output YES. If it says NO, then output NO.

This algorithm is clearly in $O(n^{2-\epsilon})$ time. We show that it is correct.

If alg says YES then there is a triple in $A$ that sums to 0.
Assume that there is an $x, y, z \in A'$ such that $ax + by + cz = 0$. Let

$x = DR_1 + k'$,
$y = DY_2 + k''$,
$z = DZ_3 + k'''$.

where $X, Y, Z \in \{bc, ac, ab\}$ and $k', k'', k''' \in \{k_1, k_2, k_3\}$. In both cases
repeats are allowed.

Take the equation $ax + by + cz = 0$ mod $D$ to get

$$ak' + bk'' + ck''' \equiv 0 \pmod{D}.$$ 

Since $D, k_1, k_2, k_3$ are cool we have that $k' = k_1$, $k'' = k_2$, and $k''' = k_3$. Hence we may assume that $X = bc, Y = ac,$ and $Z = ab$. So

$x = Dbr_1 + k_1$
$y = Dacr_2 + k_2$
$z = Dabr_3 + k_3$.

Since $ax + by + cz = 0$ we have

$$(Dbr_1 + ak_1) + (Dacr_2 + bk_2) + (Dabr_3 + ck_3) = 0$$

$$Dabc(r_1 + r_2 + r_3) + (ak_1 + bk_2 + ck_3) = 0$$

Since $D, k_1, k_2, k_3$ is cool, $ak_1 + bk_2 + ck_3 = 0$. Hence

$$r_1 + r_2 + r_3 = 0.$$
So we have a triple in $A$ that sums to 0.

**If there is triple in $A$ that sums to 0 then alg says YES.**

If $r_1, r_2, r_3 \in A$ and $r_1 + r_2 + r_3 = 0$ then

\[
x = Dbc r_1 + k_1
\]

\[
y = Dac r_2 + k_2
\]

\[
z = Dabr_3 + k_3.
\]

are all in $A'$ and

\[
ax + by + cz = Dabc(r_1 + r_2 + r_3) + ak_1 + bk_2 + ck_3 = 0.
\]

Hence the algorithm will output YES.

\[
\]

## 5 Open Questions

If $a + b + c = 0$ then $(a, b, c)$ is not cool (we leave this proof to the reader). Hence Theorem 4.2 will not cover all $(a, b, c)$.

1. Show that for all $(a, b, c)$ with $a + b + c = 0$, $3\text{SUM} \leq abc\text{SUM}$.

2. Show that if $abc \neq 0$ and $(a, b, c)$ is not cool then $a + b + c = 0$?

3. Disproof either of the above.

## References


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