Open Problems Column
Monochromatic Unit Squares: Exposition and Open Problems

William Gasarch*, Auguste Gezalyan†Ryan Parker‡

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Open Problem Column Edited by William Gasarch

1 Request for Columns!

I invite any reader who has knowledge of some area to contact me and arrange to write a column about open problems in that area. That area can be (1) broad or narrow or anywhere inbetween, and (2) really important or really unimportant or anywhere inbetween.

2 The Set Up

BILL: I have a nice problem to tell you about. First, the setup. Say you have a finite coloring of $\mathbb{R}^n$. A *mono unit square* is a set of four points that are (a) all the same color, and (b) form a square of side 1. The square does not need to be parallel to any of the axes.

DARLING: Okay. What is the problem?

BILL: It is known that for all 2-colorings of $\mathbb{R}^6$ there is a mono unit square.

DARLING: $\mathbb{R}^6$? Really! Thats hilarious! Surely, better is known.

BILL: Yes better is known. And stop calling me Shirley.

DARLING: Okay, so what else is known.

BILL: An observation about the $\mathbb{R}^6$ result gives us the result for $\mathbb{R}^5$. (The $\mathbb{R}^5$ result also follows from a different technique.) Then a much harder proof gives us the result for $\mathbb{R}^4$. It is easy to show a coloring of $\mathbb{R}^2$ without a mono unit square. The problem for $\mathbb{R}^3$ is open.

DARLING: That's to bad. We live in $\mathbb{R}^3$.

*University of Maryland College Park
†University of Maryland College Park
‡University of Maryland College Park

3 Introduction

Notation 3.1 Let $a, n \in \mathbb{N}$.

1. $[n] = \{1, \ldots, n\}$.
2. If A is a set then $\binom{A}{a}$ is the set of all a-subsets of A. Henceforth, let $\binom{[n]}{2}$ be the complete graph on n vertices.
3. C_n is the cycle on n vertices, defined by:
   $C_n = (V, E)$ where $V = [n]$ and $E = \{(i, i+1 \pmod{n}) \colon 1 \leq i \leq n\}$.

Definition 3.2 Let $c, d \geq 2$. Let $k \geq 3$.

1. Let $\text{COL}\colon \mathbb{R}^d \rightarrow [c]$ be a given coloring. A *monochromatic unit square* (henceforth *mono unit square*) is a square in $\mathbb{R}^d$ whose vertices are all the same color and sides all have length 1.
2. $d(c)$ is the least d such that the following is true: *For all* $\text{COL}\colon \mathbb{R}^d \rightarrow [c]$ *there is a mono unit square.*
3. $R_c(C_k)$ is the least n such that, for all $\text{COL}\colon \binom{[n]}{2} \rightarrow [c]$ there exists a monochromatic (henceforth mono) cycle of length k.

The following are known:

1. Burr proved that $d(2) \leq 6$. He did not publish the result; however, it appears (crediting him) in a paper by Erdős et al. [3]. The proof uses the following theorem: $R_2(C_4) = 6$. The accounts of Burr's result that we have seen say that $R_2(C_4) = 6$ is either *well-known* or *easy* and do not give a reference, which is Chvátal & Harary [2]. It may be well-known and easy for some people; however, for others it can be inaccessible.
2. Cantwell [1] claims that a trivial change in the $d(2) \leq 6$ proof yields $d(2) \leq 5$. No reference or proof is given. We note that the change may be trivial for some, and is perhaps trivial *once you see it.*
3. Cantwell [1] showed $d(2) \leq 4$. This proof is very different from the proofs of $d(2) \leq 6$ and $d(2) \leq 5$.

In this article, we do the following:

1. Present the complete proof that $R_2(C_4) = 6$. Our presentation includes pictures that will make it easier to follow than the Chvátal & Harary paper.
2. Present the complete proof of $d(2) \leq 6$ and $d(2) \leq 5$ including the parts that are allegedly easy.
3. Present bounds on $d(c)$. These results appear to be new.
4. Present Cantwell's proof that $d(2) \leq 4$. Our presentation includes pictures that will make it easier to follow than Cantwell's paper.

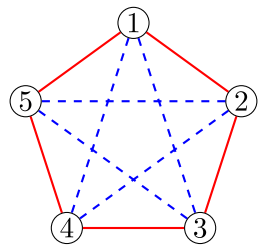


Figure 1: $R_2(C_4) \geq 6$

5. Present some open problems.

Convention 3.3 We use **RED** and **BLUE** for the actual colors. So we might say COL$(1,2) =$ **RED**. We use the terms *red* and *blue* in prose. So we might say *Since* COL$(1,2) =$ **BLUE** *we have a blue* C_4. The symbol **RED** (**BLUE**) will appear red (blue) if you are reading this paper in color and in normal font (black) if you are not.

4 A Lemma Needed To Prove $d(2) \leq 6$

In the figures in this section, a dotted line is a blue line. That is only important if you are reading this paper in black and white.

The following is due to Chvátal & Harary [2].

Theorem 4.1 $R_2(C_4) = 6$.

Proof:
1) $R_2(C_4) \geq 6$:

We present a COL: $\binom{[5]}{2} \rightarrow [2]$ with no mono C_4. Figure 1 is a 2-coloring of $\binom{[5]}{2}$ with no mono C_4. (If you are reading this in black and white instead of color, then the coloring is that the cycle $1-2-3-4-5$ is all red edges and the rest of the edges are blue.)

Note 4.2 The coloring in Figure 1 has a mono C_5 but not a mono C_4. The study of $R_c(C_k)$ is very different from the study of the usual Ramsey numbers. While a graph with a mono K_n also has a mono K_{n-1}, a graph that has a mono C_n does not guarantee the existence of a mono C_{n-1}.

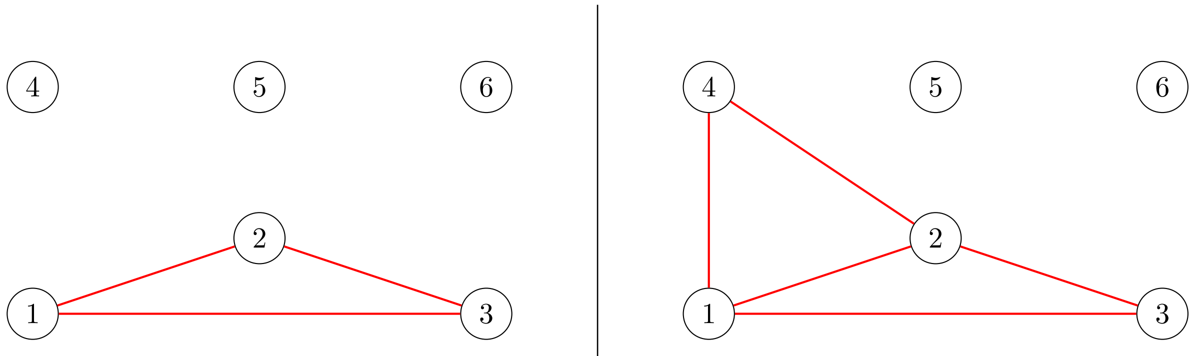
2) $R_2(C_4) \leq 6$:

Let COL: $\binom{[6]}{2} \rightarrow [2]$. By standard Ramsey Theory there is a mono K_3. We assume that it is red on the vertices $\{1,2,3\}$. See Figure 2a.

We view $\{1,2,3\}$ and $\{4,5,6\}$ as the sides of a bipartite graph.

Notation 4.3

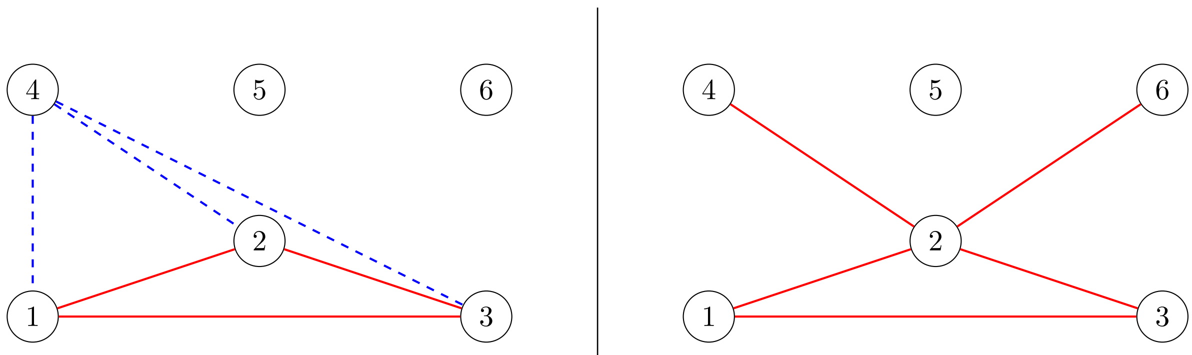
1. If $a \in \{1,2,3\}$ then $\deg_{\mathbf{RED}}(a)$ is the number of red edges between a and $\{4,5,6\}$.
2. If $a \in \{4,5,6\}$, then $\deg_{\mathbf{RED}}(a)$ is the number of red edges between a and $\{1,2,3\}$.



(a) Red Triangle

(b) $\deg_{\textbf{RED}}(v) \geq 2$ yields a mono C_4

Figure 2



(a) $\deg_{\textbf{BLUE}}(4) = 3$

(b) $\deg_{\textbf{RED}}(v) \geq 2$ yields a mono C_4

Figure 3

There are several cases. Each case assumes the negation of the prior ones.

Case 1 $(\exists v \in \{4,5,6\})[\deg_{\textbf{BLUE}}(v) \leq 1]$. The situation is pictured in Figure 2b where v with $\deg_{\textbf{BLUE}}(v) \leq 1$ is vertex 4. Note that $1-4-2-3-1$ is a mono C_4.

Case 2 $(\exists v \in \{4,5,6\})[\deg_{\textbf{BLUE}}(v) = 3]$. The situation is pictured in Figure 3a with the relevant vertex being vertex 4.

From the negation of Case 1, $\deg_{\textbf{BLUE}}(5) \geq 2$. If $\mathrm{COL}(5,1) = \mathrm{COL}(5,2) =$ **BLUE** then there is a blue C_4: $5-1-4-2-5$. The cases of $\mathrm{COL}(5,2) = \mathrm{COL}(5,3)$ and $\mathrm{COL}(5,1) = \mathrm{COL}(5,3)$ are symmetric.

By the negation of Case 1 and Case 2 we have that, for all $v \in \{4,5,6\}$, $\deg_{\textbf{BLUE}}(v) = 2$, hence $\deg_{\textbf{RED}}(v) = 1$. We will use this in Case 4.

Case 3 $(\exists v \in \{1,2,3\})[\deg_{\textbf{RED}}(v) \geq 2]$. We can assume $v = 2$. The situation is pictured in Figure 3b. From the negation of Case 2 we also know that $\deg_{\textbf{BLUE}}(4) = \deg_{\textbf{BLUE}}(6) = 2$. Hence there is a blue C_4: 4-1-6-3-4.

Case 4 Negation of Cases 1,2,3. So we have the following.

1. $(\forall v \in \{1,2,3\}[\deg_{\textbf{RED}}(v) = 1]$.

2. $(\forall v \in \{4,5,6\}[\deg_{\textbf{RED}}(v) = 1]$.

3. Hence we can assume

   (a) $\mathrm{COL}(1,4) = \mathrm{COL}(2,5) = \mathrm{COL}(3,6) =$ **RED**.

   (b) All other edges between $\{1,2,3\}$ and $\{4,5,6\}$ are blue. (We will find some other edges that must be blue.)

The situation is pictured in Figure 4. If any of $(4,5)$, $(5,6)$, or $(4,6)$ is **RED** then there will be a red C_4. This implies that those three are blue, creating a blue C_4: $4-5-6-2-4$.

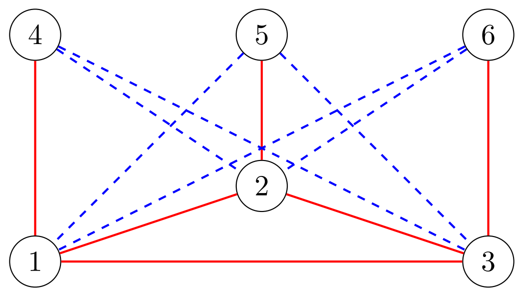


Figure 4: Negation of Cases 1,2,3,4

▮

Open Problem 4.4 The proof of Theorem 4.1 took four cases. Some of the cases had sub-cases. Is there a proof with fewer cases. Perhaps we begin with the fact that any 2-coloring of the edges of K_6 has *two* mono triangles. That result is possibly folklore; however, a general theorem from which it follows is Goodman [5]. An easier proof was obtained by Schwenk [10]. Also see an open problems column on Ramsey multiplicity by Gasarch [4].

5 $d(2) \le 6$ and $d(2) \le 5$

The following result is due to Burr (unpublished but credited by Erdős et al. [3]).

Theorem 5.1

1. $d(2) \le 6$.

2. $d(2) \le 5$.

Proof:
1) $d(2| \le 6$.

Let $\mathrm{COL}\colon \mathbb{R}^6 \to [2]$. We form a coloring $\mathrm{COL}'\colon \binom{[6]}{2} \to [2]$.
Let
$p_{1,2} = (\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0, 0, 0, 0)$.
$p_{1,3} = (\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}}, 0, 0, 0)$.

$\vdots \quad \vdots \quad \vdots$
$p_{5,6} = (0,0,0,0,\frac{1}{\sqrt{2}},\frac{1}{\sqrt{2}})$.

Define $\mathrm{COL}'(i,j) = \mathrm{COL}(p_{i,j})$.
By Theorem 4.1 there exists a mono C_4. Let the vertices be a,b,c,d and the color be red. Then

$$\mathrm{COL}'(a,b) = \mathrm{COL}'(b,c) = \mathrm{COL}'(c,d) = \mathrm{COL}'(d,a) = \textbf{RED}$$

hence

$$\mathrm{COL}(p_{a,b}) = \mathrm{COL}(p_{b,c}) = \mathrm{COL}(p_{c,d}) = \mathrm{COL}(p_{d,a}) = \textbf{RED}.$$

It it easy to see that

$$d(p_{a,b},p_{b,c}) = d(p_{b,c},p_{c,d}) = d(p_{c,d},p_{d,a}) = d(p_{d,a},p_{a,b}) = 1.$$

It is also easy to see that the lines formed by consecutive pairs are orthogonal as the dot product of the induced vectors is 0. Hence we have a mono unit square.

2) $d(2) \le 5$
This follows from the proof of Part 1 since all of the $p_{i,j}$ are on a hyperplane which is $\mathbb{R}^5$. ∎

Tóth [11] has proven a theorem that implies Theorem 5.1.

Theorem 5.2 *Let T be rectangle and $c \in \mathbb{N}$.*

1. *For all 2-colorings of $\mathbb{R}^5$ there is a mono rectangle that is congruent to T.*
2. *For all c-colorings of $\mathbb{R}^{c^2+3c^{3/2}}$ there is a mono rectangle that is congruent to T. (Tóth stated the result as $\mathbb{R}^{c^2+o(c^2)}$ but the proof clearly shows $\mathbb{R}^{c^2+3c^{3/2}}$*

Tóth's proof restricted to unit mono squares is very different from the proof of Theorem 5.1.

6 $d(2) \le 4$

The following theorem was proven by Kent Cantwell [1].

Theorem 6.1 *For all* $\mathrm{COL}\colon \mathbb{R}^4 \to [2]$ *there exists a mono unit square.*

We present his proof.
We defer the proof of Theorem 6.1 until after we prove many lemmas.
We make the following assumptions throughout the proofs of the lemmas.

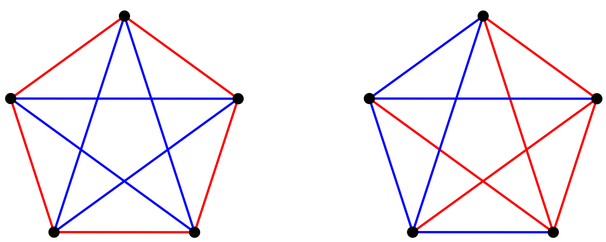
1. There is a given coloring $\mathrm{COL}\colon \mathbb{R}^4 \to [2]$. If we refer to a color of a *point in* $\mathbb{R}^4$, we are referring to COL.

2. We define $\binom{[5]}{2}$ points in $\mathbb{R}^5$, however, we will soon see that they fall on a $\mathbb{R}^4$ hyperplane.
   $p_{1,2} = (\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0, 0, 0)$.
   $p_{1,3} = (\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}}, 0, 0)$.
   $\vdots \quad \vdots \quad \vdots$
   $p_{4,5} = (0, 0, 0, \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}})$.
   These are a subset of the points in $\mathbb{R}^5$ such that $x_1 + x_2 + x_3 + x_4 + x_5 = \sqrt{2}$, hence we will regard these point as being in $\mathbb{R}^4$.
3. We define $\mathrm{COL}' \colon \binom{[5]}{2} \to [2]$ by $\mathrm{COL}'(i,j) = \mathrm{COL}(p_{i,j})$. We will refer to this graph as K_5. If we refer to the color of *an edge*, we are referring to COL'.

Definition 6.2 A *mono unit tetrahedron* are four points in $\mathbb{R}^4$ that (a) are the same color, (b) form a tetrahedron where each edge is length 1.

We start with the following observation (without proof) that has no relation to COL or COL'.

Lemma 6.3 *There are exactly two 2-colorings of the edges of K_5 that do not have a mono C_4. (They are shown in Figure 5).*



(a) (b)

Figure 5: The two colorings on 5 vertices with no mono C_4.

Lemma 6.4 *If there are four mono points (with respect to COL) that form a unit tetrahedron, then in the K_5 graph mapped on top of those four points, there is a vertex that has four mono edges coming out of it (with respect to COL').*

Proof: We can view the four mono points as being $p_{a,b}, p_{a,c}, p_{a,d}, p_{a,e}$.
Since

$$\mathrm{COL}(p_{a,b}) = \mathrm{COL}(p_{a,c}) = \mathrm{COL}(p_{a,d}) = \mathrm{COL}(p_{a,e}),$$

This maps onto the K_5 graph where

$$\mathrm{COL}'(a,b) = \mathrm{COL}'(a,c) = \mathrm{COL}'(a,d) = \mathrm{COL}'(a,e).$$

(see Figure 6).

∎

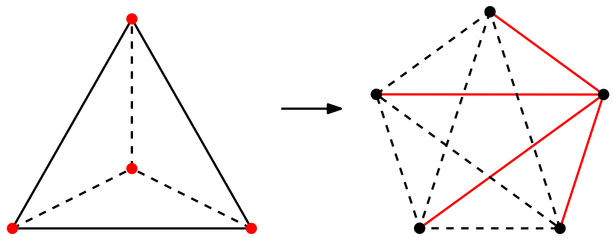


Figure 6: The tetrahedron formed by a vertex incident to 4 mono edges on the standard configuration (All 4 points are equidistant to each other since they share a vertex on the graph).

Now we are ready to make the following observation.

Lemma 6.5 *If the K_5 graph has four mono edges coming out of a single vertex, then there exists a mono unit square.*

Proof: We start with a K_5 graph and renumber such that

$$\mathrm{COL}'(a,b) = \mathrm{COL}'(a,c) = \mathrm{COL}'(a,d) = \mathrm{COL}'(a,e).$$

Lemma 6.3 gives the only two ways that the edges of K_5 can be colored and not have a mono C_4. Neither of those colorings has a vertex with 4 mono edges that use it. Hence the coloring of K_5 has a mono C_4. With renumbering (using numbers intentionally in order to separate from the letters assigned above):

$$\mathrm{COL}'(1,2) = \mathrm{COL}'(2,3) = \mathrm{COL}'(3,4) = \mathrm{COL}'(4,1).$$

Therefore

$$\mathrm{COL}(p_{1,2}) = \mathrm{COL}(p_{2,3}) = \mathrm{COL}(p_{3,4}) = \mathrm{COL}(p_{4,1}).$$

Clearly $p_{1,2}, p_{2,3}, p_{3,4}, p_{4,1}$ form a mono unit square. ∎

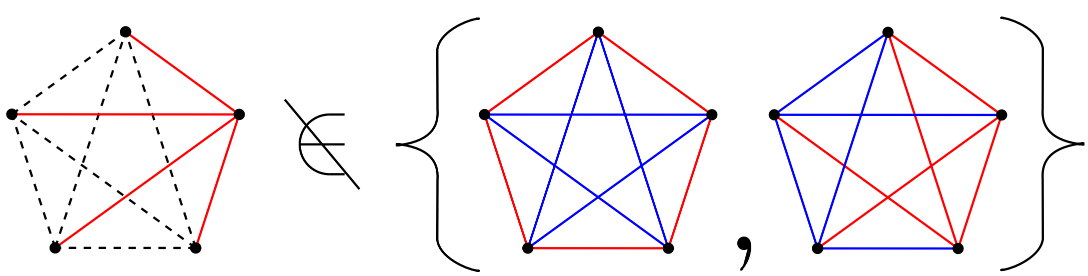


Figure 7: Neither graph in Lemma 6.3 can be constructed if a vertex has four mono edges.

Now that we know that we cannot have a mono tetrahedron, we can further eliminate situations in which we get a mono square.

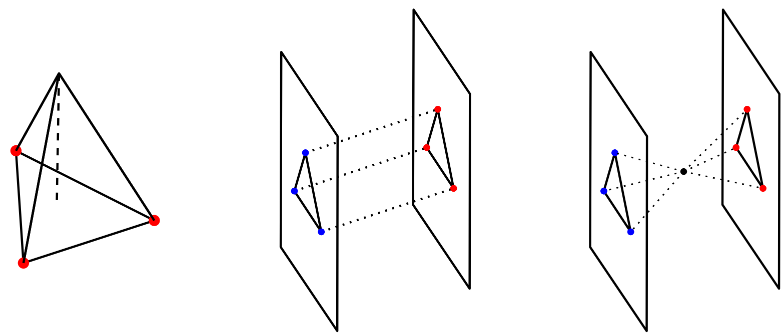
Lemma 6.6 *If there exists a mono unit tetrahedron, then there exists a mono unit square.*

Proof: We start with a mono unit tetrahedron. As stated in Lemma 6.4, this tetrahedron can be represented by a K_5 graph with four mono edges coming from a single vertex.

In Lemma 6.5, we show that this graph implies the existence of a mono unit square.

As such, the existence of a mono unit tetrahedron implies that there is a mono unit square ∎

Lemma 6.7 *Given two differently colored mono triangles (vertices), T_1 and T_2 in $\mathbb{R}^4$, which exist in two different $\mathbb{R}^2$ planes that are parallel to each other and orthogonal to each other with respect to the line-segment connecting their centroids (they can be reflected), their distance must be greater than $2\sqrt{2}/\sqrt{3}$ apart.*



(a) (b) (c)

Figure 8: (a) A tetrahedron with a monochromatic triangle base, (b) Representation of parallelism as described in Lemma 6.7, (c) No matter how the center point is colored, a mono tetrahedron is formed. Note, these figures are not geometrically accurate.

Proof: Suppose we are given two triangles $T_1, T_2 \subset \mathbb{R}^4$, such that both lie in parallel planes to each other (see Figure 8 b). Then there exists a circle C_1 centered at the center of T_1 in a plane perpendicular to it and a circle C_2 centered at the center of T_2 in a plane perpendicular to it such that C_1 is equidistant to all the vertices of T_1, and C_2 to the vertices of T_2.

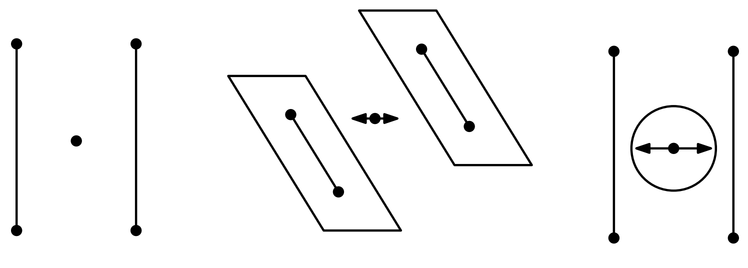
We proceed by calculating the radii of these circles. Since the centers of the circles are necessarily the centroids of the triangles, it suffices to calculate the height of the tetrahedron formed by adding a point in $\mathbb{R}^3$ to one of these triangles, 1 away from the triangles vertices (see Figure 8 (a)). This point must necessarily lie on the circle. The height of a unit tetrahedron is $\sqrt{2}/\sqrt{3}$.

If the triangles are closer than $2\sqrt{2}/\sqrt{3}$ then these circles intersect at least 1 point. No matter how this point is colored, a monochrome tetrahedron can be formed by connecting it to the vertices of one of the triangles (see Figure 8 (c)). By Lemma 6.6, this necessitates a mono unit square. ∎

We now briefly describe the set of points equidistant from two parallel line segments. This will let us address when we have two parallel triangles with a pair of vertices of opposite colors.

Lemma 6.8 *Given two unit segments in $\mathbb{R}^4$ that exist in parallel planes and are translates of each other by some distance d in an orthogonal direction, for sufficiently small d, there exists a circle of points that are 1 away from the vertices of both segments.*

Proof: If we confine the segments above to the $\mathbb{R}^2$ plane they lie in, we have a single point, p, that is equidistant to the vertices of both segments see Figure 9 (a). When we add a third dimension, we now have a line of points through p (see Figure 9 (b)) all equidistant to the vertices. Adding a fourth spatial dimension creates a plane such that any point on the plane is equidistant to the vertices. Given a specific distance, the points on the plane equidistant to the vertices form a circle (see Figure 9 (c)). ∎

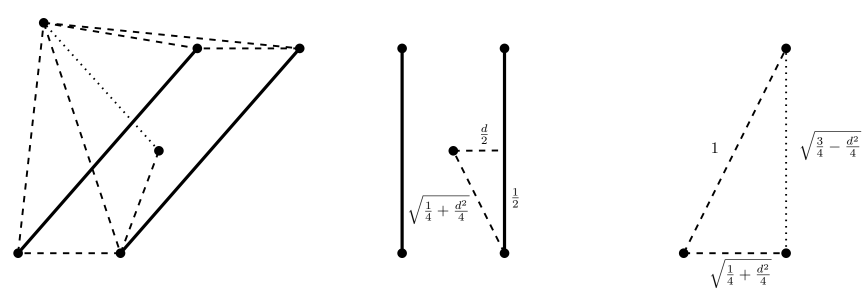


(a) (b) (c)

Figure 9: (a)) Equidistant point between two vertices in 2D, (b) Equidistant line between two vertices in 3D, (c) Circle of equidistant points to the segments at a specific distance (not geometrically accurate)

Lemma 6.9 *The radius of the circle described in Lemma 6.8. is* $\sqrt{3/4 - d^2/4}$, *where* d *is the distance between the segments.*

Proof: To compute the radius of the circle described in Lemma 6.8., it suffices to compute the height of the rectangular pyramid generated by the point 1 away from the endpoint of the segments in $\mathbb{R}^3$ (see Figure 10 (a)). Computing the distance from a corner of the rectangular base to the center of the base is $\sqrt{1/4 + d^2/4}$ (see Figure 10 (b)). Using that, we can compute the height of the pyramid since the length of its edges are 1. Using the pythagorean theorem, we therefore get the radius of the circle is $\sqrt{3/4 - d^2/4}$ (see Figure 10 (d)). ∎



(a) (b) (c)

Figure 10: (a) The rectangular pyramid, (b) center of the base of the pyramid, (c) the height of the pyramid (not geometrically accurate).

Using this, we can determine that there is a dense set of distances between two mono parallel line segments of different colors such that if those segments are that distance apart, there is a mono square

Lemma 6.10 *There exists a dense set S, so that if two parallel unit mono segments of different colors are a distance $d \in S$ apart, there is a mono unit square.*

Proof: By Lemma 6.8., given a distance in $d \in (0, \sqrt{3})$, there exists a circle of points 1 away from the vertices of both segments. By Lemma 6.9., the radius of this circle is:

$$\sqrt{3/4 - d^2/4}.$$

If $d < \sqrt{2}$, There is some angle θ, such that if we take a point on the circle and rotate it by θ, the point is 1 away from the original. The length of a chord in this circle with a given angle θ is:

$$2 * r * \sin(\theta/2),$$

where r is the radius. Solving for θ gives us:

$$\theta = 2 \arcsin\left(\left(2\sqrt{3/4 - d^2/4}\right)^{-1}\right).$$

Whenever θ is a rational multiple of π, rotating a point around the circle at that angle will return us to our original position in a finite number of rotations. If the base of the fraction is an odd number, it returns to the original position after an odd number of rotations. In this situation, we have a set of points of odd length that are two colored, and so two adjacent ones are the same color. This implies that two points on the circle 1 apart are the same color (see Figure 11).

Since all the points on the circle are 1 away from the vertices of the segments, we necessarily have 4 points of the same color all one away from each-other. This gives us a mono tetrahedron, and by Lemma 6.6 this gives a mono square. Hence $d \in (0, \sqrt{2})$ since we need the radius to be large enough for the chord to exist. We call this set of distances S. ∎

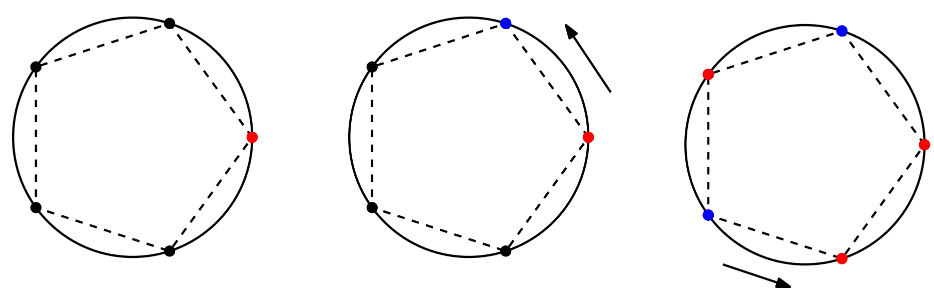


Figure 11: An odd rotation of a point around a circle, resulting in adjacent vertices the same color.

Lemma 6.11 *Given a pair of unit triangles in parallel planes that are translates in some direction orthogonal to the planes they lie on and a distance $s \in S$ apart, the triangles cannot each have 2 or more vertices of opposite colors without creating a mono unit square.*

Proof: If this is true, then we have two parallel segments that fit the criterion of 6.10. ∎

Lemma 6.12 *If two parallel mono equilateral unit triangles that are translates in some direction orthogonal the planes they lie on of the same color are closer than $\sqrt{2}$ to each other, there is a mono unit square.*

Proof:

Consider two parallel mono unit triangles A_1 and A_2 with centroids a_1 and a_2 respectively (see Figure 12 (a)). Suppose the distance from a_1 to a_2 is less than $\sqrt{2}$. Consider the circle C_2, the unit radius circle around a_2 in a plane perpendicular to the plane of A_2. Let C_1 be a circle centered at a_1 in a plane perpendicular to A_1 whose radius is in S that intersects C_2. Let an intersection point of the circles be q. Consider the triangle that is a translate of A_2 centered (centroid) at q parallel to A_1 and A_2. This triangle is a unit translate of A_2 in a direction perpendicular to the plane it is on, thus if any pair of vertices of this triangle are the same color as A_2 they will form a mono unit square with the corresponding vertices of A_2 (see Figure 12 (b)).

Now suppose they don't, then we have two vertices of opposite colors to the vertices of A_1 and A_2. So, if we look at our new triangle and A_1 we see that we have two distinctly colored mono segments with distance d $\in S$, leading to a square monochromatic unit according to Lemma 6.11 (see Figure 12 (c)). ∎

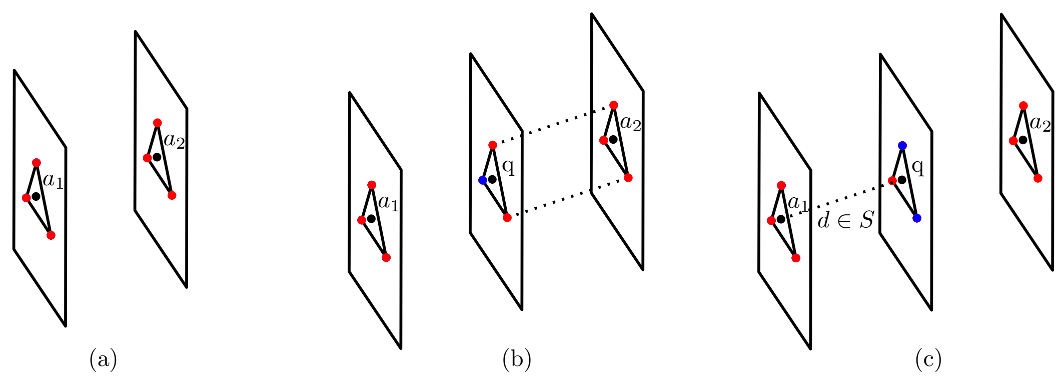


Figure 12: (a) The two triangles, (b) The mono unit square formed with the vertices of A_2, (c) $A1$ and the triangle centered at q are a distance $d \in S$ (not geometrically accurate).

We now quickly introduce the concept of a unit cross-polytope. The unit cross polytope in $\mathbb{R}^d$ is the polytope defined by going $1/\sqrt{2}$ on every axis and connecting them (see Figure 13.).

Lemma 6.13 *In an $\mathbb{R}^4$ space without a mono unit square, the four-dimensional unit cross-polytope will have 32 equilateral unit triangular faces, 4 of which will be mono (as seen in Figure 14. (b)).*

Proof:

The graph of a 4-dimensional unit cross-polytope is represented by 8 nodes, where each node is connected to every node other than its opposite (the vertex on the same coordinate axis as itself). Each axis contributes a mono or dichromatic pair. We show that the only way to color this without getting a mono unit square uses 2 dichromatic pairs of opposite vertices and 2 mono pairs of

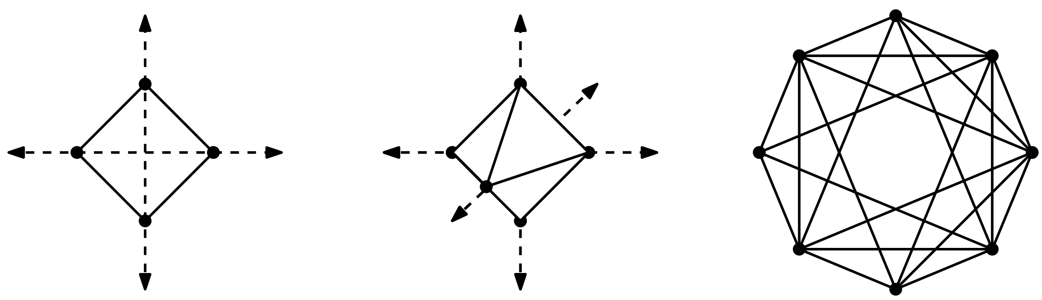


Figure 13: (a) A $\mathbb{R}^2$ cross-polytope (b) A $\mathbb{R}^3$ cross-polytope (c) The $\mathbb{R}^4$ cross-polytopes graph.

opposite vertices of differing color to each other by using a variety of cases. Each case represents a state that leads to a mono unit square.

Case 1: Number of mono pairs ≥ 3
In the event that there are 3 or more mono pairs, you are guaranteed to have two mono pairs that are the same color. This forms a mono unit square as seen in Figure 14(c).

Case 2: Number of dichromatic pairs ≥ 3
If there are three or more dichromatic pairs, it will form two equilateral unit triangles on parallel planes of distance $\sqrt{2}$, which is less than the required $2\sqrt{2}/\sqrt{3}$ from Lemma 6.7 (see Figure 14d).

Case 3: Number of mono pairs $= 2$ and they share the same color.
Same situation as Case 1, where they form the square.

Therefore, the graph must be made up of 2 dichromatic pairs and 2 distinct mono pairs as seen in Figure 14b. This gives us the 4 mono equilateral unit triangular faces. ▮

Theorem 6.14 *For every COL* $\mathbb{R}^4 \to [2]$*, there exists a mono unit square.*

Proof: Consider the hyper-cube $[0, \sqrt{2}]^4$ in $\mathbb{R}^4$. We subdivide each of the axis of the hyper-cube into intervals of size $\sqrt{2}/m$. This gives us the following lattice (see Figure 15 (a) and (b)):

$$\left(\frac{m_1\sqrt{2}}{m}, \frac{m_2\sqrt{2}}{m}, \frac{m_3\sqrt{2}}{m}, \frac{m_4\sqrt{2}}{m}\right), \text{when } 0 \leq m_i \leq m.$$

At each of our m^4 lattice points, center a unit cross-polytope. By Lemma 6.13, if there are no mono unit squares, each of these must have 4 mono unit triangular faces. Hence, we have at least $4m^4$ mono unit triangles in our hypercube.

Notice that there are 32 possible types of these mono unit triangles up to translation. We claim that each type of triangle can only appear at most $O(m^3)$ times without creating a mono unit square.

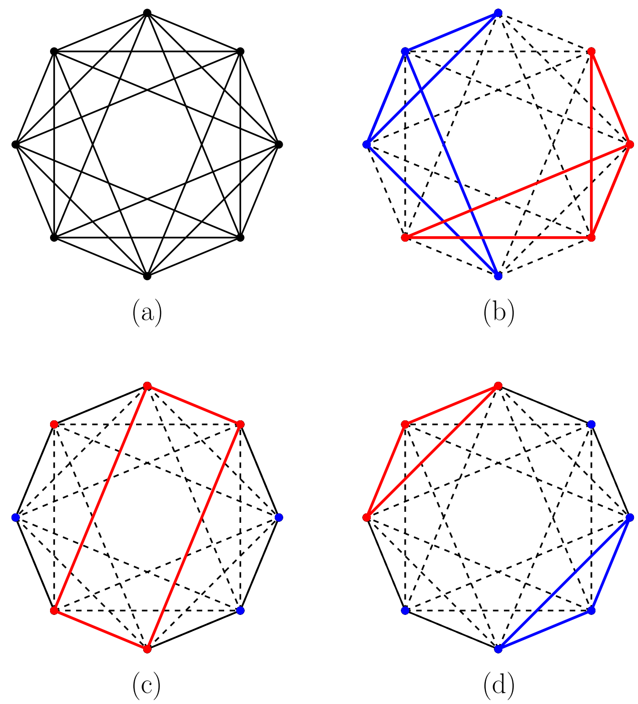


Figure 14: (a) Graph of a 4D Cross-Polytope (b) The 4 equilateral triangles (c) Square formed by >1 mono pair of same color (d) Triangles formed by ≥ 3 mono pairs

This is because if two of our triangles are parallel and translates in a direction perpendicular to the planes they are on we have a monochromatic unit square by Lemma 6.12.

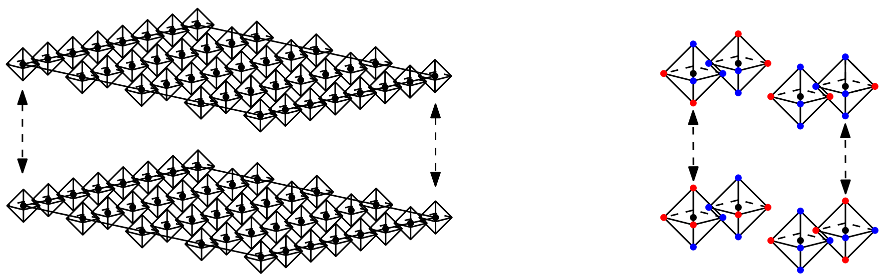
However this would necessitate that we have at most $32 * O(m^3) = O(m^3)$ mono unit triangles which is not true for all m as we have $O(m^4)$ mono unit triangles. Hence, we must have a mono unit square. ∎

Open Problem 6.15 Find an easier proof that $d(2) \leq 4$. Here are some possible directions:

1. Find a proof that is more graph-theoretic (like the proof of Theorem 5.1) and less geometric.
2. Recall that we had a proof that $d(2) \leq 6$ and used it to easily get a proof that $d(2) \leq 5$. Find a different proof of $d(2) \leq 5$ from which you can get a proof that $d(2) \leq 4$.
3. Recall that Tóth [11] has a proof that $d(2) \leq 5$ that was very different from the proof of Theorem 5.1. Perhaps this proof could be modified to get an proof that $d(2) \leq 4$ which is easier than Cantwell's proof.

We state the open problem suggested at the beginning of this paper and name it.

Open Problem 6.16
1. *The Darling Problem:* We know that $d(2) \in \{3, 4\}$. Determine $d(2)$. We do not have a conjecture as to whether $d(2) = 3$ or $d(2) = 4$.
2. *The More Colors Darling Problem* Find he value of c such that the following both hold:



(a) (b)

Figure 15: (a) An illustration of a lattice of cross-polytopes in 3D (b) Coloring the vertices of cross-polytopes on a 3D lattice.

- There exist COL: $\mathbb{R}^3 \to [c]$ with no mono unit square.
- For all COL: $\mathbb{R}^3 \to [c-1]$ there is a mono unit square.

7 If $\chi = 7$ Then $d(2) \le 4$ with an Easy Proof

The Theorem and proof in this section were emailed to us by Dömötör Pálvölgyi.

Notation 7.1 χ is the chromatic number of the following graph:
$V - \mathbb{R}^2$
$E = \{(p,q) \colon |p-q| = 1\}$. χ is called *the chromatic number of the plane.*

It is known that $5 \le \chi \le 7$. A popular conjecture is that $\chi = 7$.

Theorem 7.2 *If* $\chi = 7$*, then* $d(2) \le 4$.

Proof:

Let COL: $\mathbb{R}^4 \to [2]$. Let $t_1 = (0,0)$, $t_2 = (1,0)$, and $t_3 = (\frac{1}{2}, \frac{\sqrt{3}}{2})$. Note that t_1, t_2, t_3 is a unit equilateral triangle. If $p \in \mathbb{R}^2$, and $i \in \{1,2,3\}$, then (p, t_i) is the obvious point in $\mathbb{R}^4$.

We create a coloring $\mathrm{COL}' \colon \mathbb{R}^2 \to [6]$ as follows:

Take a point $p \in \mathbb{R}^2$. Look at $\mathrm{COL}(p,t_1)$, $\mathrm{COL}(p,t_2)$, $\mathrm{COL}(p,t_3)$. Let $i < j$ and c be such that $\mathrm{COL}(p,t_i) = \mathrm{COL}(p,t_j) = c$ (if all three are equal, take $i = 1, j = 2$). Then $\mathrm{COL}'(p) = (i,j,c)$.

Since we are assuming $\chi = 7$, there exists $p, q \in \mathbb{R}^2$ and i,j,c such that p and q are a unit apart and $\mathrm{COL}'(p) = \mathrm{COL}'(q) = (i,j,c)$. Then (p,t_i), (p,t_j), (q,t_i), (q,t_j) form a mono unit square. ∎

Open Problem 7.3 Prove from $\chi = 7$, or some other reasonable hypothesis, that $d(2) = 3$.

8 What If We Use More Colors?

Theorem 8.1

1. $d(c) \le R_c(C_4)$.

2. $d(c) \le R_c(C_4) - 1$.

Proof:
1) $d(c) \le R_c(C_4)$. Let $d = R_c(C_4)$.

Let COL: $\mathbb{R}^d \to [c]$. We form a coloring COL$'$: $\binom{[d]}{2} \to [c]$.

We define $\binom{[d]}{2}$ points in $\mathbb{R}^d$.

$p_{1,2} = (\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0, \ldots, 0)$.

$p_{1,3} = (\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}}, 0, \ldots, 0)$.

$\vdots \quad \vdots \quad \vdots$

$p_{d-1,d} = (0, \ldots, 0, \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}})$.

Define COL$'(i,j) = $ COL$(p_{i,j})$.

Since $d = R_c(4)$ there exists a mono C_4. The rest of the proof is identical to the proof of Theorem 5.1.

2) This follows from the proof of Part 1 since all of the $p_{i,j}$ are on a hyperplane which is $\mathbb{R}^{d-1}$.
■

To use Theorem 8.1 we need to know upper bounds on $R_c(C_4)$. The following lemma is obtained from a variety of results, which are stated without proof in Section 6.3.2 of Radziszowski's survey of small Ramsey numbers [9]. References to papers with proofs are in that survey.

Lemma 8.2

1. *$R_3(C_4) = 11$.*

2. *$R_4(C_4) = 18$.*

3. *For all $c \ge 1$, $R_c(C_4) \le c^2 + c + 1$.*

4. *For all $c \ge 2$, c even, $R_c(C_4) \le c^2 + c$.*

By combining Theorem 8.1 and Lemma 8.2 we obtain the following:

Theorem 8.3

1. *$d(3) \le 10$.*

2. *$d(4) \le 17$.*

3. *For $c \ge 5$, $d(c) \le c^2 + c$.*

4. *For $c \ge 6$, c even, $d(c) \le c^2 + c - 1$.*

Open Problem 8.4

1. Find better upper bounds on $d(c)$. This may require proofs similar to that of Theorem 6.1.

2. Find lower bounds on $d(c)$ by finding colorings with no mono unit square.

9 What About Other Polygons

We have discussed unit squares. What about other regular polygons with all sides of length 1? The following results are known:

1. Erdős et al. [3] showed that (1) (Page 342) there is a 2-coloring of $\mathbb{R}^2$ with no mono equilateral triangle, (2) (Page 344, Theorem 6) for all 2-colorings of $\mathbb{R}^3$ there is a mono equilateral triangle, (3) (Page 347) for all c-colorings of $\mathbb{R}^{2c}$ there is a mono equilateral triangle with side 1. By points (1) and (2), the result in point (3) is not optimal in the case of $c = 2$. For $c \geq 3$ is it open to determine the minimum value of d such that *for all c-colorings of $\mathbb{R}^d$ there exists a mono equilateral triangle.*

2. Kříž [7] (see also Graham [6]) showed that, for all c, s, there exists $d = d(s, c)$ such that, for all c-colorings of $\mathbb{R}^d$, there is a mono unit regular s-gon. No bounds on d are given though they could be derived from the proof. Obtaining the optimal value of d is open in most cases.

3. Kupavskii-Sagdeev-Zakharov [8] showed that, the $d(s, c)$ is logarithmic in c (they used a different terminology).

Open Problem 9.1

1. Determine some values $d(s, c)$ for small values of s, c.

2. The proofs in the literature about $d(s, c)$ are actually about more general sets and are somewhat difficult. Obtain proofs about $d(s, c)$ that are easier and perhaps less general.

10 Acknowledgments

We thank Dömötör Pálvölgyi for (a) pointing us many papers ([1], [6], [8], [11]), (b) giving us many good suggestions, both pedagogical and mathematical, that we incorporated into the paper, and (c) the theorem and proof in Section 7. We also thank Alexander Soifer who encourages us to make an article out of our thoughts.

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