A Known Problem in Ramsey Theory: Ramsey Multiplicity

by William Gasarch

1 Introduction

In this column we state a class of open problems that are well known in Ramsey Theory but probably not to my non-Ramsey readers. Nothing I present is original.

The problem is as follows: Let $G$ be a graph. Fill in the blank: For every 2-coloring of the edges of $K_n$ (the complete graph on $n$ vertices) there exist BLANK monochromatic copies of $G$. The key words to Google are Ramsey Multiplicity.

In Section 2 we state a motivating question. In Section 3 we look at $K_n$ such that you always get 2 monochromatic copies of $K_3$. In Section 4 we show that, for all 2-colorings of the edges of $K_n$ there exist $\frac{n^3}{24} - O(n^2)$ monochromatic $K_3$’s. Our upper bound is precise and matches the lower bound; however we do not prove this. In Section 5 we state some of what is known. In Section 6 we state a class of open problems.

2 A Motivating Question

We abbreviate monochromatic by mono throughout.

Recall the first theorem one usually hears in Ramsey Theory:

For all 2-colorings of the edges of $K_6$ there is a mono triangle.
From this we obtain a trivial theorem:

For all 2-colorings of the edges of $K_{12}$ there are 2 mono triangles.

How big does $n$ have to be such that any 2-coloring of $K_n$ has 2 mono triangles? The answer is in the next section and may surprise you.

3 Two Monochromatic $K_3$

Theorem 3.1 For all 2-colorings of edges of $K_6$ there are 2 mono triangles.

Proof: Let COL be a 2-coloring of the edges of $K_6$. Let $R$, $B$, $M$, be the sets of RED, BLUE, and MIXED (having both RED and BLUE edges) triangles, respectively. Clearly

$$|R| + |B| + |M| = \binom{6}{3} = 20.$$

We show that $|M| \leq 18$, so $|R| + |B| \geq 2$.

Let $T$ be a mixed triangle. It looks like this:

Note that there will be two vertices that have both a RED and a BLUE edge coming out of them.

- $(v_2, v_1)$ is red, $(v_2, v_3)$ is blue. View this as $(v_2, \{v_1, v_3\})$.
- $(v_3, v_1)$ is red, $(v_3, v_2)$ is blue. View this as $(v_3, \{v_1, v_2\})$.

Def 3.2 A Zan is an element $(v, \{u, w\}) \in V \times \binom{V}{2}$ such that $v \notin \{u, w\}$ and COL$(v, u) \neq$ COL$(v, w)$. ZAN is the set of all Zans.

Map ZAN to $M$ by mapping $(v, \{u, w\})$ to triangle $(v, u, w)$. This mapping is exactly 2-to-1: every element of $M$ has two Zans mapping to it. The Zans that map to
are \((v_2, \{v_1, v_3\})\) and \((v_3, \{v_1, v_2\})\).

Since there is a 2-to-1 map from ZAN to \(M\), \(|M| = |ZAN|/2\). Now we want to bound \(|ZAN|\). Look at how much each vertex can contribute to ZAN. Note that each vertex has degree 5.

**Cases:**

1. \(v\) has \(\deg_R(v) = 5\) and \(\deg_B(v) = 0\): \(v\) contributes 0.
2. \(v\) has \(\deg_R(v) = 4\) and \(\deg_B(v) = 1\): \(v\) contributes 4.
3. \(v\) has \(\deg_R(v) = 3\) and \(\deg_B(v) = 2\): \(v\) contributes 6. Max.

There are 6 vertices, each contribute \(\leq 6\), \(|M| \leq |ZAN|/2 \leq 6 \times 6/2 = 18\)

\(|R| + |B| = 20 - |M| \geq 2\)

4 **Many Mono Triangles**

The following theorem was first proven by Goodman [5]; however, we give an easier proof given by Schwenk [9], as presented by Dorwart and Finkbeiner [2].

**Theorem 4.1**

1. Assume \(n \equiv 1 \pmod{4}\). For all 2-colorings of the edges of \(K_n\) there are at least \(\frac{n^3}{24} - \frac{n^2}{4} + \frac{5n}{24}\) mono triangles.

2. Assume \(n \equiv 3 \pmod{4}\). For all 2-colorings of the edges of \(K_n\) there are at least \(\frac{n^3}{24} - \frac{n^2}{4} + \frac{5n}{24} + \frac{1}{2}\) mono triangles.

3. Assume \(n \equiv 0 \pmod{2}\). For all 2-colorings of the edges of \(K_n\) there are at least \(\frac{n^3}{24} - \frac{n^2}{4} + \frac{n}{3}\) mono triangles.
Proof:
A porism to the proof of Theorem 3.1 is that
\[ |R| + |B| = \binom{n}{3} - |ZAN|/2. \]
Hence we will upper bound $|ZAN|$.

Case 1: $n \equiv 1 \pmod{2}$. The degree of each vertex is $n - 1 \equiv 0 \pmod{2}$.
To maximize $|ZAN|$ we would, at each vertex, color half of the edges RED and half BLUE. So each vertex contributes $(\frac{n-1}{2})^2$, and there are $n$ vertices, so we have $|ZAN| \leq \frac{(n-1)^2n}{4}$. Since $|ZAN|/2 = M \in \mathbb{N}$ we have
\[ |ZAN|/2 \leq \left\lfloor \frac{(n-1)^2n}{8} \right\rfloor \]
hence
\[ |R| + |B| \geq \frac{n(n-1)(n-2)}{6} - \left\lfloor \frac{(n-1)^2n}{8} \right\rfloor \]
Case 1a: $n \equiv 1 \pmod{4}$ so $(n-1)^2 \equiv 0 \pmod{16}$. Hence:
\[ \left\lfloor \frac{(n-1)^2n}{8} \right\rfloor = \frac{(n-1)^2n}{8} \]
Hence
\[ |R| + |B| \geq \frac{n(n-1)(n-2)}{6} - \frac{(n-1)^2n}{8} = \frac{n^3}{24} - \frac{n^2}{4} + \frac{5n}{24} \]
Case 1b: $n \equiv 3 \pmod{4}$. One can easily show that
\[ \left\lfloor \frac{(n-1)^2n}{8} \right\rfloor = \frac{(n-1)^2n-4}{8} \]
Hence
\[ |R| + |B| \geq \frac{n(n-1)(n-2)}{6} - \frac{(n-1)^2n-4}{8} = \frac{n^3}{24} - \frac{n^2}{4} + \frac{5n}{24} + \frac{1}{2} \]
Case 2: $n \equiv 0 \pmod{2}$. The degree of each vertex is $n - 1 \equiv 1 \pmod{2}$.
To maximize $|ZAN|$ we would, at each vertex, color $\frac{n^2}{2}$ of the edges RED
and color \( \frac{n}{2} \) of the edges BLUE. So each vertex contributes \( \frac{n(n-2)}{4} \), and there are \( n \) vertices, so we have \( |ZAN| \leq \frac{n^2(n-2)}{4} \). Since \( |ZAN|/2 = M \in \mathbb{N} \) we have

\[
|ZAN|/2 \leq \left[ \frac{n^2(n-2)}{8} \right].
\]

Since \( n \equiv 0 \pmod{2} \), \( n^2(n-2) \equiv 0 \pmod{8} \), so the bound on \( |ZAN|/2 \) is always in \( \mathbb{N} \). Hence we do not need the floor. Hence

\[
|ZAN|/2 \leq \frac{n^2(n-2)}{8}
\]

\[
|R| + |B| \geq \frac{n(n-1)(n-2)}{6} - \frac{n^2(n-2)}{8} = \frac{n^3}{24} - \frac{n^2}{4} + \frac{n}{3}.
\]

\[\text{Note 4.2}\] The bounds given in Theorem 4.1 are tight. This was proven by Goodman [5] and Savvy [8]. They view the problem differently. They asked given that you want \( t \) triangles, how big must \( n \) be?

5 What is Already Known

\textbf{Def 5.1} \( R(k) \) is the least number \( n \) such that any 2-coloring of the edges of \( K_n \) has a mono \( K_k \). By Ramsey’s theorem for graphs, for all \( k \), \( R(k) \) exists. It is known that \( R(3) = 6 \) and \( R(4) = 18 \); however, all that is known about \( R(5) \) is \( 43 \leq R(5) \leq 48 \). It is a standard result that \( R(k) \leq 4^{k-c\log k} \) for some constant \( c \).

1. Stanislaw Radziszowski and Konrad Piwakowski [6] proved the following: All 2-colorings of \( K_{18} \) have 9 mono \( K_4 \)’s. This is tight. This was proven with the help of a computer.

2. Erdős [3] proved the following:

\textbf{Theorem 5.2} Let \( k \in \mathbb{N} \). Let \( R = R(k) \). Let \( N \) be large, so large that \( R \ll N \). Let \( \text{COL} \) be a 2-coloring of the edges of \( K_N \). Then there are \( \geq \frac{N^k}{4^{(1+o(1))R^2}} \) mono \( K_k \)’s.
We give the proof because we can.

**Proof:** Let $A_1, \ldots, A_{\binom{N}{R}}$ be a list of all the $R$-subsets of $[N]$.

Note that by the definition of $R$ and the $A_i$’s, every $A_i$ has a mono $K_k$.

We can’t just say there are $\binom{N}{R}$ mono $K_k$’s since it may be that two $A_i$’s produce the same mono $K_k$.

We now produce many monochromatic $K_k$’s.

(a) $X = \{A_i : 1 \leq i \leq \binom{N}{R}\}$

(b) $Y = \emptyset$. $Y$ will contain many mono $K_k$’s.

(c) If $X \neq \emptyset$ do the following (else terminate). Let $i$ be the least number such that $A_i \in X$. It has a mono $K_k$. Call it $C$.

i. Add $C$ to $Y$ (we will soon see that $C$ is not already in $Y$).

ii. Remove from $X$ all $A_j$’s that have $C$ in them. Hence we are removing $\binom{N-k}{R-k}$ $A_j$’s.

iii. Goto Step c

In every iteration $X$ loses $\binom{N-k}{R-k}$ $A_j$’s. Hence the number of mono $K_k$’s that this process produces is at least

$$\frac{\binom{N}{R}}{\binom{N-k}{R-k}} = \frac{N!}{R!(N-R)!} \times \frac{(N-R)!(R-k)!}{(N-k)!} = \frac{N!}{(N-k)!} \times \frac{(R-k)!}{R!}.$$  

We need to lower bound this quantity. We use $\frac{(R-k)!}{R!} \geq \frac{1}{R^k} \geq \frac{1}{4^k}$. The other inequality we need will be more delicate.

$$\frac{N!}{(N-k)!} \geq (N-k+1)^k = \frac{N^k}{(N/(N-k+1))^k}$$

We look at the denominator

$$\left(\frac{N}{N-k+1}\right)^k = \left(1+\frac{k-1}{N-k+1}\right)^k \sim e^{((k-1)k)/(N-k+1)} \sim 4^{c(k-1)/(N-k+1)}$$

for some constant $c$. Hence the number of mono $K_k$’s is at least
\[
\frac{N^k}{4k^2 + (c(k-1)k/(N-k+1))} = \frac{N^k}{4k^2(1+o(1))}.
\]

3. David Conlon [1] proved the following: Fix \( t \). For \( n \) large, for any 2-coloring of \( K_n \) there are \( \frac{n^t}{C(1+o(1))n^2} \) mono \( K_t \)'s where \( C \sim 2.18 \). Note that Conlon’s result is an improvement of Theorem 5.2.

4. Jacob Fox [4] looks at the problem for target graphs \( G \) other than \( K_t \).

5. For more results (1) look at the bibliographies of the papers above, (2) Google Ramsey Multiplicity, and (3) watch this cool lecture by David Conlon:
   

6 Open Problems

Here is a large class of open problems with the same theme as Theorem 4.1. First we need some notation.

**Notation 6.1** Let \( k \geq 1 \).

1. \( C_k \) is the cycle graph on \( k \) vertices. Vertex \( i \) has an edge to vertices \( i-1 \mod k \) and \( i+1 \mod k \) but no other vertices.

2. \( P_k \) is the path graph on \( k \) vertices. If \( i \leq k - 1 \) then vertex \( i \) has an edge to vertex \( i+1 \).

3. \( W_k \) is the wheel graph on \( k \) vertices. It is \( C_{k-1} \) with one more vertex that has an edge to all other vertices.

4. \( K_{1,k} \) is the star graph.

5. If \( G \) is a graph on \( k \) vertices, \( R(G) \) is the least number \( n \) such that any 2-coloring of the edges of \( K_n \) has a mono \( G \). By Ramsey’s theorem for graphs, for all \( G \), \( R(G) \) exists; however, for many graphs, \( R(G) \) is much lower than \( R(k) \).
6. The values of $R(G)$ are known for $G \in \{C_k, P_k, W_k, K_{1,k}\}$. See Stanislaw Radziszowski's survey of Small Ramsey Numbers [7].

We now state some open problems.

1. Fix $k$. Find the function $f$ such that, for all 2-colorings of $K_n$ there are $f(n)$ mono $K_k$'s. Try to make $f(n)$ as large as possible. The $k = 3$ case is solved. The $k = 4$ case is not solved, but it is plausible that it will be. The $k = 5$ case is harder than finding $R(5)$ and hence is unlikely to be solved . . . ever.

2. Replace $K_k$ in the last item with $C_k$, $P_k$, $W_k$, $K_{1,k}$ or whatever your favorite parameterized set of graphs is. When $R(G)$ is known there is hope of solving this problem. If $R(G)$ is unknown, then note that the problem of finding $f(n)$ is harder than finding $R(G)$.

3. Let $G$ be your favorite graph and $L$ be your favorite number. Find the least $n$ (or at least a non-obvious $k$) such that every 2-coloring of the edges of $K_n$ yields $L$ copies of a mono $G$.

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References


