Open Problems Column
Edited by William Gasarch

This Issues Column! This issue's Open Problem Column is by William Gasarch and is *The Busy Beaver Function, Small Turing Machines, and Open Problems*

Request for Columns! I invite any reader who has knowledge of some area to contact me and arrange to write a column about open problems in that area. That area can be (1) broad or narrow or anywhere inbetween, and (2) really important or really unimportant or anywhere inbetween.

The Busy Beaver Function, Small Turing Machines, and Open Problems
by William Gasarch

1 Introduction

Def 1.1 BB(n) is the largest number of steps that an n-state Turing machine that halts will take to halt (BB stands for Busy Beaver). Note that to define BB(n) rigorously you need to specify the details of the Turing machine model you are using. See Scott Aaronson [Aar20] for these details.

It is easy to see that BB(n) is not computable. Indeed, it grows faster than any computable function.

The first five Busy Beaver numbers are known. Lower bounds for BB(6) and BB(7) are known. We list out this information.

n	BB(n)	Reference
1	1	Trivial
2	6	Lin & Rado 1963 [LR65]
3	21	Lin & Rado 1963 [LR65]
4	107	Brady 1983 [Bra83]
5	$47,176,870$	The bbchallenge Collaboration [bC26]
6	$\geq 2 \uparrow\uparrow\uparrow 5$	Reported by Pascal Michel
7	$\geq 2 \uparrow^{11} 2 \uparrow^{11} 3$	Reported by Pascal Michel

(We will discuss BB(6) and BB(7) more in Section 13.)

The lower bound on BB(6) is EEEENORMOUS. We believe BB(6) will never be found. There is also a philosophical issue that was pointed out to me by Shawn Ligocki. We quote his email:

*As far as "truly understanding" the number [*BB(6)*], I think this is a philosophically deep question that I cannot hope to answer! Short story: My current feeling is that even comparatively small numbers (like a billion) are large enough that I cannot completely understand* *them in many ways, but for these truly huge numbers my current feeling is that I "understand" them when I can comfortably answer arithmetic questions about them. For example, given two numbers in up-arrow notation, can I figure out which one is bigger? This is not trivial, but I think I have the algorithm for tetration at least.*

For more background on the Busy Beaver function see (1) an open problems column by Scott Aaronson [Aar20] which inspired the work that lead to BB(5) being discovered, (2) blog posts by Scott Aaronson on the topic (go to his blog, Shtetl-optimized, and search for *Busy Beaver*), (3) Ben Brubaker's superb article in Quanta Magazine [Bru24], (4) bbchallenge's own announcement [CW24] of the BB(5) result.

In this open problems column I will discuss open problems in mathematics and their relation to the Busy Beaver Function.

2 Cryptids, a Subtle Point, and a Not-So-Subtle-Point

2.1 Cryptids

In email with Shawn Ligocki about an earlier draft of this paper he wrote

The size of BB(6) *is not the biggest roadblock to finding the exact value, the biggest roadblock is Cryptids.*

I went to the web and found two definitions of Cryptids.

Def 2.1

1. From Google AI:

   Cryptids are animals or beings whose existence is suggested by folklore, legends, or anecdotal reports but unproven by mainstream science, studied through the often-criticized field of cryptozoologc. Common types include large apes (Bigfoot, Yeti), lake monsters (Lock Ness), and hybrids (Chupacabra), usually reported in remote areas.

   While the definition reminds me of large cardinals (the sets, not the birds, and not the people in the Vatican), this is probably not what Shawn was referring to.

2. From the bbchallenge website:

   Cryptids are Turing machines whose behaviour (when started on a blank tape) can be described completely by a relatively simple mathematical rule, but where that rule falls into a class of unsolved (and presumed hard) mathematical problems.

   To say that this is what Shawn had in mind is to get history backwards. He coined the term on his website.

   `https://www.sligocki.com/2023/10/16/bb-3-3-is-hard.html`

   Here is the money quote in the section titled *Cryptids*:

These machines seem to be a sort of legendary creature, rumors have it that they either halt or do not, but nobody has been able to provide any concrete evidence to support either conclusion. I propose calling them "Cryptids", drawing parallels to the legendary creatures like the Loch Ness Monster or Chupacabra.

This paper is mostly a survey of Cryptids. It is not complete. It overlaps with the website of Cryptids where I found the definition, which is here:

`https://wiki.bbchallenge.org/wiki/Cryptids`

2.2 A Subtle Point

Here is one of the results we will talk about:

There is a TM on 25 states such that the following are equivalent:
(a) TM halts, (b) Goldbach's conjecture is false.

I can beat the 25-state bound in a stupid way:

- Let M_1 be the TM that always halts. That will take 1 state. If Goldbach's conjecture is false then the following are equivalent: (a) TM M_1 halts, (b) Goldbach's conjecture is false.
- Let M_2 be the TM that never halts. That will take 1 state. If Goldbach's conjecture is true then the following are equivalent: (a) TM M_1 halts, (b) Goldbach's conjecture is false.

Hence there is a 1-state machine that works. The problem is that we do not know which machine satisfied the desired behaviour. In this article when we write (say) *Professor X constructed an s-state TM such that TM halts iff Goldbach's Conjecuture is false* we mean that we constructed *one* machine M and it has been proven to have the desired property.

2.3 A Not-So-Subtle-Point

Some of the results I discuss have not been verified. Note that it would be hard for a human to do so.

Some of the result have been verified by Lean or other systems.

I have emailed many people who work on the Busy Beaver Problems connection to other math problems. The general consensus is that all of the results discussed in this article are either true or very close to being true.

I will assume that the results discussed are correct.

Open Problem 2.2 For all of the results stated in this paper, and others of the same type, verify that the TM's behave as advertised in Lean or some other verifiers.

3 Goldbach's Conjecture

On June 7, 1742, Christian Goldbach wrote a letter to Leonard Euler in which he made the following conjecture (in modern language):

Conjecture 3.1 *(Goldbach's Conjecture) Every even number* $n \geq 4$ *can be written as the sum of two primes.*

The following are known.

- In 1924 Godfrey Hardy and John Littlewood showed that, assuming the generalized Riemann Hypotheses, the number of even numbers in $\{1, \ldots, X\}$ for which Goldbach's conjecture is false is $\leq X^{0.5+c}$ for a small c.

- In 1951 Linnik proved that there exists K such that every sufficiently large even number is the sum of two primes and K powers of 2. In 2002 Heath-Brown and Schlage-Puchta showed that, assuming the generalized Riemann hypothesis, $K = 7$ works. In 2020 Pintz & Rusza showed (without any hypothesis) that $K = 8$ works.

- In 1973 Chen Jinru [Jin73] (also see simpler proof by Peter Ross [Ros75]) showed that every sufficiently large even number can be written as either (1) the sum of two primes, or (2) the sum of a prime and the product of two primes. In 2015 Tomohrio Yamada [Yam15] gave explicit bounds on Chen's theorem. In 2022 Matteo Bordignon [Bor22] (the 2025 arxiv version adds co-authors Daniel Johnson and Valeriia Starichkova) improved those bounds.

- In 1975 Hugh Montgomery and Robert Vaughn [MV75] showed that there exists positive constants c and C such that, for large X, the number of even numbers in $\{1, \ldots, X\}$ for which Goldbach's conjecture is false is $\leq CX^{1-c}$. (Note that the conclusion is weaker than what Hardy and Littlewood prove, but the proof does not need the generalized Riemann Hypothesis.)

- T. Oliveira e Silva has verified Goldbach's conjecture for $n \leq 4 \times 10^{18}$ by computer (I am writing this in April 2026). See

  `https://sweet.ua.pt/tos/goldbach.html`

  The website is still active. Hence, by the time you look at it, the number may be improved.

- Goldbach's conjecture is widely believed to be true.

Lengyijun, see `https://github.com/lengyijun/goldbach`. constructed a 25-state TM M, verified in Lean, such that the following are equivalent:

1. M halts.
2. Goldbach's conjecture is false.

Open Problem 3.2 Construct an s-state TM M, with $s < 25$, such that M halts iff Goldbach's conjecture is false. Shawn Ligocki has conjectured that this can be improved but notes that it may be hard since this problem has been worked on a lot. Nicholas Drozd has conjectured that the number of states may be as low as 10.

Lets say that someone proved $\mathrm{BB}(25) = x$. Then one can (in theory) solve the Goldbach conjecture by running M for x steps and seeing what happens: If M does not halt within x steps then its not going to halt, so the Goldbach conjecture is true. If it has halted by x steps then Goldbach conjecture is false. This discussion points to how hard it will be to find BB(25) since it will lead to cracking Goldbach's conjecture.

Similar comments apply to all of the open problems discussed in this paper. Since brevity is the soul of wit, I shant bring up this point again.

4 Fermat Primes

Notation 4.1 Let $n \in \mathbb{N}$. Then $F_n = 2^{2^n} + 1$.

Def 4.2 p is a *Fermat Prime* if p is prime and there exists n such that $p = F_n$.

1. Only 5 Fermat Primes are known:
   - $F_0 = 2^{2^0} + 1 = 3$.
   - $F_1 = 2^{2^1} + 1 = 5$.
   - $F_2 = 2^{2^2} + 1 = 17$.
   - $F_3 = 2^{2^3} + 1 = 257$.
   - $F_4 = 2^{2^4} + 1 = 65,537$.
2. F_5, F_6, …, F_{32} are composite. F_{33} is the first F_n for which it is open if it is prime or not.

3. As of December 2025 there are 330 F_n's that are known to be composite. See

   `http://www.prothsearch.com/fermat.html#Summary`

   for a survey.

4. It is currently believed that F_1, F_2, F_3, F_4 are the only Fermat primes; however, this is not known. Proving that there are a finite number of Fermat primes seems to be a hard problem.

Brown & Gonzalez-Hendrix & Tandi presented the outline for a proof that there is a 76-state TM M such that the following are equivalent:

1. M halts.

2. There is a Fermat prime larger than F_4.

Open Problem 4.3 Construct an s-state TM M, with $s < 76$ states, such that M halt iff there is a Fermat primes larger than F_4. Jonathan Brown has conjectured that s may be as low as 45.

5 Brocard's Problem

Def 5.1 A *Brocard Number* is an $n \in \mathbb{N}$ such that $n! + 1$ is a square.

1. Only three Brochard numbers are known:
   - $n = 4$: $4! + 1 = 25 = 5^2$
   - $n = 5$: $5! + 1 = 121 = 11^2$
   - $n = 7$: $7! + 1 = 5041 = 71^2$

2. It is not know if there are an infinite number of Brocard numbers.

3. Clever computer searches have shown there is no Brocard number $\le 10^7$ (see David Wells [Wel87] Page 70). Bruce Berndt and William Galway [BG00] have further searched up to 10^9.

4. Andrzej Dabrowski [Dab96] showed that a weak form of the ABC conjecture implies that, for all A, the number of n such that $n! + A$ is a square is finite.

5. It is widely believed that $4, 5, 7$ are the only Brocard numbers.

Jonathan Brown, Josue Gonzalez-Hendrix, and Gurpreet Tandi have constructed a 43-state TM M such that the following are equivalent:

1. M halts.
2. There is a Brocard number $n \geq 8$.

Open Problem 5.2 Construct an s-state TM M, with $s < 43$, such that, M halts iff there is a Brocard number ≥ 8. Jonathan Brown has conjectured that s may be as low as 30.

6 Riemann's Hypothesis

6.1 Background

Def 6.1 The *Riemann zeta function* is a function from $\mathbb{C}$ to $\mathbb{C}$ defined by

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}.$$

This function converges when $\mathrm{Re}(s) > 1$ and would seem to diverge otherwise. However, ζ can be analytically continued (we omit details) so that it converges on all complex numbers except $s = 1$.

The following are known.

- The only known roots of $\zeta(s)$ are of the following types:
  - If s is a negative even integers then s is a root of ζ;
  - There are some roots on the line $\mathrm{Re}(s) = \frac{1}{2}$.
- If s is a root of ζ and s is not a negative even number then $0 \leq \mathrm{Re}(s) \leq 1$.

Conjecture 6.2 *(The Riemann's Hypothesis (RH)) The only roots of ζ that are not negative even numbers are on the line* $\mathrm{Re}(s) = \frac{1}{2}$.

RH has consequences for the estimates of $\pi(x)$, the number of primes in $\{1, \ldots, x\}$:

- Without assuming RH one can show $\pi(x) = \frac{x}{\ln x} + \Theta(\frac{x}{\ln^2 x})$.
- Assuming RH one can show $\pi(x) = \frac{x}{\ln x} + \Theta(\sqrt{x} \ln x)$.

- The RH is the only problem which is both a Hilbert Problem and a Millennium Problem.

The RH *seems* like a continuous problem and hence *not* the type that can be formulated in terms of Turing Machines. However, Jeffrey Lagarias [Bro76] showed that the following are equivalent.

1. For all $n \geq 1$

$$\left(\left(\sum_{k \leq \delta(n)} \frac{1}{k}\right) - \frac{n^2}{2}\right)^2 < 36n^2.$$

where $\delta(n)$ is defined in terms of η, as below.

$$\eta(x) = \begin{cases} p & \text{If } x \text{ is a prime power and that power is } p \\ 1 & \text{If } x \text{ is not a prime power} \end{cases} \tag{1}$$

$$\delta(x) = \prod_{n<x} \prod_{j \leq n} \eta(j).$$

2. RH is true.

6.2 Yedida and Aaronson's TM related to RH

Using Lagarias's formulation, in 2016, Adam Yedidia and Scott Aaronson [YA16] constructed a 5372-state TM M such that the following are equivalent:

1. M halts
2. The RH is false.

Note that they have a paper explaining what they did.

6.3 Matiyasevich and O'Rear

In 2016 Matiyasevich and O'Rear claimed to have constructed a 744-state TM M such that the following are equivalent:

1. M halts
2. The RH is false.

On the bb-logic site,

`https://wiki.bbchallenge.org/wiki/Logical_independence`

There is a pointer to a github site for the result; however, as of April 2026 the link was broken.

We will assume the result is true since (a) it probably is, and (b) even if its not we are sure that there is an s-state TM with the property where s is not that big, say $s \leq 800$.

Open Problem 6.3

1. The result of Matiyasevich and O'Rear needs a writeup and a verification.

2. (Assuming that the result of Matiyasevich and O'Rear is correct.) Construct an s-state TM M, with $s < 744$, such that M halts iff RH is false. Scott Aaronson has conjectured that a few dozen states is plausible. Hence, far less than 744.

7 An Erdős Problem About Powers of 2 in Base 3

We denote a number m's base 3 representation as $(m)_3$. We list the first 13 $(2^n)_3$'s.

n	2^n in Base 10	2^n in Base 3
0	1	1
1	2	2
2	4	11
3	8	22
4	16	121
5	32	1012
6	64	2101
7	128	11202
8	256	100111
9	512	200222
10	1024	1102101
11	2048	2210212
12	4096	12121201
13	8196	1020202102

If you compute further you will find that, for all $n \geq 9$, $(2^n)_3$ contains a 2.

Erdős (see `https://www.erdosproblems.com/406`) asked

Is it true that there are only finitely many powers of 2 which have only the digits 0 and 1 when written in base 3?

This question is still open.

1. Jeffrey Lagarias [Lag09] showed that the number of n such that $(2^n)_3$ does not have a 2 is small in some sense.

2. Vassil Dimitrov and Everett Howe [DH25] use elementary methods to show (a) which powers of 3 can be written as the sum of ≤ 22 powers of 2, and (b) which powers of 2 can be written as the sum of ≤ 25 powers of 3. This seems relevant to the Erdős problem.

Tristan Stérin and Damien Woods [SW21] have constructed a 15-state TM M such that the following are equivalent.

- M halts.
- There exists an $n \ge 9$ such that $(2^n_3$ does not have a 2.

Open Problem 7.1 Construct an s-state TM M, with $s < 15$, such that M halts iff there is an $n \ge 9$ where $(2^n)_3$ has no 2's. Tristan Stérin has conjectured that this bound can be improved; however, it will be difficult.

8 Collatz-Like Sequences

Lothar Collatz studied the following sequence in the 1930's (see Collatz's account [Col86]).

Let a_0 be any element of $\mathbb{N}$. Then let

$$a_{n+1} = \begin{cases} \frac{a_n}{2} & \text{if } a_n \equiv 0 \pmod 2 \\ 3a_n + 1 & \text{if } a_n \equiv 1 \pmod 2 \end{cases} \tag{2}$$

Example 8.1 $a_0 = 100$ $a_1 = 50$ $a_2 = 25$ $a_3 = 76$ $a_4 = 38$
$a_5 = 19$ $a_6 = 58$ $a_7 = 29$ $a_8 = 88$ $a_9 = 44$ $a_{10} = 22$
$a_{11} = 11$ $a_{12} = 34$ $a_{13} = 17$ $a_{14} = 52$ $a_{15} = 26$ $a_{16} = 13$
$a_{17} = 40$ $a_{18} = 20$ $a_{19} = 10$ $a_{20} = 5$ $a_{21} = 16$ $a_{22} = 8$
$a_{23} = 4$ $a_{24} = 2$ $a_{25} = 1$ $a_{26} = 4$ $a_{27} = 2$ $a_{28} = 1$

For all $n \ge 25$ the sequence cycles $1, 4, 2, 1, 4, 2, \ldots$.

Conjecture 8.2 *The Collatz conjecture (CC) is that, for all $a_0 \in \mathbb{N}$, the Collatz sequence eventually hits 1.*

1. David Barina [Bar25] showed that, for all $1 \le a_0 \le 2^{71}$, CC is true.

2. CC is widely believed to be true.
3. Shaolom Eilauhou [Eil93] showed that if there are nontrivial cycles then they are enormous.
4. Ilia Krasikov and Jeffrey Lagarias [KL03] showed that, for sufficiently large x, the number of integer values $a_0 \in [1, x]$ for which CC is true is greater than $x^{0.84}$.
5. Terrence Tao [Tao15] showed that (roughly) most cycles have a small minimum element.
6. Paul Erdős said *Mathematics may not be ready for such problems.*
7. Richard Guy agrees with Paul Erdős: In 1983 Guy [Guy83] wrote a paper titled *Don't try to solve these problems*, and CC was one of them.
8. Jeffrey Lagarias has an annotated bibliography of the problem [Lag11] from 1963 to 1999.

Formally CC is
For all a_0 there exists n such that if the sequence starts at a_0 then $a_n = 1$.
Currently there is no ∀ or ∃ version of CC. Hence, for now, the classic CC is not amenable to TMs. The next open question is to find such a version. The rest of this section is about variants of CC that are either of the form ∀ or can be shown to be equivalent to a statement of the form ∀.

Open Problem 8.3

1. This is a straight-up math problem that may be hard or even impossible. Find some ∀ statement that CC (or its negation) is equivalent to.
2. Assume Part 1 has been done. Construct an s-state TM M, with s small, such that M halts iff CC is true (or false).

I quote an email from Tristan Stérin about *The Collatz Conjecture Challenge*
Recently [email in from March 2026] I created a new collaborative project called **The Collatz Conjecture Challenge***, whose aim is to formalize results from the Collatz conjecture using proof assistants (Rocq, Lean, etc….). Hopefully, interesting open problems, easier that CC itself, will arise from this project. The website for it is*
`https://ccchallenge.org/`,

8.1 An Aside About The Generalized Collatz Problem (GCP)

This section is an aside about Collatz type functions.

The CC used the functions $a_n/2$ and $3a_n + 1$ and had cases mod 2. Conway generalized this notion by allowing rational coefficients and more mods.

Def 8.4 The *Generalized Collatz Problem (GCP)* is as follows: Given $m \in \mathbb{N}$, $b_0, \ldots, b_{m-1} \in \mathbb{Q}$, $c_0, \ldots, c_{m-1} \in \mathbb{Q}$ and $a_0 \in \mathbb{N}$ consider the sequence that begins with a_0 and, for all $n \geq 0$, is defined by

$$a_{n+1} = \begin{cases} b_0 a_n + c_0 \text{ if } a_n \equiv 0 \pmod{m} \\ b_1 a_n + c_1 \text{ if } a_n \equiv 1 \pmod{m} \\ \vdots \quad \vdots \quad \vdots \\ b_{m-1} a_n + c_{m-1} \text{ if } a_n \equiv m-1 \pmod{m} \end{cases} \tag{3}$$

(you are promised that, for all i, a_i is is always an integer)
determine if this sequence ever hits 1.

John Conway [Con72] showed that GCP is undecidable.

Does Conway's result indicated that CC is hard? Not quite since CC is only one instance. But Conway's result does indicate that there will be an infinite number of instances that are hard.

8.2 Weaker Versions of CC

According to Wikipedia and other sources:

Conjecture 8.5 *The Weak Collatz Conjecture (WCC) is that, for all a_0, the Collatz sequence does not go to infinity. This is equivalent to saying, for all a_0, there is a cycle (it need no be $4-2-1-4-2-1-\cdots$).*

Currently there is no $\forall$ or $\exists$ version of WCC. Hence, for now, the classic WCC is not amenable to TMs. The next open question is to find such a version. The rest of this subsection is about variants of WCC that either of the form $\forall$ or can be shown to be equivalent to a statement of the form $\forall$.

Open Problem 8.6

1. This is a straight-up math problem that may be hard or even impossible. Find some $\forall$ statement that WCC (or its negation) is equivalent to.
2. Assume Part 1 is done. Construct an s-state TM M, with s small, such M halts iff WCC is true (or false).

Conjecture 8.7 *The Cycle Collatz Conjecture (CCC) is that, for all a_0, the only possible cycle is $4-2-1-4-2-1-\cdots$. (There could also be no cycle.)*

The CCC can be expressed as a $\forall$ statement:

$$(\forall a_0)[(\exists n)[(a_n, a_{n+1}, a_{n+2}) = (4,2,1)] \vee (\forall m)(\exists n)[a_n \geq m]].$$

The negation of this statement is

$$(\exists a_0)[(\forall n)[(a_n, a_{n+1}, a_{n+2}) \neq (4,2,1)] \wedge (\exists m)(\forall n)[a_n \leq m]].$$

In the last sentence, since the sequence is bounded it must hit some cycle; however, that cycle cannot be $(4,3,2)$. Hence the statement is equivalent to:

There exists a_0 so that if the Collatz sequence starts there, it will go into a cycle that is not (4,2,1).

AH-HA! this is a search problem (which might not halt) so a TM can do that search.

Tristan Stérin has constructed a 124-state TM such that the following are equivalent:

- M halts.
- CCC is false.

Open Problem 8.8 Construct an s-state TM M, with $s \leq 124$, such that M halts iff CCC is false. Tristan Stérin thinks this bound can easily be improved.

8.3 Collatz-Like Problems

The bbchallenge team has a website of Collatz-Like Problems and associated small-state TMs. The website is:

`https://wiki.bbchallenge.org/wiki/Beaver_Math_Olympiad`

The bbchallenge team refer humorously these type of problems as *Beaver Math Olympiad* problems.

In the next two sections we discuss two of their results. The first one involves a sequence called *Antihydra* See the bb-Collatz website stated above, `https://ccchallenge.org/`, for why this problem has that name. The second one was the first one studied so its simply called. *The First BMO Problem.*

8.4 Antihydra: A Collatz-Like Sequences

The bbchallenge team has defined the following problem, which they call *Antihydra*. (See the bb-Collatz website stated above, `https://ccchallenge.org/`, for why this problem has that name).

$a_0 = 8.$
$(\forall n \geq 1)[a_{n+1} = \lfloor \frac{3a_n}{2} \rfloor].$

Note that this sequence is Collatz-like since the floor function makes a_{n+1} depend on the parity of a_n.

The bbchallenge team have constructed a 6-state TM such that the following are equivalent

1. M halts
2. There exist a k such that $a_0, a_1, \ldots, a_k$ contains more than twice as many odd numbers as even numbers.

This result is different than the others in this paper since it is not known if the problem the TM construction is equivalent to is hard. However, that leads to an interesting open question.

Open Problem 8.9 (This is a straight-up math problem.) Is there a k such that $a_0, a_1, \ldots, a_k$ contains more than twice as many odd numbers as even numbers? Here are possible outcomes

1. (The most likely IMHO.) No progress. This is a hard problem! That makes finding a 6-state (6 is small!) TM that is equivalent to it very impressive.
2. It is proven that no such k exists. The proof will probably be very interesting and may apply to other sequences.
3. It is proven that such a k exists. Then the question will be, are there an infinite number of such k?

8.5 The First BMO Problem

Consider the following piecewise function

$$f(a,b) = \begin{cases} (a-b, 4b+2) & \text{if } a > b \\ (2a+1, b-a) & \text{if } a < b \\ \text{STOP} & \text{if } a = b \end{cases} \tag{4}$$

The bbchallenge team showed that there is a 6-state TM such that the following are equivalent.

1. M halts
2. There exist a k and an n such that $f^k(1,2) = (n,n)$.

Open Problem 8.10 (These are straight-up math problem.)

1. Is there a (k,n) such that $f^k(1,2) = (n,n)$. The possible outcomes are similar to those in Open Problem 8.9.
2. Is there an (a,b) such that, fora all k, $f^k(a,b)$ is never STOP. The possible outcomes are similar to those in Open Problem 8.9.

Nicholas Drozd has done work on functions similar to f. See his website `https://nickdrozd.github.io/2022/04/13/does-this-function-terminate.html`. He has related one of them to the Beeping busy beaver numbers.

9 Is BB a Measure of the Hardness of an Open Math Problem?

We list the results we've discussed in order of number-of-states.

1. Let a_n be the antihypdra sequence in Section 8.4. There is a TM with 6 states such that *TM halts iff there exists k such that $a_0, a_1, \ldots, a_k$ contains more than twice as many odd numbers as even numbers.*
2. Let f be the first BMO problem in Section 8.5. There is a TM with 6 states such that *TM halts iff there exists k, n such that $f^k(1,2) = (n,n)$*
3. There is a TM with 25 states such that *TM halts iff Goldbach's Conjecture is false.*
4. There is a TM with 43 states such that *TM halts iff there is a Brocard number larger than 8.*
5. There is a TM with 76 states such that *TM halts iff there is a Fermat prime larger than F_4.*
6. There is a TM with 124 states such that *TM halts iff the Cycle Collatz Conjecture is false.*
7. There is a TM with 744 state such that *TM halts iff the RH is false.*

Note that these numbers are probably not optimal (except for the 6-state TMs). No non-trivial lower bounds are known. Even so, we tend to think that the best TM for RH will be larger than the best TM for Fermat primes.

Open Problem 9.1 (This problem is vague.) Determine if harder math problems lead to larger TMs. One can look at math problems that took a long time to resolve (e.g., Fermat's Last Theorem) and write TM's associated to them, and see if the harder ones tend to have more states.

10 ZFC: Consistency

The following website is the best reference on the topic of this section.

`https://wiki.bbchallenge.org/wiki/Logical_independence`

10.1 7918 States

ZFC is a set of axioms from which most of mathematics can be derived. It is widely believed to be consistent.

Def 10.1 If T is a set of axioms then CON(T) means that from the axioms of T you will not get a contradiction. CON stands for *consistent*.

In 2016 Adam Yedidia and Scott Aaronson [YA16] constructed a a 7918-state TM M such that the following are equivalent:

1. M halts
2. $\neg$CON(ZFC).

Their paper used the following result of Harvey Friedman [Fri24]: There is a statement S in graph theory (the statement is explicit though we do not state it here) such that $S \implies$ CON(ZFC) (actually S implies more, we discuss this later). The statement S is a $\forall$ statement. Hence it can be refuted by a witness. The TM of Yedida and Aaronson searches for that witness.

Friedman's actually showed $S \implies$ CON(SRP) where SRP is Stationary Ramsey Property, a system stronger than ZFC.

10.2 A Different Approach: Use Variants of ZFC Directly

Yedidia & Aaronson used a statement S that has the property: $\implies$ CON(ZFC.

A different approach would be to use ZFC directly. One *could* build a TM M such that M halts iff $\neg$CON(ZFC) by generating sequences of statements in ZFC and, for each one, check if (a) they are a proof, and (b) their last line is a contradiction (e.g., $0 \neq 1$). If this contradiction is ever encountered then the machine halts. Since ZFC is complicated this would likely not lead to small TMs.

There are weaker systems T such that CON(ZFC) iff CON(T).

Def 10.2

1. The *Axiom of regularity (AR)* is the following:

   If $A \neq \emptyset$ then there exists $y \in A$ such that $y \cap A = \emptyset$.

2. The *axiom of foundation (AF)* is the following:

   The $\in$ relation on sets is well founded.

3. ZF is the axioms of ZFC minus AC.

From AR and the axiom of pairing, one can derive AF. From AF one can derive AR. Usually ZFC is presented with AF as an axiom; however, nothing would change if instead AR was an axiom. We take ZFC to have AR and not AF.

We state the following theorem without proof.

Theorem 10.3 *The following statements are equivalent over PRA (Primitive Recursive Arithmetic, a theory far weaker than any of the theories below.)*

- CON(ZFC).
- CON(ZF).
- CON(ZF − AR).

One could build a TM M such that M halts iff $\neg$CON(ZF − AR) by generating sequences of statements in ZF − AR and, for each one, check if (a) they are a proof, and (b) their last line is a contradiction (e.g., $0 \neq 1$). If this contradiction is ever encountered then the machine halts. Note that this TM has the behaviour we want: M halts iff $\neg$CON(ZFC).

All results about ZFC in the rest of the subsections of this section use this approach. They also build on the work of Yedidia and Aaronson.

10.3 748 states! 745 states! 636 states! 432 states!

In 2017 Stefan O'Rear (unpublished) constructed a 748-state TM M such that the following are equivalent:

1. M halts
2. ¬CON(ZFC).

In 2023 Johannes Riebel, in his Bachelor's Thesis [Rie23], wrote up O'Rear's construction and improved it by 3 states. Hence he showed that there is a a 745-state TM, M, such that the following are equivalent:

1. M halts
2. ¬CON(ZFC).

In 2024 Rohan Ridenour constructed a 636-state TM M such that the following are equivalent:

1. M halts.
2. ¬CON(ZFC).

In 2025 Andrew Wade constructed a 432-state TM M such that the following are equivalent:

1. M halts.
2. ¬CON(ZFC).

Open Problem 10.4 Construct an s-state TM, with $s < 432$, such that M halts iff ¬CON(ZFC). Scott Aaronson has conjectured that this size of the TM can be brought down to *a few dozen states*. Hence far less than 432.

Open Problem 10.5 For each axiom AX of ZFC look at ZFC − AX and ZF − AX. Determine if CON(ZFC − AX) implies CON(ZFC) (this is probably already known). If not then construct a small TM M such that M halts iff ¬CON(ZFC − AX). One can also remove sets of axioms.

11 Other Systems of Axioms

The following website is the best reference on the topic of this section.

`https://wiki.bbchallenge.org/wiki/Logical_independence`

We look at one system weaker than ZFC and one system strong than ZFC.

11.1 Peano Arithmetic (PA)

In 2026 Matthew House (referred to as @LegionMammal978 on the webpage about systems of axioms and TMs) constructed a 372-state TM M such that the following are equivalent.

- M halts
- $\neg$CON(PA).

Open Problem 11.1 For weaker variants T of PA determine if CON(T) implies CON(PA) (this is probably already known). If not then construct a small TM M such that M halts iff $\neg$CON(T). The following are weaker variants of PA: Robinson's Arithmetic, Primitive Recursive Arithmetic, $I\Sigma_n$, $I\Pi_n$.

11.2 Subtle Cardinals

A *Subtle Cardinal* is a type of large cardinal. See the the Wikipedia entry

`https://en.wikipedia.org/wiki/Subtle_cardinal`

In 2026 Matthew House constructed a 493-state TM such that the following are equivalent:

- M halts.
- $\neg$CON(ZFC + *There exists arbitrary large subtle cardinals.*)

Open Problem 11.2

1. Construct an s-state TM M, with $s < 493$, so that M halts iff
   $\neg$CON(ZFC + *There exists arbitrary large subtle cardinals*).
2. There is a chart of large cardinals at the website on logical ind. and BB (this chart appears other places on the web as well). For every large cardinal hypothesis H find a small TM M such that M halts iff $\neg$CON(ZFC + H).

12 Values of $\mathrm{BB}(n)$ that are Independent of ...

The following website is the best reference on the topic of this section.

https://wiki.bbchallenge.org/wiki/Logical_independence

Def 12.1 Let ϕ be a statement in mathematics. Let T be a logical system e.g., ZFC. ϕ is *independent of* T if the following both hold assuming CON(T):
(1) $\mathrm{CON}(\mathrm{T} \cup \{\phi\})$,
(2) $\mathrm{CON}(\mathrm{T} \cup \{\neg\phi\})$.

Theorem 12.2 *Let* T *be a recursively enumerable and arithmetically sound axiomatic theory. (All of the axiom systems in the last section are r.e. and are believed to be sound.) Let $s \in \mathbb{N}$. Assume the following.*

- *There exists an s-state TM M such that M halts iff* $\neg\mathrm{CON}(\mathrm{T})$.
- *The proof of M* halts iff $\neg\mathrm{CON}(\mathrm{T})$ *can be carried out in* T. *(All of the systems in the last section have this property.)*
- *If a TM M halts in $\le x$ steps then this can be proven in* T. *That is,* T *is able to run a TM for x steps and see if it halts. (All of the systems in the last section have this property.)*

We can now finally tell you the conclusion. Let x be the actual value of $\mathrm{BB}(s)$. *Then the statement*

$$\mathrm{BB}(s) = x$$

cannot be proven in T.

Proof: Assume, BWOC, that T proves $\mathrm{BB}(s) = x$.
Here is a proof in T that CON(T).

- Prove that M halts iff CON(T). This proof can be carried out in T by one of the premises of the Theorem.
- Prove that $\mathrm{BB}(s) = x$. This proof can be carried out in T by hypothesis.
- Prove that M halts by running it for x steps and seeing it halt. This proof can be carried out in T by one of the premises of the Theorem.
- The three statements above show CON(T).

By Godel's second incompleteness theorem no recursively enumerable and arithmetically sound axiomatic theory can prove its own consistency. Hence we have a contradiction. ∎

Def 12.3 Let T be a recursively enumerable and arithmetically sound axiomatic theory. $\mathrm{N_T}$ is the least number such that T cannot prove the value of $\mathrm{BB}(n)$ for any $n \geq \mathrm{N_T}$. Such a value must exist since one can build a TM M that halts iff $\neg$CON(T); if M has s states then $\mathrm{N_T} \leq s$.

From the results of Sections 10, 11, and Theorem 12.2 we have the following (which was one of the motivations for constructing small TM's whose halting is equivalent to theories being inconsistent).

Theorem 12.4

1. $\mathrm{N_{PA}} \leq 372$.
2. $\mathrm{N_{ZFC}} \leq 432$.
3. $\mathrm{N}_{\mathrm{ZFC+}\textit{There exists arbitrary large subtle cardinals}} \leq 493$

Open Problem 12.5

1. All three statements in Theorem 12.4 were proven by construction an s-state TM M such that M halt iff $\neg$CON(T) for appropriate $s \in \mathbb{N}$ and theories T. Prove these three statements, and more, using some other methods. Such a method may lead to smaller number-of-states.
2. (This is not an open problem. Its a reminder that some prior open problems will lead to more theorems like Theorem 12.4.) In Section 10 and 11 we stated open questions to find more s, T such that there is an s-state TM with M *halt iff* $\neg$CON(T). Once this is carried out, there will be more statements like Theorem 12.4.

13 What is the Smallest n Such that $\mathrm{BB}(n)$ is $\geq \cdots$

For a variety of computable fast growing functions f we ask

what is the smallest n *such that* $\mathrm{BB}(n) > f(n)$.

We present several well known, and non-so-well-known, fast growing functions.

Def 13.1

1. Ackermann's Function is defined as follows:

   ACK$(0, n) = n + 1$

   ACK$(m + 1, 0) = A(m, 1)$

   ACK$(m + 1, n + 1) = A(m, A(m + 1, n))$

   This function is not primitive recursive.

2. (This function has gotten a lot of attention. There is a numberphile video on it here:

   `https://www.youtube.com/watch?v=3P6DWAwwViU`

)

   For two colored graphs H and G we say $H \leq_{ie} G$ (called H *is inf-embedded in* G) if there is an injection from H to G such that, if x, y have greatest common ancestor z then $f(x)$, and $f(y)$ have greatest common ancestor $f(z)$.

   By a variant of Kruskal's Tree Theorem, for all infinite sequences of colored trees $T_1, T_2, \ldots$, there exists $i < j$ such that $T_i \leq_{ie} T_j$.

   TREE(n) is the length of the longest sequence of n-colored trees $T_1, \ldots, T_L$ such that (1) there is no i, j such that $T_i \leq_{ie} T_j$, (2) T_i has at most i vertices. The proof that TREE(n) exists is nonconstructive and depends on the Kruskal Tree Theorem.

   TREE(3) is enormous.

3. (This function has gotten no attention.)

   For two colored graphs H and G we say $H \leq_m G$ (often called H *is a minor* G) if you can obtain H from G be a finite sequence of the following operation: remove an edge and merge the two endpoints (Often the following operations are also allowed: delete edge, delete vertex. We do not allow that here since it would lead to a disconnected graph.)

   By a variant of Kruskal's Tree Theorem, for all c, for all infinite sequences of c-colored trees $T_1, T_2, \ldots$, (no restriction on the c-coloring) there exists $i < j$ such that there is $T_i \leq_m T_j$.

   MINORSTREE(c) is the length of the longest sequence of c-colored trees $T_1, \ldots, T_L$ such that (1) there is no i, j such that $T_i \leq T_j$, (2) T_i has at most i vertices. The proof that MINORSTREE(n) exists is nonconstructive and depends on the variant of the Kruskal Tree Theorem mentioned above.

   MINORSTREE(n) is likely as fast growing as TREE(n).

4. (This function has gotten no attention.) By a variant of the Graph Minor Theorem, for all c, for all infinite sequences of c-colored graphs $G_1, G_2, \ldots$, there exists $i < j$ such that $G_i \leq_m G_j$.

   MINORSGRAPH(c) is the length of the longest sequence of c-colored graphs $G_1, \ldots, G_L$ such that (1) there is no i, j such that $G_i \leq G_j$, (2) G_i has at most i vertices. The proof that MINORSGRAPH(n) exists is nonconstructive and depends on the variant of the Kruskal Tree Theorem mentioned above.

   MINORSGRAPH(n) might grow far faster than TREE(n) and MINORSTREE(n).

Open Problem 13.2

1. In Scott Aaronson's survey of the Busy Beaver function [Aar20] he states

   As I was writing this survey, my 7-year old daughter Lily raised the following question: what's the first n such that BB(n) > ACK(n, n)*.*

   For numbers this large we use Knuth's arrow notation.

   Pavel Kropitz showed that BB(6) $\geq 10 \uparrow\uparrow 15$. (This is 10 to the 10 to the 10 $\cdots$ 15 times.) See Scott Aaronson's Blog post

   `https://scottaaronson.blog/?p=6673`

   for the result and the reference.

   The lower bound on BB(6) has been improved since then. In addition, an enormous lower bound for BB(7) has been found. The new lower bounds from Pascal Michel's historical survey of champions are:

   - BB(6) $\geq 2 \uparrow\uparrow\uparrow 5$
   - BB(7) $\geq 2 \uparrow^{11} 2 \uparrow^{11} 3$

   See his website for details and results about those and other BB numbers:

   `https://bbchallenge.org/~pascal.michel/ha`

   A calculation shows that we can conclude:

   - BB(5) $\leq$ ACK($5, 5$).
   - The status of BB(6) and ACK($6, 6$) is not known.
   - BB(7) > ACK($7, 7$).

Hence the answer to Lily's question is either 6 or 7. It is plausible that someone will find a TM on 6 states that runs for more than ACK$(6,6)$ steps, hence showing that 6 is the answer. If the reality is that BB$(6) \le$ ACK$(6,6)$ this will be very hard, perhaps impossible (not in any technical sense), to prove.

2. Determine triples (x,y,z) such that BB$(z) >$ ACK(x,y). See where the open problems are.

3. What is the least n such that BB$(n) >$ TREE(n)?

4. What is the least n such that BB$(n) >$ MINORSTREE(n)?

5. What is the least n such that BB$(n) >$ MINORSGRAPH(n)?

6. How come TREE(n) is known to the public but neither MINORSTREE(n) or MINORSGRAPH(n) is?

14 Acknowledgment

We thank the following people:

1. Gurpreet Sing Tandi, Josue Gonzalez-Hendrix, and Jonathan Brown for the following: (1) their poster on the Brochard numbers and the Fermat primes and later conversations, inspired this column, (2) they proofread the paper and helped me improve it greatly

2. Scott Aaronson and Adam Yedidia whose paper also inspired this column.

3. Scott Aaronson, Zach DeStefano, Nicholas Drozd, Shawn Ligocki, Tristan Stérin and Matthew House for suggestions that lead to improvements and additions.

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