Open Problems Column
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This Issue's Column! This issue's Open Problem Column is by William Gasarch, Auguste Gezalyan, and Don Patrick. It is about permutable and compatible primes.

Request for Columns! I invite any reader who has knowledge of some area to contact me and arrange to write a column about open problems in that area. That area can be (1) broad or narrow or anywhere inbetween, and (2) really important or really unimportant or anywhere inbetween.

Permutable and Compatible Primes
by
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1 Permutable Primes

According to Wikipedia Richert [4] was the first person to define permutable primes[1].

Definition 1. A prime p is *permutable* (also called *absolute*) if all of the permutations of it are also prime. This definition assumes base 10; however, the notion can be defined for any base.

Example 2.

1. A 1-digit prime is permutable. So 2,3,5,7.
2. All 2-digit permutable primes: 11, 13, 17, 31, 37, 71, 73, 79, 97.
3. 113, 131, and 311 are permutable primes since 113, 131, 311 are all primes.

We will need the following notation.

Notation 3. If $m \in \mathsf{N}$ then $R_m = \frac{10^m-1}{9}$. When written in base 10 this is m 1's in a row. For example, $R_3 = 111$.

[1] We could not find the paper online, though its in Norwegian so it would not have enlightened us. The evidence Wikipedia gives is a link to a list of papers of which Richert's is one of them.

There is a Wikipedia page on Permutable Primes, and an entry in OEIS: A258706. The following are facts from the Wikipedia page.

1. All of the permutable primes with fewer than 49,081 digits are known.
2. The first 22 permutable primes are:

   2, 3, 5, 7, 11, 13, 17, 31, 37, 71, 73, 79, 97, 113, 131, 199, 311, 337, 373, 733, 919, 991.
3. The 23nd permutable prime is much bigger. It's $R_{19} = 1111111111111111111$.
4. The next permutable primes are $R_{23}, R_{317}, R_{1031}$.
5. R_{1031} is the largest known number of the form R_n that is prime.
6. The only known permutable primes bigger than 991 are of the form R_m.

We have found three more papers on the topic of permutable primes: Bhargaval & Doyle [1], Boal & Bevis [2], and Mavlo [3]. We will present some theorems from Mavlo.

We will need the following notation.

Notation 4. Let x, y be variables that represent elements of $\{0, 9\}^*$. Then:

- xy is, as usual, $x \times y$.
- $\overline{xy}$ is the base 10 number xy.

Theorem 5.

1. *If a number with ≥ 2 digits has in it any of the digits 0,2,4,5,6,8 then it is not a permutable prime. This is easy: (1) a perm that puts an even number at the right most place is even, and hence not prime, (2) a perm that puts 5 at the right most place is divisible by 5 and hence not prime.*
2. *(Bhargaval & Doyle [1]) If a number has all of the digits 1,3,7,9 then it is not a permutable prime. We give the proof below.*
3. *(Mavlo [3]) Let a, b be two distinct digits. If a number has 3 or more a's and 2 or more b's then it is not a permutable prime. We give the proof below.*

4. *(Mavlo [3]) If n is a permutable prime that is not of the form R_m then there exists digits $a, b,$ satisfying the conditions below, such that n is a permutation of $\overline{a\cdots ab}$.*

   Conditions:

   - $a, b \in \{1,3,7,9\}$ *and* $a \neq b$.
   - $(a,b) \notin \{(9,7),(9,1),(1,7),(7,1),(3,9),(9,3)\}$.

   We omit the proof of this; however, the proof in Mavlo [3] is elementary, well written, and uses parts 2 and 3 of this theorem.

Proof.
2) Let n be a number that has the digits 1,3,7,9. Permute the digits of n such that the last four digits are 1,3,7,9. Write n as

$$\overline{m1379} = \overline{m0000} + 1379.$$

Consider the following permutations of n.
$\overline{m1379} \equiv \overline{m0000} + 1379 \equiv \overline{m0000} \pmod 7$.
$\overline{m1793} \equiv \overline{m0000} + 1793 \equiv \overline{m0000} + 1 \pmod 7$.
$\overline{m3719} \equiv \overline{m0000} + 3719 \equiv \overline{m0000} + 2 \pmod 7$.
$\overline{m1739} \equiv \overline{m0000} + 1739 \equiv \overline{m0000} + 3 \pmod 7$.
$\overline{m1397} \equiv \overline{m0000} + 1397 \equiv \overline{m0000} + 4 \pmod 7$.
$\overline{m1937} \equiv \overline{m0000} + 1937 \equiv \overline{m0000} + 5 \pmod 7$.
$\overline{m7139} \equiv \overline{m0000} + 7139 \equiv \overline{m0000} + 6 \pmod 7$.
One of the above has to be $\equiv 0 \pmod 7$ and hence not prime.

3) Let n be a number that has at least 3 a's and at least 2 b's. Permute the digits of n such that the last five digits are $\overline{aaabb}$. Write n as

$$\overline{maaabb} = \overline{m00000} + 10^4a + 10^3a + 10^2a + 10^1b + 10^0b.$$

Consider the following permutations of n.
$\overline{mbaaba} = \overline{maaaaaa} + (b-a)(10^4+10^1) \equiv \overline{maaaaa} \pmod 7$.
$\overline{mabbaa} = \overline{maaaaaa} + (b-a)(10^3+10^2) \equiv \overline{maaaaa} + 1 \pmod 7$.
$\overline{mababa} = \overline{maaaaaa} + (b-a)(10^3+10^1) \equiv \overline{maaaaa} + 2 \pmod 7$.
$\overline{maabab} = \overline{maaaaaa} + (b-a)(10^2+10^0) \equiv \overline{maaaaa} + 3 \pmod 7$.
$\overline{maaabb} = \overline{maaaaaa} + (b-a)(10^1+10^0) \equiv \overline{maaaaa} + 4 \pmod 7$.
$\overline{mbaaab} = \overline{maaaaaa} + (b-a)(10^4+10^0) \equiv \overline{maaaaa} + 5 \pmod 7$.

$\overline{mbabaa} = \overline{maaaaaa} + (b-a)(10^4 + 10^2) \equiv \overline{maaaaa} + 6 \pmod 7$.

If $b - a \equiv 0 \pmod 7$ then one of a, b is even, which cannot be the case by Part 1 of this theorem. Hence we can assume $b - a \not\equiv 0 \pmod 7$. With that in mind, one of the above has to be $\equiv 0 \pmod 7$ and hence not prime.

□

The Wikipedia page on permutable primes lists the following conjectures:

1. There are an infinite number of permutable primes.
2. The only permutable primes bigger than 991 are of the form R_m.

We add the following open problems:

Open 6. *Let C_b be the conjectures that, in base b, there are an infinite number of permutable primes.*

1. *Prove an analog of Theorem 5 for permutable primes in base b.*
2. *Find b such that C_b holds.*
3. *Find b such that C_b does not hold.*
4. *Find pairs b_1, b_2 such that C_{b_1} implies C_{b_2}.*

2 Compatible Primes

We discuss a variant of the notion of permutable primes. It is not quite a generalization as we will see.

Definition 7. Let $k \in \mathsf{N}$. A number n is a k-*compatible prime* if the following hold: (a) n is prime, (b) there are k permutations of the digits of n that form different numbers, all of which are primes, and (c) there does not exist $k+1$ such permutations. If a number has a 0 in it and a perm of it puts the 0 as the lead digit that does count. For example, 601 is 2-compatible since 601 and 061 are both primes. Henceforth we denote *compatible* as *comp*. Note that this definition assumes base 10; however, one can define it for other bases.

Example 8.

1. 11 is 1-comp.

2. 23 is 1-comp since 23 is prime but 32 is not.

3. All 2-digit 2-comp primes: 13, 17, 31, 37, 71, 73, 79, 97.

4. 113 is 3-comp.

There is no way to define permutable primes in terms of comp primes. We show why one attempt does not work: since a prime of length L has $L!$ permutations of its digits, just use $k = L!$ This does not work since NOT all primes of length L have $L!$ permutations. Just look at 113, or any number that has repeated digits.

3 All the Comp Primes of Length 1, 2 or 3

L will always mean the length of the numbers being considered. k will always be the type of comp primes we want. When listing the k-comp primes we group the ones that are perms of each other.
$L = 1$, $k = 1$: The 1-comp primes of length 1 are: 2,3,5,7.
$L = 2$, $k = 1$: The 1-comp primes of length 2 are:

$$11, 19, 23, 29, 41, 43, 47, 53, 59, 61, 67, 83, 89.$$

$L = 2$, $k = 2$: The 2-comp primes of length 2 are:

$$(13, 31), (17, 71), (37, 73), (97, 79).$$

$L = 3$, $k = 1$: The 1-comp primes of length 3 are:

$$151, 211, 223, 227, 229, 233, 257, 263, 269, 353, 383, 409, 431, 433, 443, 449, 487, 499,$$

$$523, 541, 557, 599, 661, 677, 773, 827, 829, 853, 859, 881, 883, 887, 929, 997.$$

$L = 3$, $k = 2$: The 2-comp primes of length 3 are:

$$(251, 521), (563, 653), (239, 293), (313, 331), (061, 601), (569, 659), (587, 857), (089, 809),$$

$$(59, 509), (349, 439), (461, 641), (127, 271), (139, 193), (241, 421), (797, 977), (19, 109),$$

$$(283, 823), (067, 607), (577, 757), (191, 911), (367, 673), (683, 863), (467, 647), (769, 967),$$

$$(347,743),(277,727),(181,811),(463,643),(787,877),(041,401)(479,947),(11,101),$$
$$(53,503),(457,547),(619,691),(281,821).$$

$L=3$, $k=3$: The 3-comp primes of length 3 are:

$$(167,617,761),(359,593,953),(163,613,631),(199,919,991),(13,31,103),(337,373,733),$$
$$(113,131,311),(37,73,307),(157,571,751),(389,839,983),(137,173,317).$$

$L=3$, $k=4$: There are no 4-comp primes of length 3.
$L=3$, $k=5$: There are no 5-comp primes of length 3.
$L=3$, $k=6$: There are no 6-comp primes of length 3.

4 Data for Length 4,5, and 6

We show tables of what percent of primes of length L are 1-comp, 2-comp, etc. Following the tables we make some observations.

4.1 Length 4

$L=4$

k	percent
1	4.241281809613572
2	14.043355325164938
3	13.760603204524033
4	19.792648444863337
5	13.854853911404336
6	10.084825636192271
7	9.236569274269557
8	6.1262959472196045
9	4.3355325164938736
10	2.4505183788878417
11	2.0735155513666354

Observations

1. Only $\sim 4\%$ are 1-comp. This is *not* the lowest non-zero percent, which is ~ 2.

2. For $k = 1, 2, 3, 4$ the percent increases until $k = 4$ where it is ~ 19.

3. For $k = 4, \ldots, 11$ the percent is decreases until $k = 11$ where it is ~ 2.

4. For $k \geq 12$ there are no k-comp primes.

4.2 Length 5

k	percent
1	0.8131053449719
2	1.913189046992706
3	3.037187612100921
4	4.1372713141217266
5	5.404759057754395
6	4.6036111443261984
7	5.356929331579576
8	4.436207102714337
9	6.026545498027024
10	4.412292239626928
11	5.428673920841803
12	5.273227310773645
13	4.4840368288891547
14	4.974291522181035
15	3.539399736936506
16	6.074375224201842
17	4.866674638287696
18	2.786081549683128
19	2.8339112758579454
20	2.3675714456534738
21	2.3436565825660646
22	2.4273586033719957
23	1.4109769221571206
24	1.4348917852445295

(Continued on the next page.)

k	percent
25	1.1957431543704412
26	0.9446370919526485
27	1.8294870261867752
28	0.3348080832237235
29	1.159870859739328
30	0.0
31	0.6098290087289251
32	0.7652756187970824
33	0.3945952409422456
34	0.6815735979911515
35	0.0
36	0.43046753557335886
37	0.8011479134281956
38	0.0
39	0.4663398302044721

Observations

1. Only $\sim 0.8\%$ are 1-comp. This is *not* the lowest non-zero percent, which is ~ 0.4.

2. For $k = 1, \ldots, 16$ the percent very roughly increases until $k = 16$ where it is ~ 6.

3. For $k = 17, \ldots, 39$ the percent is mostly nonzero and decreasing until $k = 39$ where it is ~ 0.4.

4. For $k \geq 40$ there are no k-comp primes.

4.3 Length 6

We omit the tables and just give the observations.

Observations

1. Only $\sim 0.09\%$ are 1-comp. This is *not* the lowest non-zero percent but we later note that all of the nonzero percents are very low.

2. The largest percent is roughly 1.7.

3. For all $k \geq 161$ there are no k-comp primes.

4. For the following values of $k \leq 160$ there are no k-comp primes:

$$84, 90, 91, 92, 96, 100, 112, 119, 120, 122, 123, 125, 128, 129, 130, 134, 135, 138, 139, 140$$

$$142, 143, 145, 146, 147, 150, 151, \ldots, 159.$$

5 Conjectures

L is always the length of the prime.

1. As L gets large the percent of 1-comp primes will decrease. For $L \geq 5$ it will be ≤ 1.

2. There is some nonconstant function f such that, for all $k \geq (L!)/f(L)$, there are no k-comp primes of length L.

3. Look into what happens in different bases.

6 Acknowledgement

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[4] H.-E. Richert. Uber permutierbare primzahlen (Norwegian). *Norsk Mat. Tidsskr*, 33:50–53, 1951. This paper is not online so it is lost to history. Oh well.