REVERSE MATHEMATICS AND RECURSIVE GRAPH THEORY

William Gasarch
University of Maryland
Jeffry L. Hirst
Appalachian State University

October 30, 1994

Abstract. We examine a number of results of infinite combinatorics using the techniques of reverse mathematics. Our results are inspired by similar results in recursive combinatorics. Theorems included concern colorings of graphs and bounded graphs, Euler paths, and Hamilton paths.

Reverse mathematics provides powerful techniques for analyzing the logical content of theorems. By contrast, recursive mathematics analyzes the effective content of theorems. Theorems and techniques of recursive mathematics can often inspire related results in reverse mathematics, as demonstrated by the research presented here. Sections 1 and 2 analyze theorems on graph colorings. Section 3 considers graphs with Euler paths. Stronger axiom systems are introduced and applied to the study of Hamilton paths in Section 4. We assume familiarity with the methods of reverse mathematics, as described in [15]. Additional information, including techniques for encoding mathematical statements in second-order arithmetic, can be found in [4] and [16].

1. Graph Colorings.

1991 *Mathematics Subject Classification.* 03F35, 03D45.
Key words and phrases. recursion theory, proof theory, graph theory.
Gasarch's research was partially supported by NSF grant CCR9020079.

Typeset by $\mathcal{A}\mathcal{M}\mathcal{S}$-TEX

In this section we will consider theorems on node colorings of countable graphs. A (countable) graph G consists of a set of vertices $V \subseteq \mathbb{N}$ and a set of edges $E \subseteq [\mathbb{N}]^2$. We will abuse notation by denoting an edge by (x, y) rather than $\{x, y\}$. For $k \in \mathbb{N}$, we say that $\chi : V \to k$ is a k-coloring of G if χ always assigns different colors to neighboring vertices. That is, χ is a k-coloring if $\chi : V \to k$ and $(x, y) \in E$ implies $\chi(x) \neq \chi(y)$. If G has a k-coloring, we say that G is k-chromatic. Using an appropriate axiom system, it is possible to prove that a graph is k-chromatic if it satisfies the following local condition.

Definition 1 ($\mathbf{RCA_0}$). A graph G is *locally k-chromatic* if every finite subgraph of G is k-chromatic.

The following theorem is a reverse mathematics analog of Theorem 1 of Bean [2]. To prove that (1) implies (2), a tree is constructed in which every infinite path encodes a k-coloring. The proof of the reversal uses a graph whose k-colorings encode separating sets for a pair of injections. This implies (1) by a result of Simpson [14]. For a detailed proof, see Theorem 3.4 in [9].

Theorem 2 ($\mathbf{RCA_0}$). *For every $k \geq 2$, the following are equivalent:*

(1) $\mathbf{WKL_0}$.

(2) *If G is locally k-chromatic, then G is k-chromatic.*

In [2], Bean proved that there is a recursive 3-chromatic graph with no recursive coloring, regardless of the number of colors allowed. We now present a related theorem of reverse mathematics.

Theorem 3 ($\mathbf{RCA_0}$). *For each $k \geq 2$, the following are equivalent:*

(1) $\mathbf{WKL_0}$.

(2) *If G is locally k-chromatic, then G is $(2k-1)$-chromatic.*

Proof. Whenever $k \geq 2$, we have that $2k - 1 > k$, so every k-coloring is automatically a $(2k-1)$-coloring. Consequently, (1) implies (2) follows immediately from Theorem 2.

We will now prove that (2) implies (1) when $k = 2$, and then indicate how the argument can be generalized to any $k \in \mathbb{N}$. By a result of SIMPSON [14], $\mathbf{WKL_0}$ can be proved by showing that the ranges of an arbitrary pair of disjoint injections can be separated. Let $f : \mathbb{N} \to \mathbb{N}$ and $g : \mathbb{N} \to \mathbb{N}$ be injections such that for all $m, n \in \mathbb{N}$, $f(n) \neq g(m)$. We will construct a 2-chromatic graph with the property that any 3-coloring of G encodes a set S such that $y \in \mathit{Range}(f)$ implies $y \in S$, and $y \in S$ implies $y \notin \mathit{Range}(g)$.

The graph G contains an infinite complete bipartite subgraph consisting of upper vertices $\{b^u_n : n \in \mathbb{N}\}$, lower vertices $\{b^l_n : n \in \mathbb{N}\}$, and connecting edges $\{(b^u_n, b^l_m) : n, m \in \mathbb{N}\}$. Also, G contains an infinite collection of pairs of vertices, denoted by n^u and n^l for $n \in \mathbb{N}$. Each such pair is connected, so the edges $\{(n^u, n^l) : n \in \mathbb{N}\}$ are included in G. Additional connections depend on the injections f and g. If $f(i) = n$, add the edges (b^u_m, n^l) and (b^l_m, n^u) for all $m \geq i$. If $g(i) = n$, add the edges (b^u_m, n^u) and (b^l_m, n^l) for all $m \geq i$. Naively, if n is in the range of f or g, then the pair (n^u, n^l) is connected to the complete bipartite subgraph. If n is in the range of G, the pair is "flipped" before it is connected. The reader can verify that G is Δ^0_1 definable in f and g, and thus exists by the recursive comprehension axiom. Every finite subgraph of G is clearly bipartite, so G is locally 2-chromatic. Thus, by (2), G has a 3-coloring; denote it by $\chi : G \to 3$.

If χ is a 2-coloring, we can define the separating set, S, by

$$S = \{y \in \mathbb{N} : \chi(y^u) = \chi(b^u_0) \vee \chi(y^l) = \chi(b^l_0)\}.$$

When χ uses all 3 colors, we must modify the construction of S. In particular, we must find a $j \in \mathbb{N}$ such that

(a) $\forall y(\exists n(n \geq j \wedge f(n) = y) \to (\chi(y^u) = \chi(b^u_j) \vee \chi(y^l) = \chi(b^l_j)))$, and

(b) $\forall y(\exists n(n \geq j \wedge g(n) = y) \to (\chi(y^l) \neq \chi(b^l_j) \wedge \chi(y^u) \neq \chi(b^u_j)))$.

Suppose, by way of contradiction, that no such j exists. The for some m and y, either $f(m) = y \wedge \chi(y^u) \neq \chi(b^u_0) \wedge \chi(y^l) \neq \chi(b^l_0)$ or $g(m) = y \wedge (\chi(y^l) = \chi(b^l_0) \vee \chi(y^u) = \chi(b^u_0))$. If $f(m) = y$, since χ is a 3-coloring, either $\chi(y^u) = \chi(b^l_0)$ or $\chi(y^l) = \chi(b^u_0)$. By the construction of G, for every $n > m$, $\chi(b^u_m) = \chi(b^u_n)$ and $\chi(b^l_m) = \chi(b^l_n)$. Similarly, the case $g(m) = y$ also yields a point beyond which the complete bipartite subgraph of G is 2-colored. By the negation of (a) and (b), there is an $m' > m$ and a $z \in \mathbb{N}$ such that either $f(m') = z \wedge \chi(z^u) \neq \chi(b^u_m) \wedge \chi(z^l) \neq \chi(b^l_m)$ or $g(m') = z \wedge (\chi(z^l) = \chi(b^l_m) \vee \chi(z^u) = \chi(b^u_m))$. If $f(m') = z$, then since χ is a 3-coloring, either $\chi(z^u) = \chi(b^l_m)$ or $\chi(z^l) = \chi(b^u_m)$. Since $m' > m$, $\chi(b^l_m) = \chi(b^l_{m'})$ and $\chi(b^u_m) = \chi(b^u_{m'})$, so either $\chi(z^l) = \chi(b^u_{m'})$ or $\chi(z^u) = \chi(b^l_{m'})$. But $(z^l, b^u_{m'})$ and $(z^u, b^l_{m'})$ are edges of G, so χ is not a 3-coloring. Assuming $g(m') = z$ yields a similar contradiction. Thus, a j satisfying (a) and (b) exists.

Given an integer j satisfying (a) and (b), the separating set S may be defined as the union of $\{y \in \mathbb{N} : \exists n < j\, f(n) = y\}$ and

$$\{y \in \mathbb{N} : (\forall n < j\, g(n) \neq y) \wedge (\chi(y^u) = \chi(b^u_j) \vee \chi(y^l) = \chi(b^l_j))\}$$

S is Δ^0_1 definable in χ and j, so the recursive comprehension axiom assures the existence of S. If $f(n) = y$ and $n < j$, then $y \in S$. If $f(n) = y$ and $n \geq j$, then by (a) and the fact that f and g have disjoint ranges, $y \in S$. Thus $Range(f) \subseteq S$. If $g(n) = y$, and $n < j$, then since the ranges of f and g are disjoint we have $y \notin S$. If $g(n) = y$ and $n \geq j$, by (b) $y \notin S$. Thus S is the desired separating set. This completes the proof for $k = 2$.

For $k > 2$, the preceding proof requires the following modifications. Replace the complete bipartite subgraph of G by a complete k-partite subgraph with vertices $\{b^p_m : p < k \wedge m \in \mathbb{N}\}$. Each pair (n^u, n^l) is replaced by a complete graph on the vertices $\{n^p : p < k\}$. If $f(i) = n$, add the edges $(b^p_m, n^{p'})$ for all $m \geq i$ and all $p \neq p'$ less than k. If $g(i) = n$, twist the subgraph before attaching it. That is, add the edges $(b^p_m, n^{p'})$ for all $m \geq i$ and all p and p' less than k such that $p \not\equiv p' + 1 \pmod{k}$. The argument locating the integer j is similar, except that m and m' must be replaced by a sequence $m_1, \ldots, m_k$. Beyond the point m_{k-1}, the complete k-partite subgraph of G is k-colored by χ. The definition of S is very similar, except that a bounded quantifier should be used to avoid the k-fold conjunction. $\square$

Because Bean [2] constructed a recursive graph with no recursive coloring, the following conjecture seems reasonable. Unfortunately, even the case where $k = 2$ and $m = 4$ remains open.

Conjecture 4 ($\mathbf{RCA_0}$). *For each $k \geq 2$ and each $m \geq k$ the following are equivalent:*

(1) $\mathbf{WKL_0}$.

(2) *If G is locally k-chromatic, then G is m-chromatic.*

2. Bounded graphs and sequences of graphs.

As noted above, a locally k-chromatic recursive graph may not have a recursive coloring, regardless of the number of colors used. By contrast, highly recursive graphs always have recursive colorings. A proof theoretic analog of a highly recursive graph is a bounded graph.

Definition 5 ($\mathbf{RCA_0}$). A graph $G = \langle V, E\rangle$ is *bounded* if there is a function $h : V \to \mathbb{N}$ such that for all $x, y \in V$, $(x, y) \in E$ implies $h(x) \geq y$.

Schmerl [11] proved that every highly recursive k-chromatic graph has a recursive $(2k-1)$-coloring. (This result was independently rediscovered by Carstens and Pappinghaus [3].) Formalizing Schmerl's proof in $\mathbf{RCA_0}$ yields the following result.

Theorem 6 ($\mathbf{RCA_0}$). *For $k \in \mathbb{N}$, if G is a bounded locally k-chromatic graph, then G is $(2k-1)$-chromatic.*

In [11], Schmerl also showed that for each $k \geq 2$, there is a highly recursive k-chromatic graph which has no recursive $2k - 2$ coloring. Using the constructions from his proof, one can easily prove the following theorem.

Theorem 7 ($\mathbf{RCA_0}$). *For every $k \geq 2$, the following are equivalent:*

(1) $\mathbf{WKL_0}$.

(2) *If G is a bounded locally k-chromatic graph, then G is $(2k-2)$-chromatic.*

Proof. Whenver $k \geq 2$, we have $2k-2 \geq k$, so every k-coloring is automatically a $(2k-2)$-coloring. Consequently, (1) implies (2) follows immediately from Theorem 2. To prove that (2) implies (1), we imitate SCHMERL's [11] construction of a bounded locally k-chromatic graph for which any $(2k-2)$-coloring separates the ranges of a pair of disjoint injections. An application of a theorem of SIMPSON [14] yields $\mathbf{WKL_0}$. □

We will close this section with a theorem concerning *sequences* of graphs and its recursion theoretic corollary. We say that a graph G is *colorable* if there exists an integer k such that G is k-chromatic.

Theorem 8 ($\mathbf{RCA_0}$). *The following are equivalent:*

(1) $\mathbf{ACA_0}$.

(2) *Given a countable sequence of graphs, $\langle G_i : i \in \mathbb{N}\rangle$, there is a function $f : \mathbb{N} \to 2$ such that $f(i) = 1$ if G_i is colorable and $f(i) = 0$ otherwise.*

Proof. To prove that (1) implies (2), assume $\mathbf{ACA_0}$ and let $\langle G_i : i \in \mathbb{N}\rangle$ be a sequence of graphs. Define $f : \mathbb{N} \to \mathbb{N}$ by setting $f(i) = 1$ if there exists a $k \in \mathbb{N}$ such that G_i is locally k-chromatic, and setting $f(i) = 0$ otherwise. Since "G_i is locally k-chromatic" is an arithmetical sentence with parameter G_i, f exists by the arithmetical comprehension axiom. Since $\mathbf{ACA_0}$ implies $\mathbf{WKL_0}$, we may apply Theorem 2 to show that $f(i) = 1$ if and only if G_i is colorable.

To prove the converse, assume $\mathbf{RCA_0}$ and (2). To prove $\mathbf{ACA_0}$ it suffices to show that for every injection g, *Range*(g) exists [14]. Define the sequence of graphs $\langle G_i : i \in \mathbb{N}\rangle$ as follows. Let $\{v_j : j \in \mathbb{N}\}$ be the vertices of G_i. If $j < k$ and $\forall m \leq k(g(m) \neq i)$, add the edge (v_j, v_k) to G_i. $\mathbf{RCA_0}$ can prove that $\langle G_i : i \in \mathbb{N}\rangle$ exists, and G_i is colorable if and only if $i \in$ *Range*(g). Thus, the function f supplied by (2) is the characteristic function for *Range*(g). By the recursive comprehension axiom, *Range*(g) exists. □

Corollary 9. *There is a recursive sequence of recursive graphs $\langle G_i : i \in \mathbb{N}\rangle$ such that $0'$ is recursive in $\{i \in \mathbb{N} : G_i$ is colorable$\}$.*

Proof. In the proof of the reversal for Theorem 8, let g be a recursive function such that $0'$ is recursive in $Range(g)$. The sequence of graphs constructed in the proof has the desired properties. $\square$

3. Euler paths.

Now, we will turn to the study of Euler paths. A *path* in a graph G is a sequence of vertices $v_0, v_1, v_2, \ldots$ such that for every $i \in \mathbb{N}$, (v_i, v_{i+1}) is an edge of G. A path is called an *Euler path* if it uses every edge of G exactly once.

The following terminology is useful in determining when a graph has an Euler path. A graph $G = \langle V, E \rangle$ is *locally finite* if for each vertex v, the set $\{u \in V : (v, u) \in E\}$ is finite. If H is a subgraph of G, $G - H$ denotes the graph obtained by deleting the edges of H from G. Using this terminology, we can describe a condition which, from a naive viewpoint, is sufficient for the existence of an Euler path.

Definition 10 ($\mathbf{RCA_0}$). A graph G is *pre-Eulerian* if it is

(1) connected,

(2) has at most one vertex of odd degree,

(3) if it has no vertices of odd degree, then it has at least one vertex of infinite degree, and

(4) if H is any finite subgraph of G then $G - H$ has exactly one infinite connected component.

Note that the formula "G is pre-Eulerian" is arithmetical in the set parameter G. $\mathbf{RCA_0}$ suffices to prove that every graph with an Euler path is pre-Eulerian. However, $\mathbf{RCA_0}$ can only prove that bounded pre-Eulerian graphs have Euler paths. (Bounded graphs are defined in Section 3.) This result is just a formalization of BEAN'S [1] proof that every highly recursive pre-Eulerian graph has a recursive Euler path.

Theorem 11 ($\mathbf{RCA_0}$). *If G is a bounded pre-Eulerian graph, then G has an Euler path.*

If G is not bounded, additional axiomatic strength is required to prove the existence of an Euler path.

Theorem 12 ($\mathbf{RCA_0}$). *The following are equivalent:*

(1) $\mathbf{ACA_0}$.

(2) *If G is a pre-Eulerian graph, then G has an Euler path.*

(3) *If G is a locally finite pre-Eulerian graph, then G has an Euler path.*

Proof. To prove that (1) implies (2), assume $\mathbf{ACA_0}$ and let G be a pre-Eulerian graph. Let $\langle E_i : i \in \mathbb{N} \rangle$ be an enumeration of the edges of G. Let v_0 be the vertex of G of odd degree, or a vertex of infinite degree if no odd vertex exists. Imitating the proof of Theorem 3.2.1 of Ore [10], there is a finite path P containing the edge E_0 such that

- P starts at v_0,
- $G - P$ is connected, and
- P ends at the odd vertex of $G - P$, or at an infinite vertex of $G - P$ if no odd vertex exists.

Furthermore, since the finite paths of G can be encoded by integers, we can pick the unique path P_0 satisfying the conditions above and having the least code. Similarly, any path P_i satisfying the three conditions can be extended to a unique path P_{i+1} which contains the edge E_{i+1}, satisfies the three conditions, and has the least code among all paths with these properties. Note the P_{i+1} extends P_i by including P_i as an initial segment. The reader may verify that the sequence of paths $\langle P_i : i \in \mathbb{N} \rangle$ is arithmetically definable in G, and so exists by arithmetical comprehension. Let v_i denote the i^{th} vertex of P_i. Then the sequence $\langle v_i : i \in \mathbb{N} \rangle$ exists by recursive comprehension and includes each P_i as an initial segment. Consequently, $\langle v_i : i \in \mathbb{N} \rangle$ defines an Euler path through G.

Since (3) is a special case of (2), showing that (3) implies (1) will complete the proof of the theorem. Assume $\mathbf{RCA_0}$ and fix an injection $f : \mathbb{N} \to \mathbb{N}$. We will construct a locally finite pre-Eulerian graph G such that every Euler path through G encodes $Range(f)$.

Define the vertices of G by

$$V = \{a_n, b_n, c_n : n \in \mathbb{N}\}.$$

For each n, include the edges (a_n, a_{n+1}) and (b_n, c_n) in G. Additionally, for each i and n, if $f(i) = n$ then include the edges (a_n, b_i) and (c_i, a_n) in G. $\mathbf{RCA_0}$ suffices to prove that G exists, and is both locally finite and pre-Eulerian. By (3), G has an Euler path. Note that $n \in \mathit{Range}(f)$ if and only if the first occurrence of a_n in the Euler path is not followed immediately by a_{n+1}. By the recursive comprehension axiom, $\mathit{Range}(f)$ exists. Since f was an arbitrary injection, this suffices to prove $\mathbf{ACA_0}$ [14]. $\square$

Corollary 13. *There is a recursive pre-Eulerian graph G such that $0'$ is recursive in every Euler path through G.*

Proof. Let f be a recursive function such that $0'$ is recursive in $\mathit{Range}(f)$. Construct the graph G as in the proof of the reversal in Theorem 12. Then G is recursive, and $\mathit{Range}(f)$ is recursive in every Euler path through G. $\square$

$\mathbf{ACA_0}$ also suffices to address the problem of determining which elements of a sequence of graphs have Euler paths. This contrasts sharply with the situation for Hamilton paths, as described in Theorem 20.

Theorem 14 ($\mathbf{RCA_0}$). *The following are equivalent:*

(1) $\mathbf{ACA_0}$.

(2) *Given a countable sequence of graphs, $\langle G_i : i \in \mathbb{N}\rangle$, there is a set $Z \subseteq \mathbb{N}$ such that $i \in Z$ if and only if G_i has an Euler path.*

Proof. To prove that (1) implies (2), note that $\{i \in \mathbb{N} : G_i \text{ is pre-Eulerian}\}$ is arithmetically definable, and by the previous results contains only those i such that G_i has an Euler path.

To prove the converse, one constructs a sequence of graphs so that the set provided by (2) encodes the range of a given injection. $\square$

Theorem 14 can be used to establish rough upper and lower bounds for the complexity of the problem of determining which graphs in a sequence have Euler paths. Sharper bounds can be found in the work of Beigel and Gasarch (see [6]).

Remark. A *two-way* or *endless* Euler path is a bijection between the integers (both positive and negative) and the set of edges of G such that each edge shares one vertex with its predecessor and its other vertex with its successor. Theorems 11, 12, and 14 can be modified to address the existence of two-way Euler paths.

4. Hamilton paths.

Now we will consider theorems on the existence of Hamilton paths. A path through a graph G is called a (one way) *Hamilton path* if it uses every vertex of G exactly once. There is no arithmetical analog of the characterization "pre-Eulerian" for graphs containing Hamilton paths. Consequently, all the results of this section concern sequences of graphs.

The proofs of the theorems in this section are all reasonably straightforward, and have been omitted for the sake of brevity. The proofs of the graph theoretic statements from the axiom systems require only formalization, followed by direct application of the axioms. Each reversal relies on the construction of a sequence of graphs from a sequence of trees, coupled with an application of Lemma 3.14 of [5]. The existence of the desired constructions follows from the following lemma, which can be proved by imitating the proof of Theorem 1 of Harel [5].

Lemma 15 ($\mathbf{RCA_0}$). *Given a sequence of trees $\langle T_i : i \in \mathbb{N}\rangle$, there is a sequence of graphs $\langle G_i : i \in \mathbb{N}\rangle$ such that*

(1) *for each $i \in \mathbb{N}$, T_i has a (unique) path if and only if G_i has a (unique) Hamiltonian path, and*

(2) *if there is a sequence $\langle P_i : i \in \mathbb{N}\rangle$ such that P_i is a Hamiltonian path through G_i for each $i \in \mathbb{N}$, then there is a sequence $\langle P'_i : i \in \mathbb{N}\rangle$ such that P'_i is a path through T_i for each $i \in \mathbb{N}$.*

The next three theorems analyze the following tasks:

(1) finding Hamilton paths through graphs known to have such paths,

(2) determining whether graphs that have at most one Hamilton path have such a path, and

(3) determining whether arbitrary graphs have Hamilton paths.

Using proof theoretic strength as a measure of difficulty, we shall see that these tasks are strictly increasing in order of difficulty.

Theorem 16 ($\mathbf{RCA_0}$). *The following are equivalent:*

(1) Σ^1_1–$\mathbf{AC_0}$.

(2) *If $\langle G_i : i \in \mathbb{N}\rangle$ is a sequence of graphs such that each G_i has a Hamilton path, then there is a sequence $\langle P_i : i \in \mathbb{N}\rangle$ such that for each i, P_i is a Hamilton path through G_i.*

From Theorem 16, together with the fact that ω together with the hyperarithmetical sets is a model of Σ^1_1–$\mathbf{AC_0}$ [16], we can draw the following recursion theoretic conclusion.

Corollary 17. *If $\langle G_i : i \in \mathbb{N}\rangle$ is a hyperarithmetical sequence of graphs, each of which has a hyperarithmetical Hamilton path, then there is a hyperarithmetical sequence $\langle P_i : i \in \mathbb{N}\rangle$ such that for each i, P_i is a Hamilton path through G_i.*

Using Theorem 5.2 of [16] to provide an appropriate characterization of $\mathbf{ATR_0}$, it is easy to prove:

Theorem 18 ($\mathbf{RCA_0}$). *The following are equivalent:*

(1) $\mathbf{ATR_0}$.

(2) *If $\langle G_i : i \in \mathbb{N}\rangle$ is a sequence of graphs each of which has at most one Hamilton path, then there is a set $Z \subseteq \mathbb{N}$ such that for all $i \in \mathbb{N}$, $i \in Z$ if and only if G_i has a Hamilton path.*

The following corollary is a recursion theoretic consequence of Theorem 18, together with the fact that ω together with the hyperarithmetical sets is not a model of $\mathbf{ATR_0}$ [16].

Corollary 19. *There is a hyperarithmetical sequence of graphs* $\langle G_i : i \in \mathbb{N}\rangle$*, each of which has at most one hyperarithmetical Hamilton path, such that the set* $\{i \in \mathbb{N} : G_i$ *has a hyperarithmetical Hamilton path*$\}$ *is not hyperarithmetical.*

Now we will analyze the third and most difficult task. Theorem 20 is closely related to Harel's proof [7] that the problem of finding a Hamiltonian path is Σ^1_1 complete.

Theorem 20 $(\mathbf{RCA_0})$**.** *The following are equivalent:*

(1) $\mathbf{\Pi^1_1}-\mathbf{CA_0}$.

(2) *If* $\langle G_i : i \in \mathbb{N}\rangle$ *is a sequence of graphs, then there is a set* $Z \subseteq \mathbb{N}$ *such that* $i \in Z$ *if and only if* G_i *has a Hamilton path.*

Theorem 20 contrasts nicely with Theorem 14. Since $\mathbf{\Pi^1_1}-\mathbf{CA_0}$ is a much stronger axiom system than $\mathbf{ACA_0}$, we can conclude that it is more difficult to determine if certain graphs have Hamilton paths than to determine if they have Euler paths. Determining which finite graphs have Hamilton paths is an NP-complete problem, while determining which finite graphs have Euler paths is polynomial time computable. It would be nice to know if this sort of parallel is common, and exactly what it signifies.

References

1. D. R. Bean, *Recursive Euler and Hamiltonian paths*, Proc. Amer. Math. Soc. **55** (1976), 385–394.
2. D. R. Bean, *Effective Coloration*, J. Symb. Logic **41** (1976), 469–480.
3. H.-G. Carstens and P. Papinghaus, *Recursive coloration of countable graphs*, Ann. Pure Appl. Logic **25** (1983), 19–45.
4. H. Friedman, *Some systems of second order arithmetic and their use*, Proceedings of the International Congress of Mathematicians, Vol. 1, (Vancouver, Canada, 1974), Canadian Mathematical Congress, 1975, pp. 235–242.
5. H. Friedman and J. Hirst, *Weak comparability of well orderings and reverse mathematics*, Ann. Pure Appl. Logic **47** (1990), 11–29.
6. W. Gasarch, *A survey of recursive combinatorics*, (in preparation).
7. D. Harel, *Hamiltonian paths in infinite graphs*, Israel J. of Math. **76** (1991), 317–336.
8. J. Hirst, *Combinatorics in subsystems of second order arithmetic*, Ph.D. Thesis, The Pennsylvania State University, 1987.
9. J. Hirst, *Marriage theorems and reverse mathematics*, Logic and Computation, Contemporary Math. 106 (W. Sieg, ed.), AMS, Providence, RI, 1990, pp. 181–195.
10. O. Ore, *Theory of graphs*, AMS Colloquium Publications Vol. 38, AMS, Providence, RI, 1962.
11. J. Schmerl, *Recursive colorings of graphs*, Can. J. Math. **32** (1980), 821–830.
12. D. Scott, *Algebras of sets binumerable in complete extensions of arithmetic*, Proc. Symp. Pure Math. **5** (1962), 117–121.

13. J. Shoenfield, *Degrees of models*, J. Symb. Logic **25** (1960), 233–237.
14. S. Simpson, *Which set existence axioms are needed to prove the Cauchy/Peano theorem for ordinary differential equations?*, J. Symb. Logic **49** (1984), 783–802.
15. ———, *Subsystems of Z_2 and reverse mathematics*, Proof Theory (G. Takeuti, ed.), North-Holland, Amsterdam New York, 1985.
16. ———, *Subsystems of second order arithmetic*, (in preparation).

Dept. of Computer Science and Institute for Advanced Studies, University of Maryland, College Park, MD 20742
E-mail address: gasarch@cs.umd.edu

Dept. of Mathematical Sciences, Appalachian State University, Boone, NC 28608
E-mail address: jlh@math.appstate.edu