A Statement in Combinatorics that is Independent of ZFC
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1 Introduction

There are some statements that are independent of Zermelo-Frankl Set Theory (henceforth ZFC). Such statements cannot be proven or disproven by conventional mathematics. The Continuum Hypothesis is one such statement ("There is no cardinality strictly between $\aleph_0$ and $2^{\aleph_0}$.") Many such statements are unnatural in that they deal with objects only set theorists and other logicians care about.

We present a natural statement in combinatorics that is independent of ZFC. The result is due to Erdős. In the last section we will discuss the question of whether the statement is really natural.

Notation 1.1. We use N to denote $\{0, 1, 2, \ldots\}$. We use N^+ to denote $\{1, 2, 3, \ldots\}$. If $n \in \mathsf{N}^+$ then $[n]$ is the set $\{1, 2, \ldots, n\}$. We use R and C to denote the sets of real and complex numbers, respectively. We use Z to denote the integers. We use k-AP to refer to an arithmetic progression with k distinct elements. For a set A and $k \in \mathsf{N}$, we let $\binom{A}{k}$ denote the set of k-element subsets of A.

2 Colorings and equations

Definition 2.1. A *finite coloring* of a set S is a map from S to a finite set. An $\aleph_0$-*coloring* of a set S is a map from S to a countable set.

The following theorem is well known. We prove it for the sake of completeness.

Theorem 2.2. *For any finite coloring of* N^+*, there exists distinct monochromatic* e_1, e_2, e_3, e_4 *such that*

$$e_1 + e_2 = e_3 + e_4.$$

Proof. Let COL be a finite coloring of N^+. Let $[c]$ be image of COL.

First Proof

Recall Ramsey's theorem (6; 8; 13) on N: for any finite coloring of *unordered pairs of naturals* there exists an infinite set A such that all pairs of elements from A have the same color.

Let $COL^* : \binom{\mathsf{N}}{2} \rightarrow [c]$ be defined by $COL^*(\{a, b\}) = COL(|a - b|)$. Let A be the set that exists by Ramsey's theorem. Let $a_1 < a_2 < a_3 < a_4 \in A$. Since A is infinite we can take a_1, a_2, a_3, a_4 such that the six numbers $\left\{a_j - a_i \mid \{i, j\} \in \binom{[4]}{2}\right\}$ are distinct.

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Since all of the $COL^*(\{a_i, a_j\})$ are the same color we have that, for $i < j$, $COL(a_j - a_i)$ are all the same color. Let

$$\begin{aligned} e_1 &= a_2 - a_1 \\ e_2 &= a_4 - a_2 \\ e_3 &= a_3 - a_1 \\ e_4 &= a_4 - a_3 \end{aligned}$$

Clearly e_1, e_2, e_3, e_4 are distinct, $COL(e_1) = COL(e_2) = COL(e_3) = COL(e_4)$, and $e_1 + e_2 = e_3 + e_4$.

Second Proof

Recall van der Waerden's theorem (7; 8; 10; 15): For all k, for any finite coloring of N^+, there exists a monochromatic k-AP, that is, a k-AP all of whose elements are the same color.

Apply van der Waerden's Theorem to COL with $k = 4$. There exists $a, d \in \mathsf{N}^+$ such that a, $a + d$, $a + 2d$, $a + 3d$ are the same color. Let

$$\begin{aligned} e_1 &= a \\ e_2 &= a + 4d \\ e_3 &= a + 2d \\ e_4 &= a + 3d \end{aligned}$$

$\square$

Note 2.3. Rado's theorem characterizes which equations lead to theorems like Theorem 2.2 and which ones do not. We will discuss Rado's theorem in Section 7.

3 What if we color the reals?

What if we finitely color the Reals? Theorem 2.2 will still hold since we can just restrict the coloring to N^+. What if we $\aleph_0$-color the reals?

Let S be the following statement:

For any $\aleph_0$-coloring of the reals, there exist distinct monochromatic e_1, e_2, e_3, e_4 such that

$$e_1 + e_2 = e_3 + e_4.$$

Is S true? Before answering this we need to decide what *truth* really is. We consider a statement to be true iff it can be derived from the axioms of ZFC (Zermelo-Frankel with Choice). This encompasses most of mathematics. All of our reasoning is done in ZFC.

It turns out that S is equivalent to the negation of CH, and hence is independent of ZFC. Komjáth (9) claims that Erdős proved this result. The proof we give is due to Davies (3). The goal of our paper is to present and popularize this result.

Definition 3.1. ω is the first infinite ordinal, namely $\{1 < 2 < 3 < \cdots\}$. (Formally it is any ordering that is equivalent to $\{1 < 2 < 3 < \cdots\}$.) ω_1 is the first uncountable ordinal. ω_2 is the first ordinal with cardinality bigger than ω_1.

Fact 3.2.

1. *If CH is true, then there is a bijection between* R *and* ω_1. *For all* $\alpha \in \omega_1$ *let* α *map to* x_α. *We can picture the reals listed out as such:*

$$x_0, x_1, x_2, \ldots, x_\alpha, \ldots.$$

   Note that, for all $\alpha \in \omega_1$, *the set* $\{x_\beta \mid \beta < \alpha\}$ *is countable.*

2. *If CH is false, then there is an injection from* ω_2 *to* R.

4 CH $\implies \neg S$

Definition 4.1. Let $X \subseteq \mathsf{R}$. Then $CL(X)$ is the smallest set $Y \supseteq X$ of reals such that

$$a, b, c \in Y \implies a + b - c \in Y.$$

Lemma 4.2.

1. *If* X *is countable then* $CL(X)$ *is countable.*
2. *If* $X_1 \subseteq X_2$ *then* $CL(X_1) \subseteq CL(X_2)$.

Proof. 1) Assume X is countable. $CL(X)$ can be defined with an ω-induction (that is, an induction just through ω).

$$\begin{aligned} C_0 &= X \\ C_{n+1} &= C_n \cup \{a + b - c \mid a, b, c \in C_n\} \end{aligned}$$

One can easily show that $CL(X) = \cup_{i=0}^{\infty} C_i$ and that this set is countable.
2) This is an easy exercise. $\square$

Theorem 4.3. *Assume CH is true. There exists an* $\aleph_0$*-coloring of* R *such that there are no distinct monochromatic* e_1, e_2, e_3, e_4 *such that*

$$e_1 + e_2 = e_3 + e_4.$$

Proof. Since we are assuming CH is true, we have, by Fact 3.2.1, a bijection between R and ω_1. For each $\alpha \in \omega_1$ let x_α be the real that α maps to.

For $\alpha < \omega_1$ let

$$X_\alpha = \{x_\beta \mid \beta < \alpha\}.$$

Note the following:

1. For all α, X_α is countable.

2. $X_0 \subset X_1 \subset X_2 \subset X_3 \subset \cdots \subset X_\alpha \subset \cdots$

3. $\bigcup_{\alpha<\omega_1} X_\alpha = \mathsf{R}$.

We define another increasing sequence of sets Y_α by letting

$$Y_\alpha = CL(X_\alpha).$$

Note the following:

1. For all α, Y_α is countable. This is from Lemma 4.2.1.

2. $Y_0 \subseteq Y_1 \subseteq Y_2 \subseteq Y_3 \subseteq \cdots \subseteq Y_\alpha \subseteq \cdots$. This is from Lemma 4.2.2.

3. $\bigcup_{\alpha<\omega_1} Y_\alpha = \mathsf{R}$.

We now define our last sequence of sets:
For all $\alpha < \omega_1$,

$$Z_\alpha = Y_\alpha - \left(\bigcup_{\beta<\alpha} Y_\beta \right).$$

Note the following:

1. Each Z_α is finite or countable.

2. The Z_α form a partition of R (although some of the Z_α may be empty).

We will now define an $\aleph_0$-coloring of R: For each $\alpha < \omega_1$ we color Z_α with colors in ω making sure that every element of Z_α has a different color (this is possible since Z_α is at most countable).

Assume, by way of contradiction, that there are distinct monochromatic e_1, e_2, e_3, e_4 such that

$$e_1 + e_2 = e_3 + e_4.$$

Let $\alpha_1, \alpha_2, \alpha_3, \alpha_4 \in \omega_1$ be such that $e_i \in Z_{\alpha_i}$. Since all of the elements in any Z_α are colored differently, all of the α_i's are different. We will assume $\alpha_1 < \alpha_2 < \alpha_3 < \alpha_4$. The other cases are similar. Note that

$$e_4 = e_1 + e_2 - e_3.$$

and

$$e_1, e_2, e_3 \in Z_{\alpha_1} \cup Z_{\alpha_2} \cup Z_{\alpha_3} \subseteq Y_{\alpha_1} \cup Y_{\alpha_2} \cup Y_{\alpha_3} = Y_{\alpha_3}.$$

Since $Y_{\alpha_3} = CL(X_{\alpha_3})$ and $e_1, e_2, e_3 \in Y_{\alpha_3}$, we have $e_4 \in Y_{\alpha_3}$. Hence $e_4 \notin Z_{\alpha_4}$. This is a contradiction. $\square$

What was it about the equation

$$e_1 + e_2 = e_3 + e_4$$

that made the proof of Theorem 4.3 work? Absolutely nothing:

Theorem 4.4. *Let $n \geq 2$. Let $a_1, \ldots, a_n \in \mathsf{R}$ be nonzero. Assume CH is true. There exists an $\aleph_0$-coloring of R such that there are no distinct monochromatic $e_1, \ldots, e_n$ such that*

$$\sum_{i=1}^{n} a_i e_i = 0.$$

Proof sketch. Since this proof is similar to the last one we just sketch it.

Definition 4.5. Let $X \subseteq R$. $CL(X)$ is the smallest superset of X such that the following holds:

For all $m \in \{1, \ldots, n\}$ and for all $e_1, \ldots, e_{m-1}, e_{m+1}, \ldots, e_n$,

$$e_1, \ldots, e_{m-1}, e_{m+1}, \ldots, e_n \in CL(X) \implies -(1/a_m) \sum_{i \in \{1,\ldots,n\}-\{m\}} a_i e_i \in CL(X).$$

Let X_α, Y_α, Z_α be defined as in Theorem 4.3 using this new definition of CL. With these definitions define an $\aleph_0$-coloring like the one in the proof of Theorem 4.3.

Assume, by way of contradiction, that there are distinct monochromatic $e_1, \ldots, e_n$ such that

$$\sum_{i=1}^{n} a_i e_i = 0.$$

Let $\alpha_1, \ldots, \alpha_n$ be such that $e_i \in Z_{\alpha_i}$. Since all of the elements in any Z_α are colored differently, all of the α_i's are different. We will assume $\alpha_1 < \alpha_2 < \cdots < \alpha_n$. The other cases are similar. Note that

$$e_n = -(1/a_n) \sum_{i=1}^{n-1} a_i e_i \in CL(X)$$

and

$$e_1, \ldots, e_{n-1} \in Z_{\alpha_1} \cup \cdots \cup Z_{\alpha_{n-1}} \subseteq Y_{\alpha_{n-1}}.$$

Since $Y_{\alpha_{n-1}} = CL(X_{\alpha_{n-1}})$ and $e_1, \ldots, e_{n-1} \in Y_{\alpha_{n-1}}$, we have $e_n \in Y_{\alpha_{n-1}}$. Hence $e_n \notin Z_{\alpha_n}$. This is a contradiction. $\square$

Note 4.6. The converse to Theorem 4.4 is not true. The $s = 2$ case of Theorem 7.7 (in Section 7) states that every $\aleph_0$-coloring of N has a distinct monochromatic solution to $x_1 + 2x_2 = x_3 + x_4 + x_5$ is $\aleph_0$ iff $2^{\aleph_0} > \aleph_2$. Therefore if $2^{\aleph_0} = \aleph_2$ (so CH is false) then there is an $\aleph_0$-coloring of N such that there is no monochromatic distinct solution to $x_1 + 2x_2 = x_3 + x_4 + x_5$. This contradicts the converse of Theorem 4.4

5 $\neg$ CH $\implies S$

Theorem 5.1. *Assume CH is false. For any $\aleph_0$-coloring of R there exist distinct monochromatic e_1, e_2, e_3, e_4 such that*

$$e_1 + e_2 = e_3 + e_4.$$

Proof. By Fact 3.2 there is an injection of ω_2 into R. If $\alpha \in \omega_2$, then x_α is the real associated to it.

Given an $\aleph_0$-coloring COL of R we show that there exist distinct monochromatic e_1, e_2, e_3, e_4 such that $e_1 + e_2 = e_3 + e_4$.

We define a map F from ω_2 to $\omega_1 \times \omega_1 \times \omega_1 \times \omega$ as follows:

1. Let $\beta \in \omega_2$.
2. Define a map from ω_1 to ω by
$$\alpha \mapsto COL(x_\alpha + x_\beta).$$
3. Let $\alpha_1, \alpha_2, \alpha_3 \in \omega_1$ be distinct elements of ω_1, and $i \in \omega$, such that $\alpha_1, \alpha_2, \alpha_3$ all map to i. Such $\alpha_1, \alpha_2, \alpha_3, i$ clearly exist since $\aleph_0 + \aleph_0 = \aleph_0 < \aleph_1$. (There are $\aleph_1$ many elements that map to the same element of ω, but we do not need that.)
4. Map β to $(\alpha_1, \alpha_2, \alpha_3, i)$.

Since F maps a set of cardinality $\aleph_2$ to a set of cardinality $\aleph_1$, there exists some element that is mapped to twice by F (actually there is an element that is mapped to $\aleph_2$ times, but we do not need this). Let $\alpha_1, \alpha_2, \alpha_3$ be distinct elements of ω_1, $i \in \omega$, and β, β' be distinct elements of ω_2, such that
$$F(\beta) = F(\beta') = (\alpha_1, \alpha_2, \alpha_3, i).$$

Choose distinct $\alpha, \alpha' \in \{\alpha_1, \alpha_2, \alpha_3\}$ such that $x_\alpha - x_{\alpha'} \notin \{x_\beta - x_{\beta'}, x_{\beta'} - x_\beta\}$. We can do this because there are at least three possible values for $x_\alpha - x_{\alpha'}$.

Since $F(\beta) = (\alpha_1, \alpha_2, \alpha_3, i)$, we have
$$COL(x_\alpha + x_\beta) = COL(x_{\alpha'} + x_\beta) = i.$$

Since $F(\beta') = (\alpha_1, \alpha_2, \alpha_3, i)$, we have
$$COL(x_\alpha + x_{\beta'}) = COL(x_{\alpha'} + x_{\beta'}) = i.$$

Let
$$\begin{aligned} e_1 &= x_\alpha + x_\beta \\ e_2 &= x_{\alpha'} + x_{\beta'} \\ e_3 &= x_{\alpha'} + x_\beta \\ e_4 &= x_\alpha + x_{\beta'}. \end{aligned}$$

Then
$$COL(e_1) = COL(e_2) = COL(e_3) = COL(e_4) = i$$
and
$$e_1 + e_2 = e_3 + e_4 = x_\alpha + x_{\alpha'} + x_\beta + x_{\beta'}.$$

Since $x_\alpha \neq x_{\alpha'}$ and $x_\beta \neq x_{\beta'}$, we have $\{e_1, e_2\} \cap \{e_3, e_4\} = \emptyset$.

Moreover, the equation $e_1 = e_2$ is equivalent to
$$x_\alpha - x_{\alpha'} = x_{\beta'} - x_\beta,$$
which is ruled out by our choice of α, α', and so $e_1 \neq e_2$.

Similarly, $e_3 \neq e_4$.

Thus e_1, e_2, e_3, e_4 are all distinct. $\square$

6 A generalization

All the results of the last two sections about $\aleph_0$-colorings of R hold practically verbatim with R replaced by R^k, for any fixed integer $k \geq 1$. In this more geometrical context, e_1, e_2, e_3, e_4 are vectors in k-dimensional Euclidean space, and the equation $e_1 + e_2 = e_3 + e_4$ says that e_1, e_2, e_3, e_4 are the vertices of a parallelogram (whose area may be zero). In particular, we have the following two theorems:

Theorem 6.1. *Fix any integer $k \geq 1$. The following are equivalent:*

1. $2^{\aleph_0} > \aleph_1$.
2. *For any $\aleph_0$-coloring of R^k, there exist distinct monochromatic vectors $e_1, e_2, e_3, e_4 \in \mathsf{R}^k$ such that $e_1 + e_2 = e_3 + e_4$.*

Theorem 6.2. *Fix any integers $k \geq 1$ and $n \geq 2$, and let $a_1, \ldots, , a_n \in \mathsf{R}$ be nonzero. If CH is true, then there exists an $\aleph_0$-coloring of R^k such that there are no distinct monochromatic vectors $e_1, \ldots, e_n \in \mathsf{R}^k$ such that*

$$\sum_{i=1}^{n} a_i e_i = 0.$$

7 More is known

Theorem 2.2 is a special case of a general theorem about colorings and equations.

Definition 7.1. Let $\vec{b} = (b_1, \ldots, b_n) \in \mathsf{Z}^n$

1. $\vec{b}$ is *regular* if the following holds: *For all finite colorings of N^+ there exist monochromatic $e_1, \ldots, e_n \in \mathsf{N}^+$ such that*
$$\sum_{i=1}^{n} b_i e_i = 0.$$
2. $\vec{b}$ is *distinct regular* if the following holds: *For all finite colorings of N^+ there exist monochromatic $e_1, \ldots, e_n \in \mathsf{N}^+$, all distinct, such that*
$$\sum_{i=1}^{n} b_i e_i = 0.$$

In 1916 Schur (14) (see also (7; 8)) proved that, for any finite coloring of N^+, there is a monochromatic solution to $x+y=z$. Using the above terminology he proved that $(1, 1, -1)$ was regular. For him this was a Lemma en route to an alternative proof to the following theorem of Dickson (4):

For all $n \geq 2$ there is a prime p_0 such that, for all primes $p \geq p_0$, $x^n + y^n = z^n$ has a nontrivial solution mod p.

For an English version of Schur's proof of Dickson's theorem see either the book by Graham-Rothschild-Spencer (8) or the free online book by Gasarch-Kruskal-Parrish (7).

Schur's student Rado (11; 12) (see also (8; 7)) proved the following generalization of Schur's lemma:

Theorem 7.2.

1. *$\vec{b}$ is regular iff some subset of $b_1, \ldots, b_n$ sums to 0.*

2. *$\vec{b}$ is distinct-regular iff some subset of $b_1, \ldots, b_n$ sums to 0 and there exists a vector $\vec{\lambda}$ of distinct reals such that $\vec{b} \cdot \vec{\lambda} = 0$.*

Note 7.3. Rado's theorem is about any finite coloring. What about any (say) 3-coloring? An equation is k-regular if for any k-coloring of N there is a monochromatic solution. There is no known characterization of which equations are k-regular. Alexeev and Tsimmerman (1) have shown that there are equations that are $(k-1)$-regular that are not k-regular.

We want to summarize the equivalence of S and $\neg CH$ using the notion of regularity.

Definition 7.4. $\vec{b}$ is $\aleph_0$*-distinct regular* if the following holds: *For all $\aleph_0$-colorings of R there exist distinct monochromatic $e_1, \ldots, e_n \in \mathsf{R}$ such that*

$$\sum_{i=1}^{n} b_i e_i = 0. \tag{1}$$

Notation 7.5. We may also say that an equation is $\aleph_0$-distinct-regular. For example, the statement $x_1 + x_2 = x_3 + x_4$ is $\aleph_0$-distinct-regular means that $(1, 1, -1, -1)$ is $\aleph_0$-distinct-regular.

If we combine Theorems 4.3 and 5.1 and use this definition of regular we obtain the following.

Theorem 7.6. *$x_1 + x_2 = x_3 + x_4$ is $\aleph_0$-distinct-regular iff $2^{\aleph_0} > \aleph_1$.*

What about other linear equations over the reals? Jacob Fox (5) has generalized Theorem 7.6 to prove the following.

Theorem 7.7. *Let $s \in \mathsf{N}$. The equation*

$$x_1 + s x_2 = x_3 + \cdots + x_{s+3} \tag{2}$$

is $\aleph_0$-distinct regular iff $2^{\aleph_0} > \aleph_s$.

Since we are considering coloring the reals, one can easily define what it means for tuples $(b_1, \ldots, b_n) \in \mathsf{R}^n$ or other domains to be $\aleph_0$-distinct regular. A theorem of Ceder (2, Theorem 4) gives us some information about when (b_1, b_2, b_3) is $\aleph_0$-distinct regular for arbitrary complex b_1, b_2, b_3 (not just integers). It uses no assumptions outside of ZFC.

Theorem 7.8 (Ceder)**.** *If P is any polygon with at least 3 vertices in R^1 (or R^2), then there exists an $\aleph_0$-coloring of R^1 (resp R^2) and such that there is no monochromatic polygon similar to P.*

Corollary 7.9. *For any $b_1, b_2, b_3 \in \mathsf{C}$ not all zero, if $b_1 + b_2 + b_3 = 0$, then (b_1, b_2, b_3) is not $\aleph_0$-distinct regular.*

Proof. Suppose $b_1 + b_2 + b_3 = 0$ and at least one of the b_i is nonzero. We can assume *all* the b_i are nonzero. (If any of the b_i is zero—say b_1—then Equation (1) reduces to $e_2 = e_3$, which cannot be satisfied with distinct e_2, e_3.) Dividing by b_3 and rearranging, Equation (1) becomes

$$e_3 - e_1 = \gamma(e_2 - e_1), \tag{3}$$

where $\gamma := -b_2/b_3$. Note that $\gamma \notin \{0, 1\}$, which forces the e_i to be distinct. Identify C with R^2 in the usual way. Any $e_1, e_2, e_3 \in \mathsf{C}$ satisfying (3) form the vertices of a triangle similar to the triangle $P = \{0, 1, \gamma\}$. By Theorem 7.8, we get an $\aleph_0$-coloring of C—based on the partition $\{A_n\}_{n=1}^{\infty}$—that monocolors no such $\{e_1, e_2, e_3\}$. $\square$

There is nothing special about the field C of Ceder's result. Any field will do.

Theorem 7.10. *Let F be any field. For any $\gamma \in F - \{0, 1\}$, there exists an $\aleph_0$-coloring of F such that there are no distinct monochromatic $x, y, z \in F$ such that*

$$z - x = \gamma(y - x). \tag{4}$$

Proof. This proof is essentially Ceder's, generalized to F. Let K be some countable subfield of F containing γ. Choose a basis $\{b_i\}_{i \in I}$ of F as a vector space over K, where I is some index set with a linear order $<$. Then for any $w \in F$, there are unique coordinates $\{w_i\}_{i \in I}$ where each $w_i \in K$, only finitely many of the w_i are nonzero, and

$$w = \sum_{i \in I} w_i b_i.$$

Define the *support* of w as

$$\operatorname{supp}(w) := \{i \in I : w_i \neq 0\} = \{i_1 < i_2 < \cdots < i_k\}$$

for some k. Then define the *signature* of w as the k-tuple of the nonzero coordinates of w, namely

$$\operatorname{sig}(w) := (w_{i_1}, w_{i_2}, \ldots, w_{i_k}).$$

Note that $\operatorname{supp}(w)$ and $\operatorname{sig}(w)$ together uniquely determine w. Also note that there are only countably many possible signatures, so coloring each element of F by its signature gives us an $\aleph_0$-coloring of F.

Suppose $x, y, z \in F$ are distinct and satisfy Equation (4). We show that x, y, and z cannot all have the same signature. Equation (4) is equivalent to

$$z = \gamma y + (1 - \gamma)x,$$

or equivalently, since $\gamma \in K$,

$$(\forall i \in I)[\ z_i = \gamma y_i + (1 - \gamma)x_i\ ].$$

If $\operatorname{sig}(x) \neq \operatorname{sig}(y)$, then we are done, so assume $\operatorname{sig}(x) = \operatorname{sig}(y)$. Since $x \neq y$, we must have $\operatorname{supp}(x) \neq \operatorname{supp}(y)$. Let $\ell \in I$ be the least element of $\operatorname{supp}(x) \triangle \operatorname{supp}(y)$. Then for every $j < \ell$, we have $x_j = y_j$, and so

$$z_j = \gamma y_j + (1 - \gamma)x_j = y_j = x_j.$$

We now have two cases for ℓ:

Case 1: $\ell \in$ **supp**(y)**.** Then $y_\ell \neq 0$ and $x_\ell = 0$, because $\ell \notin \mathrm{supp}(x)$. This gives

$$z_\ell = \gamma y_\ell \notin \{0, y_\ell\},$$

which puts ℓ into $\mathrm{supp}(z)$ and forces $\mathrm{sig}(z) \neq \mathrm{sig}(y)$.

Case 2: $\ell \in$ **supp**(x)**.** A similar argument, swapping the roles of x and y and swapping γ with $1-\gamma$, shows that $\mathrm{sig}(z) \neq \mathrm{sig}(x)$.

$\square$

Corollary 7.11. *Let F be any field. For any $b_1, b_2, b_3 \in F$ not all zero, if $b_1 + b_2 + b_3 = 0$, then (b_1, b_2, b_3) is not $\aleph_0$-distinct regular.*

8 Is the statement really natural?

Theorem 4.3 and 5.1 are stated as though they are about R. However, all that is used about R is that it is a vector space over Q. Hence the proof we gave really proves Theorem 8.2 below, from which Theorems 4.3 and 5.1 (as well as Theorems 6.1 and 6.2, for that matter) follow as easy corollaries.

Definition 8.1. For any vector space V over Q, let $S(V)$ be the statement, "For any $\aleph_0$-coloring of V there exist distinct monochromatic $e_1, e_2, e_3, e_4 \in V$ such that $e_1+e_2 = e_3+e_4$."

Theorem 8.2. *If V is a vector space over Q, then $S(V)$ iff V has dimension at least $\aleph_2$.*

The proof of Theorem 8.2 is in ZFC.

One can ask the following: Since the result, when abstracted, has nothing to do with R and is just a statement provable in ZFC, do we really have a *natural* statement that is independent of ZFC? We believe so.

After you know that every finite coloring of N has a distinct monochromatic solution to $e_1 + e_2 = e_3 + e_4$, it is natural to consider the following question:

Does every $\aleph_0$-coloring of R have a distinct monochromatic solution to $e_1+e_2 = e_3+e_4$?

This question can be understood by a bright high school or secondary school student with no knowledge of vector spaces. The fact that *after* you show that this question is independent of ZFC you *can then* abstract the proof to obtain Theorem 8.2 does not diminish the naturalness of the original question.

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