# Sane Bounds on Some Poly VDW Numbers

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# VDW's Theorem

**Theorem**

*For all $k$, for all $c$, there exists $W = W(k, c)$ such that for all $c$-colorings of $[W]$ there exists $a, d$*

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How to prove?

1. VDW Proof [4]. Elem. $\omega^2$-induction. Bounds INSANE!
2. Shelah's Proof [3]. Elem. Bounds HUGE.
3. Gowers's Proof [2]. Adv. Bound is *only* 9 towers.

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$$a, a+d, a+2d, \ldots, a+(k-1)d?$$

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**Theorem**

*For all* $p_1, \ldots, p_k \in Z[x]$, $(\forall i)[p_i(0) = 0]$, *for all* $c$, *there exists* $W = W(p_1, \ldots, p_k; c)$ *such that for all* $c$*-colorings of* $[W]$ *there exists* $a, d$ *such that*

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How to prove?

1. Bergelson and Leibman [1]. Adv. No bounds!
2. Walters [5]. Elem. $\omega^\omega$ induction. Bounds INSANE.

# Main Question

**Want SANE bounds on** $W(p_1, \ldots, p_k; c)$

# Forbidden Distances

**Definition**

Let $F \subseteq Z$, $c \in N$, and $W \in N$.

1. A $(F, c)$-proper coloring of $[W]$ is a coloring $COL : [W] \rightarrow [c]$ such that, for all $x, y \in [W]$ with $y - x \in F$, $COL(x) \neq COL(y)$.
2. $W = W(F; c)$ is the least num such that there is no $(F, c)$-proper coloring of $[W]$. If no such num exists, we set $W(F; c) = \infty$.

# PART I: 2-COLORABILITY

**We will get SANE bounds on** $W(p(x);2)$

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| | | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 1 | 6 | 11 | 3 | 8 | 13 | 5 | 10 | 2 | 7 | 12 | 4 | 9 | 1 |
| R | B | R | B | R | B | R | B | R | B | R | B | R | B |

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|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 1 | 6 | 11 | 3 | 8 | 13 | 5 | 10 | 2 | 7 | 12 | 4 | 9 | 1 |
| $R$ | $B$ | $R$ | $B$ | $R$ | $B$ | $R$ | $B$ | $R$ | $B$ | $R$ | $B$ | $R$ | $B$ |

AH-HA: 1 is BOTH $R$ and $B$. Contradiction.
AH-HA: Could color [12]– just omit the 1 and renum.

# If $s, t$ are relatively prime and one is even...

**Lemma**

*If $s, t$ are relatively prime and one of them is even then $W(\{s,t\};2) = s + t = m$.*

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**Proof.**

Assume $COL : [m] \rightarrow \{R, B\}$.

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**Proof.**

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KEY: $COL(x) = R \implies COL(x + s \bmod m) = B$.

| 1 | $1 + s \bmod m$ | $\cdots$ | $1 + (m-1)s \bmod m$ | $1 + ms \bmod m = 1$ |
|---|---|---|---|---|
| $R$ | $B$ | $\cdots$ | $R$ | $B$ |

AH-HA: 1 is BOTH $R$ and $B$. Contradiction.
AH-HA: Could color $[m-1]$– just omit the 1 and renum. $\square$

# Generalize to $s, t$ Not Necc. Rel Prime

Lemma

*Let* $s, t \in \mathbb{N}$. *Let* $g = \gcd(s, t)$. *Then*

$$W(\{s,t\};2) = \begin{cases} s+t-g+1 & \text{if either } s/g \text{ or } t/g \text{ is even} \\ \infty & \text{otherwise} \end{cases}$$

# BACK to POLYNOMIALS

**Lemma**

*Let* $p(x) \in Z[x]$ *such that* $p(0) = 0$*. Let* $i, j \in \mathrm{N}$*, such that* $p(i), p(j) \neq 0$*, and* $g = \gcd(p(i), p(j))$*. If either* $p(i)/g$ *or* $p(j)/g$ *is even then*

$$W(p(x); 2) \leq |p(i)| + |p(j)| - g + 1.$$

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$$W(p(x); 2) \leq |p(i)| + |p(j)| - g + 1.$$

Note: The premise is equivalent to $p(i)$ and $p(j)$ have a different number of factors of 2.

# First Upshot

**Theorem**

*If* $p(x) \in Z[x]$ *such that* $p(0) = 0$ *then*

$$W(p(x);2) \le |p(|2p(1)|)| + 1.$$

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Proof.

$p(1)$ and $p(2p(1))$ have different number of factors of 2.
Now use Lemma. □

# Second Upshot

**Theorem**

*If* $p(x) \in Z[x]$ *such that* $p(0) = 0$ *and* $p$ *has degree* $n$*. Then*

$$W(p(x); 2) < 2 \max_{1 \le i \le n+1} |p(i)|.$$

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$$p(1) + \sum_{k=1}^{n} \sum_{i=0}^{k-1} \binom{k-1}{i} (-1)^i \left(p(i+2) - p(i+1)\right) \; = \; 0.$$

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All terms paired off except the initial $p(1)$. Hence odd num of $p()$ terms. If all have same number of factors of 2 factor them out. Have odd sum of odd nums is 0. Contradiction. □

# PART II: 3-COLORABILITY

**We will get SANE bounds on $W(ax^2 + bx; 3)$**

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Show that if $\{1, 2, \ldots, 2006\}$ is 3-colored then there exists two numbers a square apart that are the same color.

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1. $COL(x + 16) \neq R$, $COL(x - 9) \neq R$.
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$COL(x + 7) = COL((x + 16) - 9) \neq COL(x + 16) = B$.
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Hence $COL(x + 7) = R = COL(x)$.

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Contradiction! 10 and $10 + 7^2$ are a square apart!

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Can replace 2006 with $10 + 7 \times 7 = 59$

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1. Approach can't work for $p(x) = x^3$.
2. We will only work with quadratics.

# (ALMOST) GENERAL QUADRATIC CASE

**Theorem**

*Let $a, b \in \mathbb{N}$ such that a divides b. Then*

$$W(ax^2 + bx; 3) \leq 72b^2/a + 1.$$

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**Proof.**

Let

$$x = 5b/a, y = 6b/a, z = 8b/a.$$

$$\begin{aligned} p(x) = &\quad a(5b/a)^2 + b(5b/a) = 25b^2/a + 5b^2/a = 30b^2/a \\ p(y) = &\quad a(6b/a)^2 + b(6b/a) = 36b^2/a + 6b^2/a = 42b^2/a \\ p(z) = &\quad a(8b/a)^2 + b(8b/a) = 64b^2/a + 8b^2/a = 72b^2/a \end{aligned}$$

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$$p(x) + p(y) = p(z).$$

Still more to do, but this is key ingredient. □

# GENERAL QUADRATIC CASE

**Theorem**

*Let* $a, b \in \mathbb{N}$. *Let* $m = \max\{a, b\}$. *Then*

$$W(ax^2 + bx; 3) \leq O(m^7).$$

**Proof.**

You don't really want to see the proof. □

# PART III: OPEN QUESTIONS

1. Obtain smaller values for $W(ax^2 + bx; 3)$ (e.g., $O(m^6)$).
2. Obtain SANE values for $W(ax^3 + bx^2 + cx; 3)$.
3. Obtain SANE values for $W(ax^2 + bx; 4)$.
4. Obtain SANE values for $W(p_1(x), \ldots, p_k(x); 2)$.
5. Obtain SANE values for $W(p_1(x), \ldots, p_k(x); 3)$
6. Start study of $W(f; c)$ where $f$ is poly with constant term.
7. Start study of $W(f; c)$ where $f$ is non poly functions.

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