Easy Parts of Quantum Graph Coloring

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Notation The least k such that G is k-colorable is denoted $\chi(G)$.

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One could have **defined** $\chi(G)$ in terms of this game.

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Are there any graphs where A&B can do better? I do not know.

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- If A&B could communicate a bits then might be able to increase their chance of winning (has not been looked at).
- If A&B could share **QUANTUM STUFF I DO NOT UNDERSTAND THAT INVOLVES ENTANGLEMENT** then there are graphs G with $\chi(G) = k$ such that A&B win with Prob 1 using $k' < k$.

One Case that is Known and Impressive

Notation if $x, y \in \{0,1\}^n$ then $d(x,y)$ is the number of places they differ. This is also called the **Hamming Distance**.

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Theorem

1. $\chi(H_n) = \Theta(2^n)$. (This is an old classical result.)
2. If A&B can share **QUANTUM STUFF I DO NOT UNDERSTAND THAT INVOLVES ENTANGLEMENT** then A&B can win with prob 1 the Game with H_n and $k = n$. So EXPONENTIAL improvement.

Easy Ramseyesque Theorem

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- There exists a finite colorings of $\{1, \ldots, 9\}$ with NO 4 homog and NO 4 rainbow.

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- For all finite colorings of $\{1, \ldots, 10\}$ there will be either 4 homog or 4 rainbow.

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Look at other Theorems in Ramsey Theory and formulate Quantum Questions.

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Then answer them!