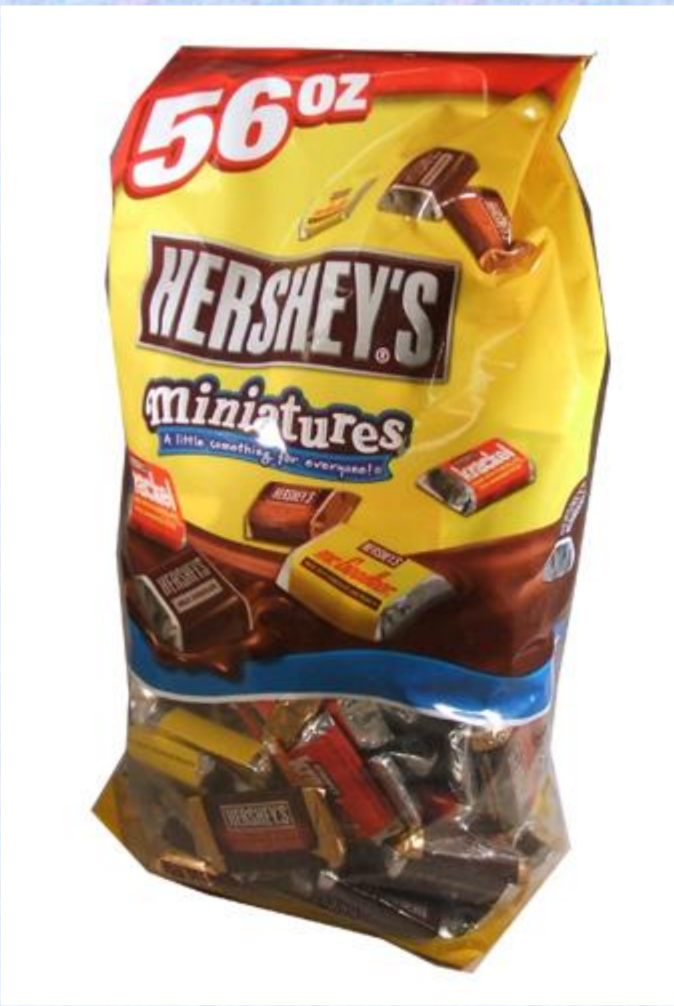
CMSC 474, Introduction to Game Theory

2. Normal-Form Games

Mohammad T. Hajiaghayi
University of Maryland

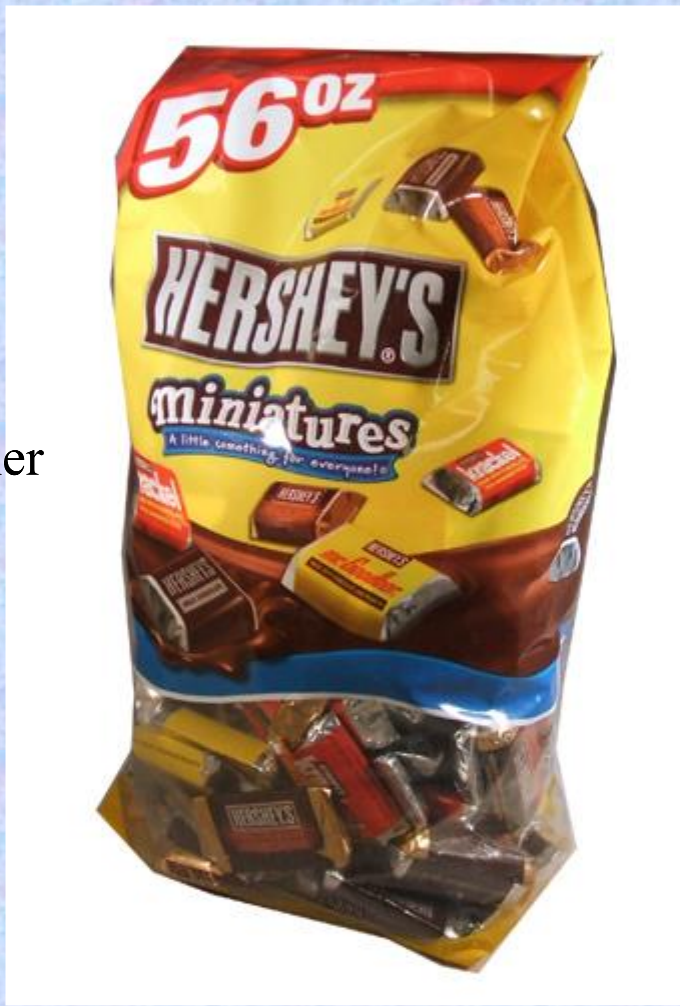
Example: Let’s Play a Game

- I need two volunteers to play a short game
  - Preferably two people who don’t know each other
  - You’ll have a chance to get some chocolate



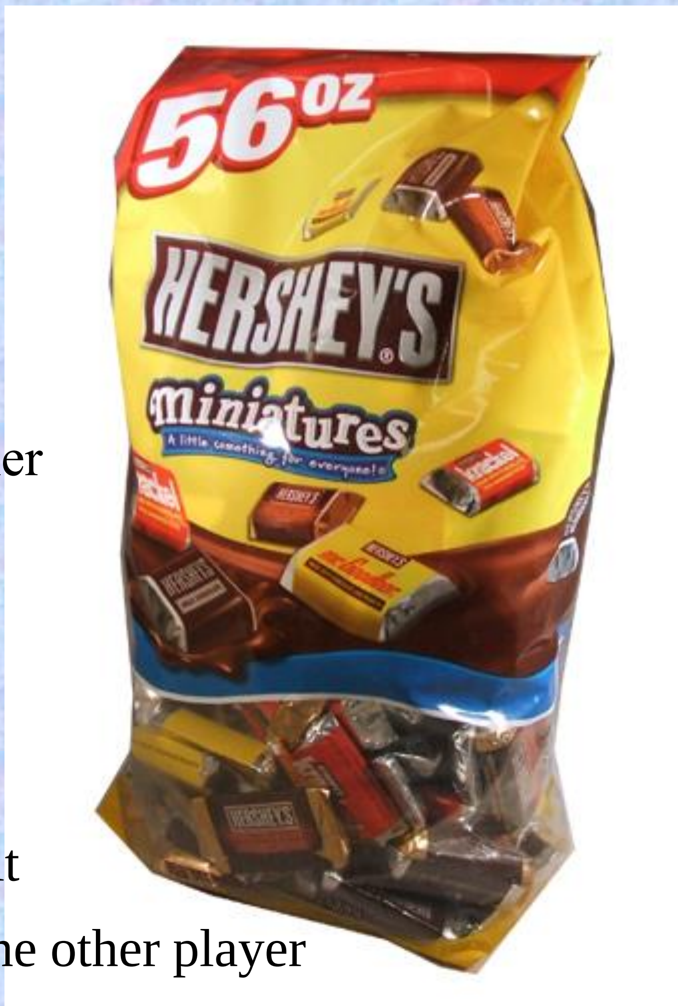
Example: Let's Play a Game

- I need two volunteers to play a short game
  - Preferably two people who don't know each other
  - You'll have a chance to get some chocolate
- Come to the front of the room
  - Face opposite directions, don't talk to each other
- You may choose one of two actions
  - Don't let the other player know which one, until I say so



Example: Let's Play a Game

- I need two volunteers to play a short game
  - Preferably two people who don't know each other
  - You'll have a chance to get some chocolate
- Come to the front of the room
  - Face opposite directions, don't talk to each other
- You may choose one of two actions
  - Don't let the other player know which one, until I say so
- The Chocolate Dilemma*
  - **Take 1** piece of chocolate, and you may keep it
  - **Take 3** pieces of chocolate, and they'll go to the other player



---

* http://theoryclass.wordpress.com/2010/03/05/the-chocolate-dilemma/

- Please go to http://www.surveymonkey.com/s/RYLDSRX and tell which Chocolate Dilemma action you would choose in each of these situations:
  - The other player is a stranger whom you'll never meet again.
  - The other player is an enemy.
  - The other player is a friend.
  - The other player is a computer program instead of a human.
  - You haven't eaten in two days.
  - "Take1" means you take two chocolates instead of just one.
  - You and the other player can discuss what choices to make.
  - You will be playing the game repeatedly with the same person.
  - Thousands of people are playing the game. None of you knows which of the others is the one you're playing with.
  - Thousands of people are playing the game. "Take3" means the three chocolates go to a collection that will be divided equally among everyone.
  - The bag is filled with money. "Take1" means you take $2500 and you can keep it. "Take3" means you take $3000 but it will go to the other player.

Some game-theoretic answers

- Suppose that—
  - Each player just wants to maximize how many chocolates he/she gets
    - Neither player cares about ***anything*** other than that
  - Both players understand all of the possible outcomes
  - All this is common knowledge to both players

- Then each player will take 1 piece of chocolate
  - If they can talk to each other beforehand,
    it won't change the outcome
  - Repeat any fixed number of times => same outcome
  - Repeat an unbounded number of times => they might take 3 instead

- **Is this realistic?** We discuss it further later

Games in Normal Form

- A (finite, n-person) **normal-form game** includes the following:
  1. An ordered set $N = (1, 2, 3, \ldots, n)$ of **agents** or **players**:
  2. Each agent i has a finite set A_i of possible actions
     - An **action profile** is an n-tuple $\mathbf{a} = (a_1, a_2, \ldots, a_n)$, where $a_1 \in A_1$, $a_2 \in A_2, \ldots, a_n \in A_n$
     - The set of all possible action profiles is $\mathbf{A} = A_1 \times \cdots \times A_n$
  3. Each agent i has a real-valued **utility** (or **payoff**) function

$$u_i(a_1, \ldots, a_n) = i\text{'s payoff if the action profile is } (a_1, \ldots, a_n)$$

- Most other game representations can be reduced to normal form
- Usually represented by an n-dimensional **payoff** (or **utility**) **matrix**
  - for each action profile, shows the utilities of all the agents

	take 3	take 1
take 3	3, 3	0, 4
take 1	4, 0	1, 1

The Prisoner's Dilemma


- Scenario: The police are holding two prisoners as suspects for committing a crime
  - For each prisoner, the police have enough evidence for a 1 year prison sentence
  - They want to get enough evidence for a 4 year prison sentence
  - They tell each prisoner,
    - "If you testify against the other prisoner, we'll reduce your prison sentence by 1 year"
  - *C* = *Cooperate* (with the other prisoner): refuse to testify against him/her
  - *D* = *Defect*: testify against the other prisoner
  - Both prisoners cooperate => both go to prison for 1 year
  - Both prisoners defect => both go to prison for 4 – 1 = 3 years
  - One defects, other cooperates => cooperator goes to prison for 4 years; defector goes free

	C	*D*
C	–1, –1	–4, 0
D	0, –4	–3, –3

Prisoner's Dilemma

We used this:

	C	D
C	–1, –1	–4, 0
D	0, –4	–3, –3

Equivalent:

	take 3	take 1
take 3	3, 3	0, 4
take 1	4, 0	1, 1

Game theorists usually use this:

	C	D
C	3, 3	0, 5
D	5, 0	1, 1

	C	D
C	a, a	b, c
D	c, b	d, d

- General form:

$$c > a > d > b$$

$$2a \geq b + c$$

Utility Functions

- Idea: the preferences of a rational agent must obey some constraints
- Constraints:

  Orderability (sometimes called **Completeness**):

  $(A \succ B) \lor (B \succ A) \lor (A \sim B)$

  Transitivity:

  $(A \succ B) \land (B \succ C) \Rightarrow (A \succ C)$

- Agent's choices are based on rational preferences
  ⇒ agent's behavior is describable as maximization of expected utility
- **Theorem** (Ramsey, 1931; von Neumann and Morgenstern, 1944).
- Given preferences satisfying the constraints above, there exists a real-valued function u such that

$$u(A) \geq u(B) \Leftrightarrow A \succeq B \qquad (*)$$

u is called a **utility function**

Utility Scales

- Rational preferences are invariant with respect to **positive affine** (or **positive linear**) transformations
- Let

$$u'(x) = c\, u(x) + d$$

where c and d are constants, and $c > 0$

  - Then u' models the same set of preferences that u does

- **Normalized utilities:**
  - define u such that $u_{max} = 1$ and $u_{min} = 0$

Utility Scales for Games

- Suppose that all the agents have rational preferences, and that this is common knowledge* to all of them
- Then games are insensitive to positive affine transformations of one or more agents' payoffs
  - Let c and d be constants, $c > 0$
  - For one or more agents i, replace every payoff x_{ij} with $cx_{ij} + d$
  - The game still models the same sets of rational preferences

	a_{21}	a_{22}
a_{11}	x_{11}, x_{21}	x_{12}, x_{22}
a_{12}	x_{13}, x_{23}	x_{14}, x_{24}

	a_{21}	a_{22}
a_{11}	$cx_{11}+d, x_{21}$	$cx_{12}+d, x_{22}$
a_{12}	$cx_{13}+d, x_{23}$	$cx_{14}+d, x_{24}$

	a_{21}	a_{22}
a_{11}	$cx_{11}+d, ex_{21}+f$	$cx_{12}+d, ex_{22}+f$
a_{12}	$cx_{13}+d, ex_{23}+f$	$cx_{14}+d, ex_{24}+f$

*Common knowledge is a complicated topic; I'll discuss it later

Examples

- Are these transformations positive affine?

	C	*D*
C	3, 3	0, 4
D	4, 0	1, 1

→

	C	*D*
C	3, –1	0, 0
D	4, –4	1, –3

→

	C	*D*
C	–1, –1	–4, 0
D	0, –4	–3, –3

- How about these?

	C	*D*
C	3, 3	0, 4
D	4, 0	1, 1

→

	C	*D*
C	3, 3	0, 5
D	4, 0	1, 1

→

	C	*D*
C	3, 3	0, 5
D	5, 0	1, 1

Decision Making Under Risk

- Which of the following lotteries would you choose?
  - A: 100% chance of receiving $3000
  - B: 80% chance of receiving $4000; 20% chance of receiving nothing

Decision Making Under Risk

- Which of the following lotteries would you choose?
  - C: 100% chance of losing $3000
  - D: 80% chance of losing $4000; 20% chance of losing nothing

Decision Making Under Risk

- Which of the following lotteries would you choose?
  - A: 100% chance of receiving $3000
  - B: 80% chance of receiving $4000; 20% chance of receiving nothing

- Which of the following lotteries would you choose?
  - C: 100% chance of losing $3000
  - D: 80% chance of losing $4000; 20% chance of losing nothing

- Kahneman & Tversky, 1979:
  - EV(A) = $3000 < EV(B) = $3200, but most people would choose A
    - For prospects involving gains, we're *risk-averse*
  - EV(C) = –$3000 > EV(D) = –$3200, but most people would choose D
    - For prospects involving losses, we're *risk-prone*
  - http://www.econport.org/econport/request?page=man_ru_advanced_prospect