CMSC 474, Introduction to Game Theory

2. Normal-Form Games

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Example: Let’s Play a Game

- I need two volunteers to play a short game
  - Preferably two people who don’t know each other
  - You’ll have a chance to get some chocolate
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  - Face opposite directions, don’t talk to each other
- You may choose one of two actions
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- Come to the front of the room
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- You may choose one of two actions
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- The Chocolate Dilemma*
  - Take 1 piece of chocolate, and you may keep it
  - Take 3 pieces of chocolate, and they’ll go to the other player

* [http://theoryclass.wordpress.com/2010/03/05/the-chocolate-dilemma/](http://theoryclass.wordpress.com/2010/03/05/the-chocolate-dilemma/)
Please go to [http://www.surveymonkey.com/s/RYLDSRX](http://www.surveymonkey.com/s/RYLDSRX) and tell which Chocolate Dilemma action you would choose in each of these situations:

- The other player is a stranger whom you'll never meet again.
- The other player is an enemy.
- The other player is a friend.
- The other player is a computer program instead of a human.
- You haven't eaten in two days.
- "Take1" means you take two chocolates instead of just one.
- You and the other player can discuss what choices to make.
- You will be playing the game repeatedly with the same person.
- Thousands of people are playing the game. None of you knows which of the others is the one you're playing with.
- Thousands of people are playing the game. "Take3" means the three chocolates go to a collection that will be divided equally among everyone.
- The bag is filled with money. "Take1" means you take $2500 and you can keep it. "Take3" means you take $3000 but it will go to the other player.
Some game-theoretic answers

- Suppose that—
  - Each player just wants to maximize how many chocolates he/she gets
    - Neither player cares about anything other than that
  - Both players understand all of the possible outcomes
  - All this is common knowledge to both players

- Then each player will take 1 piece of chocolate
  - If they can talk to each other beforehand, it won’t change the outcome
  - Repeat any fixed number of times => same outcome
  - Repeat an unbounded number of times => they might take 3 instead

- Is this realistic? We discuss it further later
Games in Normal Form

- A (finite, \(n\)-person) **normal-form game** includes the following:
  1. An ordered set \(N = (1, 2, 3, \ldots, n)\) of **agents** or **players**:
  2. Each agent \(i\) has a finite set \(A_i\) of possible actions
     - An **action profile** is an \(n\)-tuple \(a = (a_1, a_2, \ldots, a_n)\), where \(a_1 \in A_1, a_2 \in A_2, \ldots, a_n \in A_n\)
     - The set of all possible action profiles is \(A = A_1 \times \cdots \times A_n\)
  3. Each agent \(i\) has a real-valued **utility** (or **payoff**) function
     \[ u_i(a_1, \ldots, a_n) = i’s \text{ payoff if the action profile is } (a_1, \ldots, a_n) \]

- Most other game representations can be reduced to normal form
- Usually represented by an \(n\)-dimensional **payoff** (or **utility**) **matrix**
  - for each action profile, shows the utilities of all the agents

\[
\begin{array}{c|cc}
& \text{take 3} & \text{take 1} \\
\hline
\text{take 3} & 3, 3 & 0, 4 \\
\text{take 1} & 4, 0 & 1, 1 \\
\end{array}
\]
The Prisoner’s Dilemma

- Scenario: The police are holding two prisoners as suspects for committing a crime
  - For each prisoner, the police have enough evidence for a 1 year prison sentence
  - They want to get enough evidence for a 4 year prison sentence
  - They tell each prisoner,
    - “If you testify against the other prisoner, we’ll reduce your prison sentence by 1 year”
  - $C =$ Cooperate (with the other prisoner): refuse to testify against him/her
  - $D =$ Defect: testify against the other prisoner
  - Both prisoners cooperate => both go to prison for 1 year
  - Both prisoners defect => both go to prison for $4 - 1 = 3$ years
  - One defects, other cooperates => cooperator goes to prison for 4 years; defector goes free

\[
\begin{array}{c|cc}
   & C & D \\
\hline
C & -1, -1 & -4, 0 \\
D & 0, -4 & -3, -3
\end{array}
\]
Prisoner’s Dilemma

General form:

\[ c > a > d > b \]
\[ 2a \geq b + c \]

\begin{align*}
C & \quad D \\
C & \quad -1, -1 \quad -4, 0 \\
D & \quad 0, -4 \quad -3, -3
\end{align*}

We used this:

\begin{align*}
C & \quad D \\
C & \quad \{3, 3\} \quad \{0, 4\} \\
D & \quad \{4, 0\} \quad \{1, 1\}
\end{align*}

Equivalent:

\begin{align*}
C & \quad D \\
C & \quad \{a, a\} \quad \{b, c\} \\
D & \quad \{c, b\} \quad \{d, d\}
\end{align*}

Game theorists usually use this:
Utility Functions

- Idea: the preferences of a rational agent must obey some constraints
- Constraints:
  - **Orderability** (sometimes called Completeness):
    \[(A > B) \lor (B > A) \lor (A \sim B)\]
  - **Transitivity:**
    \[(A > B) \land (B > C) \Rightarrow (A > C)\]

- Agent’s choices are based on rational preferences
  \[\Rightarrow\] agent’s behavior is describable as maximization of expected utility
- **Theorem** (Ramsey, 1931; von Neumann and Morgenstern, 1944).
- Given preferences satisfying the constraints above, there exists a real-valued function \(u\) such that
  \[u(A) \geq u(B) \iff A \succeq B\]  
  \((\ast)\)
  \(u\) is called a **utility function**
Utility Scales

- Rational preferences are invariant with respect to positive affine (or positive linear) transformations.

- Let

  \[ u'(x) = c \ u(x) + d \]

  where \( c \) and \( d \) are constants, and \( c > 0 \)

  Then \( u' \) models the same set of preferences that \( u \) does.

- Normalized utilities:

  define \( u \) such that \( u_{max} = 1 \) and \( u_{min} = 0 \).
Utility Scales for Games

- Suppose that all the agents have rational preferences, and that this is common knowledge* to all of them.
- Then games are insensitive to positive affine transformations of one or more agents’ payoffs.
  - Let $c$ and $d$ be constants, $c > 0$.
  - For one or more agents $i$, replace every payoff $x_{ij}$ with $cx_{ij} + d$.
  - The game still models the same sets of rational preferences.

*Common knowledge is a complicated topic; I’ll discuss it later.
### Examples

- Are these transformations positive affine?

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- How about these?

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Decision Making Under Risk

Which of the following lotteries would you choose?

- A: 100% chance of receiving $3000
- B: 80% chance of receiving $4000; 20% chance of receiving nothing
Decision Making Under Risk

Which of the following lotteries would you choose?

- C: 100% chance of losing $3000
- D: 80% chance of losing $4000; 20% chance of losing nothing
Decision Making Under Risk

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  - A: 100% chance of receiving $3000
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- Which of the following lotteries would you choose?
  - C: 100% chance of losing $3000
  - D: 80% chance of losing $4000; 20% chance of losing nothing

- Kahneman & Tversky, 1979:
  - EV(A) = $3000 < EV(B) = $3200, but most people would choose A
    - For prospects involving gains, we’re risk-averse
  - EV(C) = −$3000 > EV(D) = −$3200, but most people would choose D
    - For prospects involving losses, we’re risk-prone