CMSC 474, Introduction to Game Theory

3. Important Normal-Form Games

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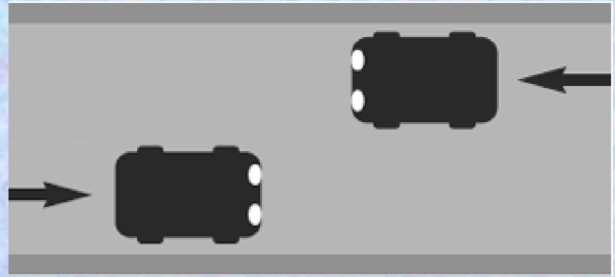
Common-payoff Games

- **Common-payoff game**:
  - For every action profile, all agents have the same payoff
- Also called a **pure coordination** game or a **team game**
  - Need to coordinate on an action that is maximally beneficial to all

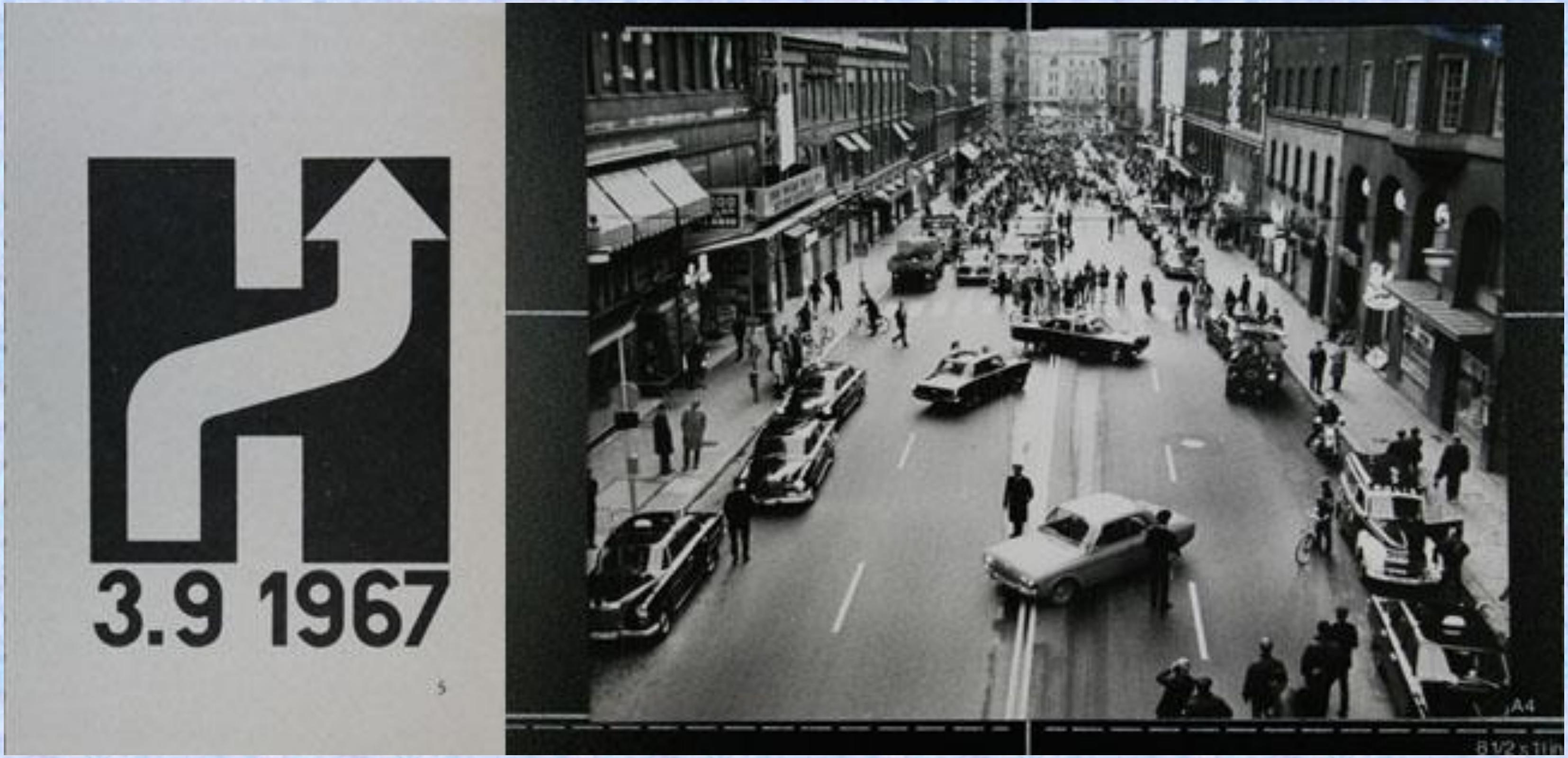
- **Which side of the road?**
  - 2 people driving toward each other in a country with no traffic rules
  - Each driver independently decides whether to stay on the left or the right
  - Need to coordinate your action with the action of the other driver



	Left	Right
Left	1, 1	0, 0
Right	0, 0	1, 1

A Brief Digression

- **Mechanism design**: set up the rules of the game, to give each agent an incentive to choose a desired outcome
- E.g., the law says what side of the road to drive on
  - Sweden on September 3, 1967:



Zero-sum Games

- These games are purely competitive

- **Constant-sum** game:
  - For every action profile, the sum of the payoffs is the same, i.e.,
  - there is a constant c such for every action profile $\mathbf{a} = (a_1, \ldots, a_n)$,
    - $u_1(\mathbf{a}) + \ldots + u_n(\mathbf{a}) = c$

- For any constant-sum game can be transformed into an equivalent game in which the sum of the payoffs is always 0
  - Positive affine transformation: subtract c/n from every payoff

- Thus constant-sum games are usually called **zero-sum** games

Examples

- **Matching Pennies**
  - Two agents, each has a penny
  - Each independently chooses to display Heads or Tails
    - If same, agent 1 gets both pennies
    - Otherwise agent 2 gets both pennies

	Heads	Tails
Heads	1, –1	–1, 1
Tails	–1, 1	1, –1

- **Penalty kicks in soccer**
  - A kicker and a goalie
  - Kicker can kick left or right
  - Goalie can jump to left or right
  - Kicker scores if he/she kicks to one side and goalie jumps to the other

Another Example:Rock-Paper-Scissors

- Two players. Each simultaneously picks an action: *Rock*, *Paper*, or *Scissors*.
- The rewards:

Rock	beats	*Scissors*
Scissors	beats	*Paper*
Paper	beats	*Rock*

- The matrices:

$$R_1 = \begin{array}{c} \\ \text{R} \\ \text{P} \\ \text{S} \end{array} \begin{array}{c} \begin{array}{ccc} \text{R} & \text{P} & \text{S} \end{array} \\ \left(\begin{array}{rrr} 0 & -1 & 1 \\ 1 & 0 & -1 \\ -1 & 1 & 0 \end{array} \right) \end{array} \qquad R_2 = \begin{array}{c} \\ \text{R} \\ \text{P} \\ \text{S} \end{array} \begin{array}{c} \begin{array}{ccc} \text{R} & \text{P} & \text{S} \end{array} \\ \left(\begin{array}{rrr} 0 & 1 & -1 \\ -1 & 0 & 1 \\ 1 & -1 & 0 \end{array} \right) \end{array}$$

Nonzero-Sum Games

- A game is **nonconstant-sum** (usually called **nonzero-sum**) if there are action profiles **a** and **b** such that
  - $u_1(\mathbf{a}) + \ldots + u_n(\mathbf{a}) \neq u_1(\mathbf{b}) + \ldots + u_n(\mathbf{b})$
  - e.g., the Prisoner's Dilemma

	C	*D*
C	3, 3	0, 5
D	5, 0	1, 1

- **Battle of the Sexes**
  - Two agents need to coordinate their actions, but they have different preferences
  - Original scenario:
    - husband prefers football, wife prefers opera
  - Another scenario:
    - Two nations must act together to deal with an international crisis, and they prefer different solutions

		Husband:	
		Opera	Football
Wife:	Opera	2, 1	0, 0
	Football	0, 0	1, 2

Symmetric Games

- In a **symmetric** game, every agent has the same actions and payoffs
  - If we change which agent is which, the payoff matrix will stay the same

Which side of the road?

	Left	Right
Left	1, 1	0, 0
Right	0, 0	1, 1

- For a 2x2 symmetric game, it doesn't matter whether agent 1 is the row player or the column player
  - The payoff matrix looks like this:

	a_1	a_2
a_1	w, w	x, y
a_2	y, x	z, z

- In the payoff matrix of a symmetric game, we only need to display u_1
  - If you want to know another agent's payoff, just interchange the agent with agent 1

	a_1	a_2
a_1	w	x
a_2	y	z

Strategies in Normal-Form Games

- **Pure strategy**: select a single action and play it
  - Each row or column of a payoff matrix represents both an action and a pure strategy
- **Mixed strategy**: randomize over the set of available actions according to some probability distribution
  - $s_i(a_j)$ = probability that action a_j will be played in mixed strategy s_i
- The **support** of s_i = {actions that have probability > 0 in s_i}
- A pure strategy is a special case of a mixed strategy
  - support consists of a single action
- A strategy s_i is **fully mixed** if its support is A_i
  - i.e., nonzero probability for every action available to agent *i*
- **Strategy profile**: an *n*-tuple $\mathbf{s} = (s_1, \ldots, s_n)$ of strategies, one for each agent

Expected Utility

- A payoff matrix only gives payoffs for pure-strategy profiles
- Generalization to mixed strategies uses expected utility
  - First calculate probability of each outcome, given the strategy profile (involves all agents)
  - Then calculate average payoff for agent *i*, weighted by the probabilities
  - Given strategy profile $\mathbf{s} = (s_1, \ldots, s_n)$
    - expected utility is the sum, over all action profiles, of the profile's utility times its probability:

$$u_i(\mathbf{s}) = \sum_{\mathbf{a} \in \mathbf{A}} u_i(\mathbf{a}) \Pr[\mathbf{a} \mid \mathbf{s}]$$

i.e.,

$$u_i(s_1, \ldots, s_n) = \sum_{(a_1, \ldots, a_n) \in \mathbf{A}} u_i(a_1, \ldots, a_n) \prod_{j=1}^{n} s_j(a_j)$$

Let’s Play another Game

- Choose a number in the range from 0 to 100
  - Write it on a piece of paper
  - Also write your name (this is optional)
  - Fold your paper in half, so nobody else can see your number
  - Pass your paper to the front of the room

- The winner(s) will be whoever chose a number that’s closest to the average of all the numbers
  - I’ll tell you the results later
  - The winner(s) will get some prize

Summary of Past Three Sessions

- Basic concepts:
  - normal form, utilities/payoffs, pure strategies, mixed strategies
- How utilities relate to rational preferences (not in the book)
- Some classifications of games based on their payoffs
  - Zero-sum
    - Rock-paper-scissors, Matching Pennies
  - Non-zero-sum
    - Chocolate Dilemma, Prisoner's Dilemma, Which Side of the Road?, Battle of the Sexes
  - Common-payoff
    - Which Side of the Road?
  - Symmetric
    - All of the above except Battle of the Sexes