CMSC 474, Introduction to Game Theory

7. Rationalizability and Price of Anarchy

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Rationalizability

- A strategy is **rationalizable** if a *perfectly rational agent* could justifiably playit against *perfectly rational opponents*
  - The formal definition complicated
- Informally:
  - A strategy for agent *i* is rationalizable if it's a best response to strategies that *i* could *reasonably* believe the other agents have
  - To be reasonable, *i*'s beliefs must take into account
    - the other agents' knowledge of *i*'s rationality,
    - their knowledge of *i*'s knowledge of *their* rationality,
    - and so on so forth recursively
- A **rationalizable strategy profile** is a strategy profile that consists only of rationalizable strategies

Rationalizability

	Left	Right
Left	1, 1	0, 0
Right	0, 0	1, 1

- Every Nash equilibrium is composed of rationalizable strategies
- Thus the set of rationalizable strategies (and strategy profiles) is always nonempty

Example: Which Side of the Road

- For Agent 1, the pure strategy $s_1 = Left$ is rationalizable because
  - $s_1 = Left$ is 1's best response if 2 uses $s_2 = Left$,
  - and 1 can reasonably believe 2 would rationally use $s_2 = Left$, because
    - $s_2 = Left$ is 2's best response if 1 uses $s_1 = Left$,
    - and 2 can reasonably believe 1 would rationally use $s_1 = Left$, because
      - $s_1 = Left$ is 1's best response if 2 uses $s_2 = Left$,
      - and 1 can reasonably believe 2 would rationally use $s_2 = Left$, because
        - … and so on so forth…

Rationalizability

	Heads	Tails
Heads	1, –1	–1, 1
Tails	–1, 1	1, –1

- Some rationalizable strategies are not part of any Nash equilibrium

Example: Matching Pennies

- For Agent 1, the pure strategy s_1 = *Heads* is rationalizable because
  - s_1 = *Heads* is 1's best response if 2 uses s_2 = *Heads*,
  - and 1 can reasonably believe 2 would rationally use s_2 = *Heads*, because
    - s_2 = *Heads* is 2's best response if 1 uses s_1 = *Tails*,
    - and 2 can reasonably believe 1 would rationally use s_1 = *Tails*, because
      - s_1 = *Tails* is 1's best response if 2 uses s_2 = *Tails*,
      - and 1 can reasonably believe 2 would rationally use s_2 = *Tails*, because
        - … and so on so forth…

Common Knowledge

- Rationalizability is closely related to the idea of ***common knowledge***
- The book mentions common knowledge in several places, but doesn't define what it means
  - Section 4.4 (centipede game)
  - Section 6.1 (dominant strategies in repeated games)
  - Start of Chapter 7 (Bayesian games)
  - Section 7.1 (epistemic types)
  - Section 7.3 (best response to an *ex interim* payoff matrix)

Common Knowledge

- The definition of common knowledge is recursive analogous to the definition of rationalizability
- A property *p* is *common knowledge* if
  - Everyone knows *p*
  - Everyone knows that everyone knows *p*
  - Everyone knows that everyone knows that everyone knows *p*
  - …

The *p*-Beauty Contest

(not in the book)

- Recall that I asked you to play the following game:

> - Choose a (rational) number in the range from 0 to 100
>   - Write it on a piece of paper
>   - Also write your name (this is optional)
>   - Fold your paper in half, so nobody else can see your number
>   - Pass your paper to the front of the room
> - The winner(s) will be whoever chose a number that's closest to 2/3 of the average

- This game is famous among economists and game theorists
  - It's called the *p*-beauty contest for $p < 1$
  - I used $p = 2/3$

The *p*-Beauty Contest

- We can show that in the *p*-beauty contest
  - the only rationalizable strategy is to choose 0
  - the only Nash equilibrium is for everyone to choose 0
  - if rationality is common knowledge, everyone will choose 0

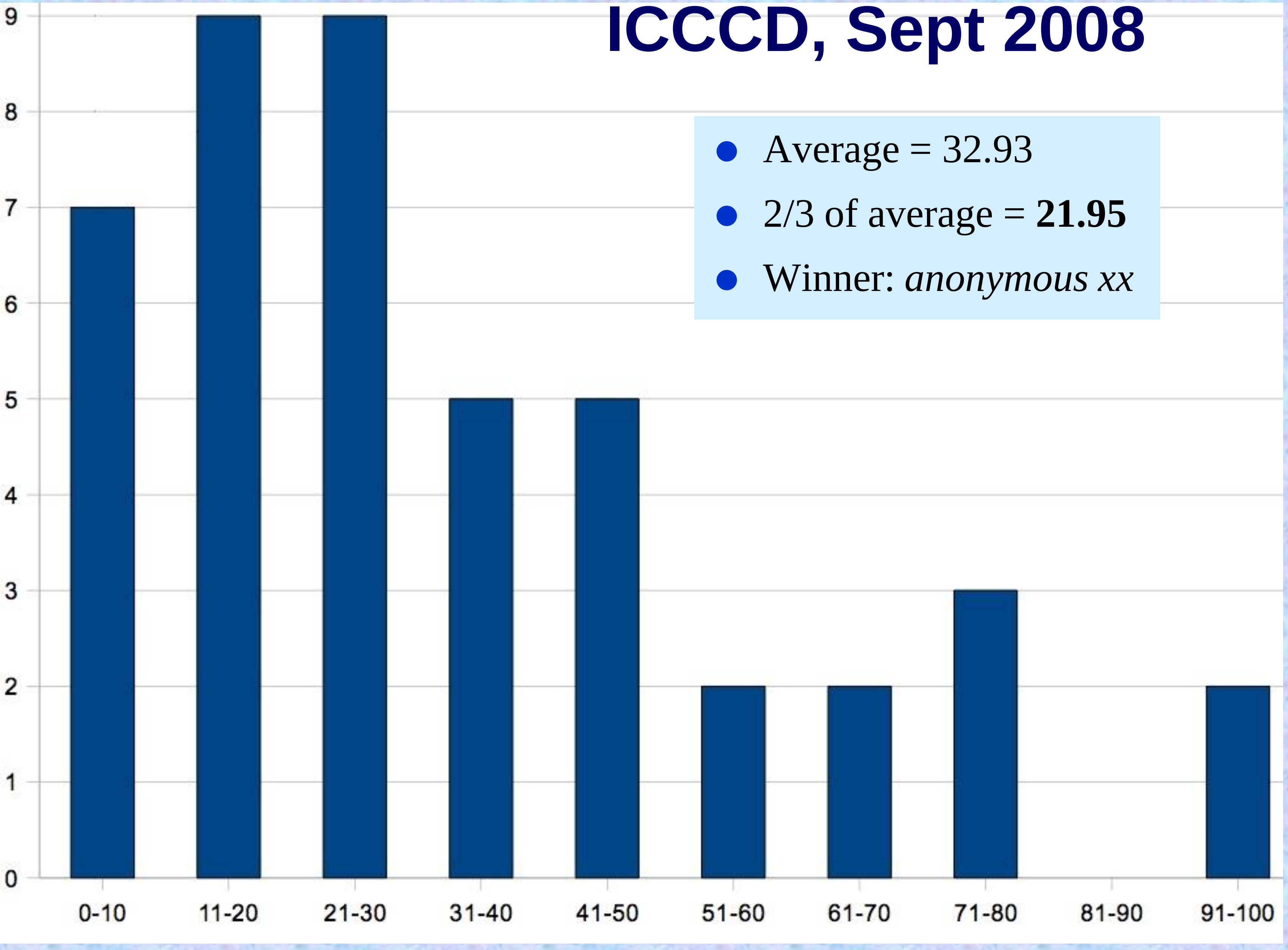
- As we saw in our class

  - Average = 27.46
  - 2/3 of average = 18.31
  - Winner: Jun Zhang with 18

- Now let's see more data

ICCCD, Sept 2008

- Average = 32.93
- 2/3 of average = **21.95**
- Winner: *anonymous xx*



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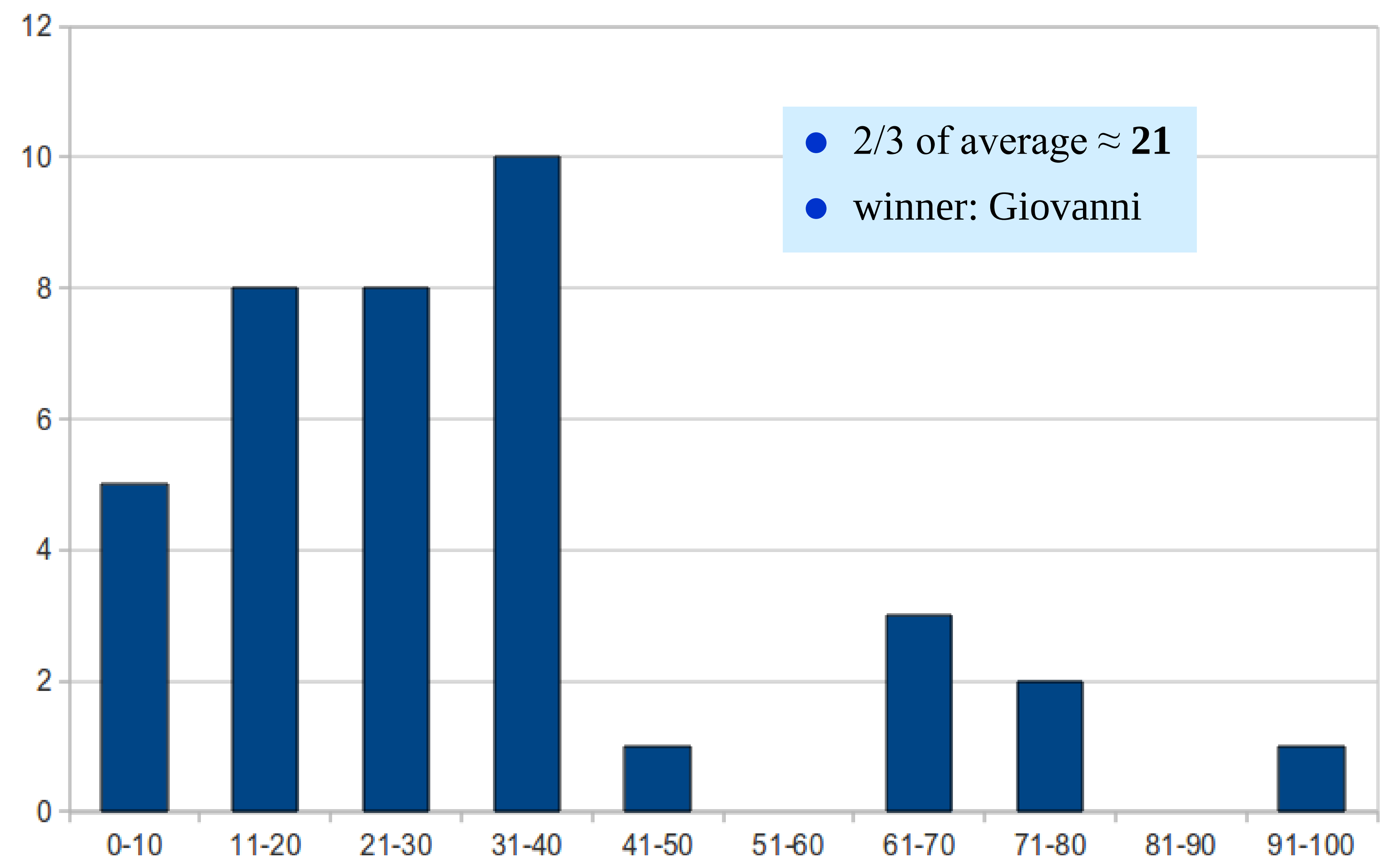

- 2/3 of average ≈ **21**
- winner: Giovanni

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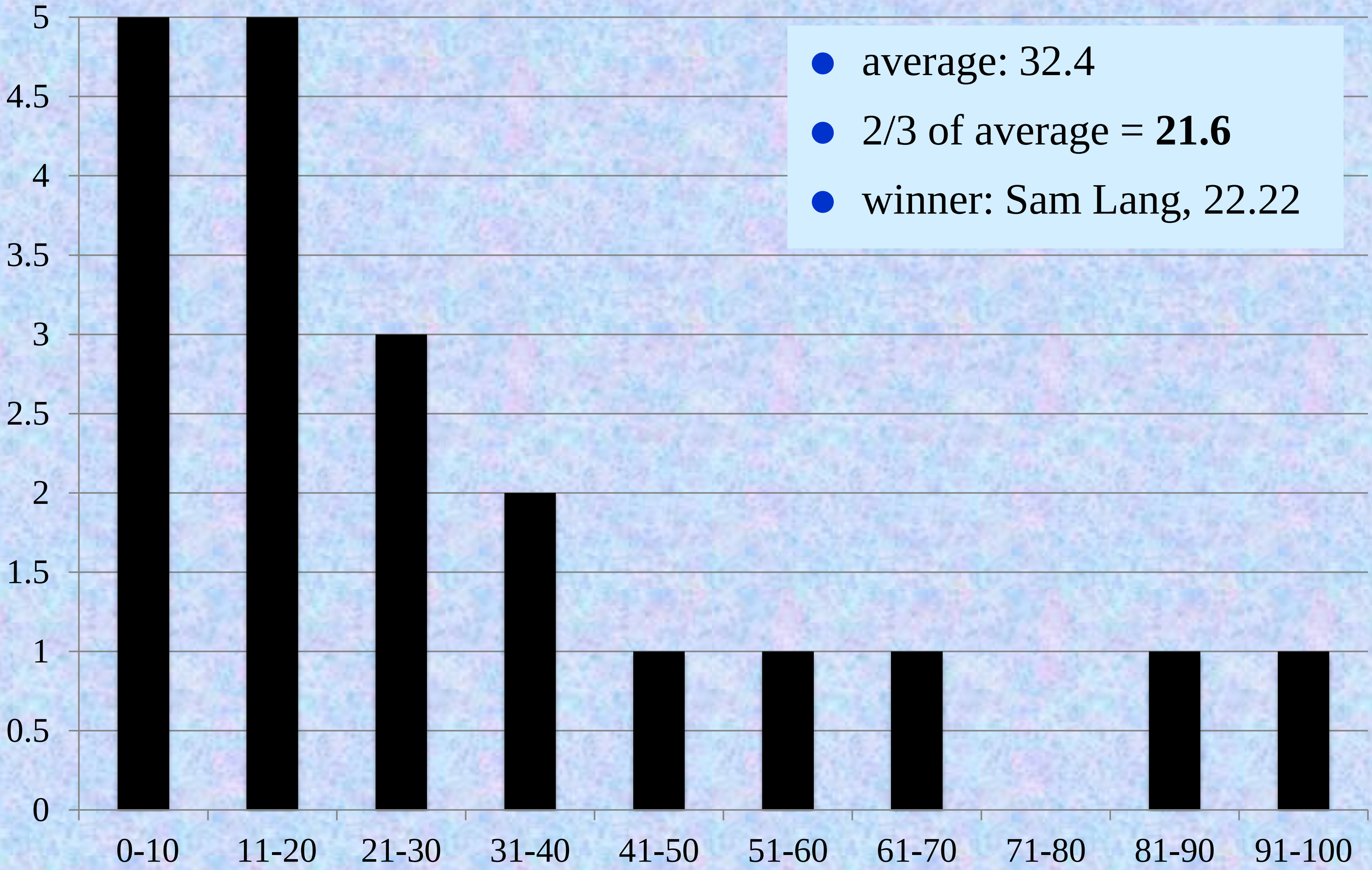
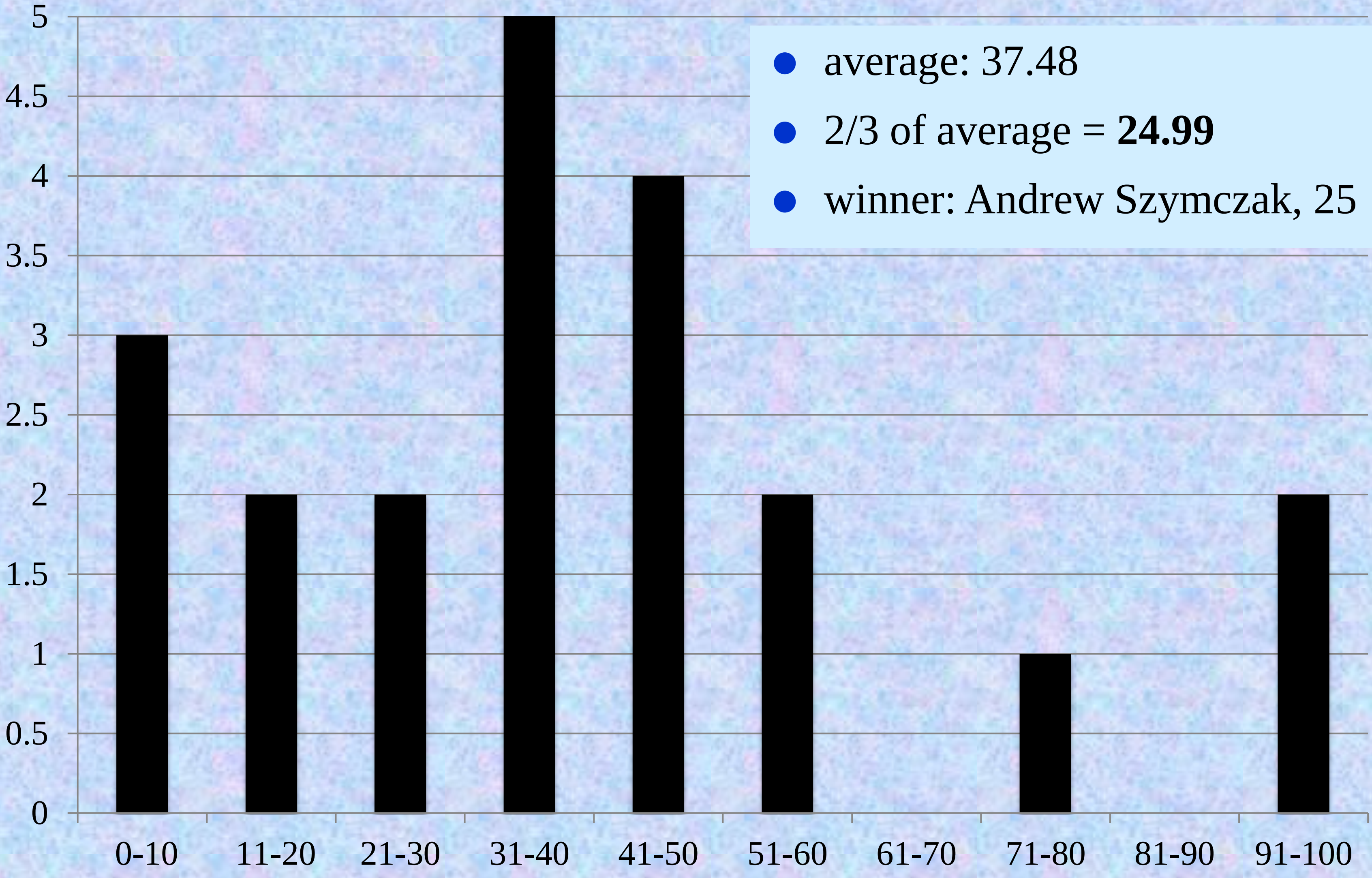


- average: 32.4
- 2/3 of average = **21.6**
- winner: Sam Lang, 22.22

CMSC 498T, September 2011



- average: 37.48
- 2/3 of average = **24.99**
- winner: Andrew Szymczak, 25

We Aren’t Rational

- More evidence that we aren’t game-theoretically rational agents

- Why choose an “irrational” strategy?
  - Several possible reasons …

Reasons for Choosing "Irrational" Strategies

(1) Limitations in reasoning ability

- Didn't calculate the Nash equilibrium correctly
- Don't know how to calculate it
- Don't even know the concept

(2) Wrong payoff matrix - doesn't encode agent's actual preferences

- It's a common error to take an external measure (money, points, etc.) and assume it's all that an agent cares about
- Other things may be more important than winning
  - Being helpful
  - Curiosity
  - Creating mischief
  - Venting frustration

(3) Beliefs about the other agents' likely actions (next slide)

Beliefs about Other Agents' Actions

- A Nash equilibrium strategy is best for you if the other agents also use their Nash equilibrium strategies
- In many cases, the other agents won't use Nash equilibrium strategies
  - If you can guess what actions they'll choose, then
    - You can compute your best response to those actions
      - maximize your expected payoff, given their actions
    - Good guess => you may do much better than the Nash equilibrium
    - Bad guess => you may do much worse

The Price of Anarchy (PoA)

- In the Chocolate Game, recall that
  - (T3,T3) is the action profile that provides the best outcome for everyone

	T3	*T1*
T3	3, 3	0, 4
T1	4, 0	1, 1

  - If we assume each payer acts to maximize his/her utility without regard to the other, we get (T1,T1)
  - By choosing (T3,T3), each player could have gotten 3 times as much

	T3	*T1*
T3	3, 3	0, 4
T1	4, 0	1, 1

- Let's generalize "best outcome for everyone"

The Price of Anarchy

- ***Social welfare function***: a function $w(\mathbf{s})$ that measures the players' welfare, given a strategy profile $\mathbf{s}$, e.g.,
  - Utilitarian function: $w(\mathbf{s})$ = average expected utility
  - Egalitarian function: $w(\mathbf{s})$ = minimum expected utility
- ***Social optimum***: benevolent dictator chooses $\mathbf{s}^*$ that optimizes w
  - $\mathbf{s}^* = \arg\max_{\mathbf{s}} w(\mathbf{s})$
- ***Anarchy***: no dictator; every player selfishly tries to optimize his/her own expected utility, disregarding the welfare of the other players
  - Get a strategy profile $\mathbf{s}$ (e.g., a Nash equilibrium)
  - In general, $w(\mathbf{s}) \leq w(\mathbf{s}^*)$
- ***Price of Anarchy (PoA)*** $= w(\mathbf{s}^*) / w(\mathbf{s})$

The Price of Anarchy

- Example: the Chocolate Game
  - Utilitarian welfare function:
    $w(\mathbf{s})$ = average expected utility

	T 3	*T1*
T3	3, 3	0, 4
T1	4, 0	1, 1

- Social optimum: $\mathbf{s}^* = (T3,T3)$
  - $w(\mathbf{s}^*) = 3$

	T3	*T1*
T3	3, 3	0, 4
T1	4, 0	1, 1

- Anarchy: $\mathbf{s} = (T1,T1)$
  - $w(\mathbf{s}) = 1$
- Price of anarchy

  $= w(\mathbf{s}^*) / w(\mathbf{s}) = 3/1 = 3$
- What would the answer be if we used the egalitarian welfare function?

The Price of Anarchy

- Sometimes instead of ***maximizing*** a welfare function *w*, we want to ***minimize*** a cost function *c* (e.g. in Prisoner's Dillema)
  - Utilitarian function: $c(\mathbf{s})$ = avg. expected cost
  - Egalitarian function: $c(\mathbf{s})$ = max. expected cost

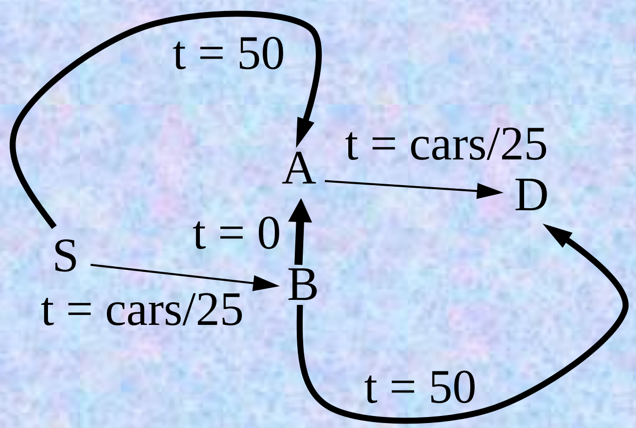
	C	*D*
C	3, 3	0, 5
D	5, 0	1, 1

- Need to adjust the definitions
  - ***Social optimum***: $\mathbf{s}^* = \arg\min_{\mathbf{s}} c(\mathbf{s})$
  - ***Anarchy***: every player selfishly tries to minimize his/her own cost, disregarding the costs of the other players
    - Get a strategy profile **s** (e.g., a Nash equilibrium)
    - In general, $c(\mathbf{s}) \geq c(\mathbf{s}^*)$
  - ***Price of Anarchy (PoA)*** = $c(\mathbf{s}) / c(\mathbf{s}^*)$
    - i.e., the reciprocal of what we had before
    - E.g. in Prisoner's dilemma PoA= 3

The Price of Anarchy

- Example: Braess's Paradox
  - Utilitarian cost function: $c(\mathbf{s})$ = average expected cost

- Social optimum:
  - $\mathbf{s}^*$ = [500 go S→A→D; 500 go S→B→D]
  - $c(\mathbf{s}^*) = 70$
- Anarchy: $\mathbf{s}$ = [1000 drivers go S→B→A→D]
  - $c(\mathbf{s}) = 80$
- Price of anarchy = $c(\mathbf{s}) / c(\mathbf{s}^*) = 8/7$



- What would the answer be if we used the egalitarian cost function?
- Note that when we talk about Price of Anarchy for Nash equilibria in general, we consider the **worst case** Nash equilibrium