CMSC 474, Introduction to Game Theory

9. Dominant Strategies and Correlated Equilibrium

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Outline

- Chapter 2 discussed two solution concepts:
 - Pareto optimality and Nash equilibrium
- Chapter 3 discusses several more:
 - Maxmin and Minmax
 - Dominant strategies
 - Correlated equilibrium
 - > Trembling-hand perfect equilibrium
 - > ε-Nash equilibrium
 - > Evolutionarily stable strategies

Dominant Strategies

• Let s_i and s_i' be two strategies for agent i

- Intuitively, s_i dominates s_i' if agent i does better with s_i than with s_i' for every strategy profile s_{-i} of the remaining agents
- Mathematically, there are three gradations of dominance:
 - > s_i strictly dominates s_i' if for every \mathbf{s}_{-i} ,

 $u_i(s_i, \mathbf{s}_{-i}) > u_i(s_i', \mathbf{s}_{-i})$

> s_i weakly dominates s_i' if for every \mathbf{s}_{-i} ,

 $u_i(s_i, \mathbf{s}_{-i}) \ge u_i(s_i', \mathbf{s}_{-i})$

and for at least one \mathbf{s}_{-i} ,

 $u_i(s_i, \mathbf{s}_{-i}) > u_i(s_i', \mathbf{s}_{-i})$

> s_i very weakly dominates s_i' if for every \mathbf{s}_{-i} ,

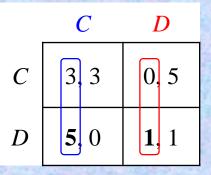
 $u_i(s_i, \mathbf{s}_{-i}) \ge u_i(s'_i, \mathbf{s}_{-i})$

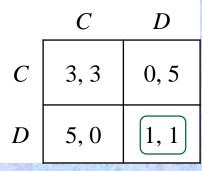
Dominant Strategy Equilibria

- A strategy is **strictly** (resp., **weakly**, **very weakly**) **dominant** for an agent if it strictly (weakly, very weakly) dominates any other strategy for that agent
- A strategy profile (s_1, \ldots, s_n) in which every s_i is dominant for agent *i* (strictly, weakly, or very weakly) is a Nash equilibrium
 - Why?
 - Such a strategy profile forms an equilibrium in strictly (weakly, very weakly) dominant strategies

Examples

- Example: the Prisoner's Dilemma
 - http://www.youtube.com/watch?v=ED9gaAb2BEw
- For agent 1, D is strictly dominant
 - ➢ If agent 2 uses C, then
 - Agent 1's payoff is higher with D than with C
 - ▶ If agent 2 uses *D*, *then*
 - Agent 1's payoff is higher with D than with C
- Similarly, D is strictly dominant for agent 2
- So (D,D) is a Nash equilibrium in strictly dominant strategies
- How do strictly dominant strategies relate to strict Nash equilibria?





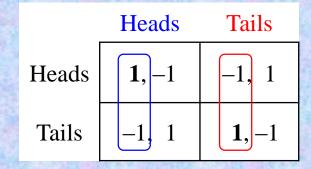
Example: Matching Pennies

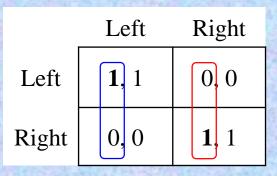
Matching Pennies

- > If agent 2 uses Heads, then
 - For agent 1, Heads is better than Tails
- ➢ If agent 2 uses Tails, then
 - For agent 1, Tails is better than Heads
- Agent 1 doesn't have a dominant strategy
 => no Nash equilibrium in dominant strategies

• Which Side of the Road

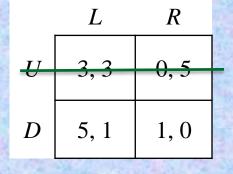
- Same kind of argument as above
- No Nash equilibrium in dominant strategies

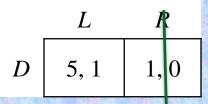




Elimination of Strictly Dominated Strategies

- A strategy s_i is strictly (weakly, very weakly) dominated for an agent *i* if some other strategy s_i' strictly (weakly, very weakly) dominates s_i
- A strictly dominated strategy can't be a best response to any move, so we can eliminate it (remove it from the payoff matrix)
 - > This gives a **reduced** game
 - Other strategies may now be strictly dominated, even if they weren't dominated before





D

L

5, 1

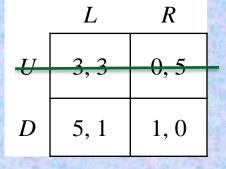
- IESDS (Iterated Elimination of Strictly Dominated Strategies):
 - Do elimination repeatedly until no more eliminations are possible
 - When no more eliminations are possible, we have the maximal reduction of the original game

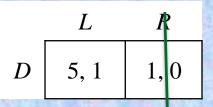
IESDS

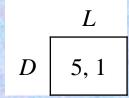
- If you eliminate a strictly dominated strategy, the reduced game has the same Nash equilibria as the original one
 - Thus

{Nash equilibria of the original game}
= {Nash equilibria of the maximally reduced game}

- Use this technique to simplify finding Nash equilibria
 Look for Nash equilibria on the maximally reduced game
- In the example, we ended up with a single cell
 - The single cell *must* be a unique Nash equilibrium in all three of the games





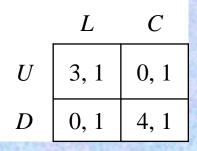


IESDS

- Even if s_i isn't strictly dominated by a pure strategy, it may be strictly dominated by a mixed strategy
- Example: the three games shown at right
 - > 1st game:
 - R is strictly dominated by L (and by C)
 - Eliminate it, get 2nd game
 - $> 2^{nd}$ game:
 - Neither U nor D dominates M
 - But $\{(\frac{1}{2}, U), (\frac{1}{2}, D)\}$ strictly dominates M
 - > This wasn't true before we removed R
 - Eliminate it, get 3rd game
 - > 3rd game is maximally reduced

	L	С	R
U	3, 1	0, 1	0, 0
М	1, 1	1, 1	5,0
D	0, 1	4, 1	0, 0

	L	С
U	3, 1	0, 1
М	1, 1	1, 1
D	0, 1	4, 1



Correlated Equilibrium: Pithy Quote

If there is intelligent life on other planets, in a majority of them, they would have discovered correlated equilibrium before Nash equilibrium.

----Roger Myerson

Correlated Equilibrium: Intuition

- Not every correlated equilibrium is a Nash equilibrium but every Nash equilibrium is a correlated equilibrium
- We have a **traffic light**: a fair randomizing device that tells one of the agents to go and the other to wait.

• Benefits:

- > easier to compute than Nash, e.g., it is polynomial-time computable
- > fairness is achieved
- > the sum of social welfare exceeds that of any Nash equilibrium

- Recall the mixed-strategy equilibrium for the Battle of the Sexes
 - > $s_w = \{(2/3, \text{Opera}), (1/3, \text{Football})\}$
 - > $s_h = \{(1/3, \text{Opera}), (2/3, \text{Football})\}$

	Husband Wife	Oper a	Football
	Opera	2, 1	0, 0
1000	Football	0, 0	1, 2

- This is "fair": each agent is equally likely to get his/her preferred activity
- But 5/9 of the time, they'll choose different activities => utility 0 for both
 - Thus each agent's expected utility is only 2/3
 - > We've required them to make their choices independently
- Coordinate their choices (e.g., flip a coin) => eliminate cases where they choose different activities
 - > Each agent's payoff will always be 1 or 2; expected utility 1.5
- Solution concept: correlated equilibrium
 - Generalization of a Nash equilibrium

- Let G be an n-agent game
- Let v_1, \ldots, v_n be random variables, one for each agent
 - > For each *i*, let D_i be the domain (the set of possible values) of v_i
- Let π be a joint distribution over v_1, \ldots, v_n
 - $\pi(d_1, ..., d_n) = \Pr[v_1 = d_1, ..., v_n = d_n]$
- "Nature" uses π to choose values $\mathbf{d} = (d_1, \dots, d_n)$ for $\mathbf{v} = (v_1, \dots, v_i)$
- "Nature" tells each agent *i* the value of v_i (privately)
 - > An agent can condition his/her action on the value of v_i
 - > An agent's strategy is a deterministic mapping $\sigma_i : D_i \to A_i$ (note that we might have $\sigma_i(d_1) = \sigma_i(d_2)$ for d_1 not equal to d_2)
 - As book says mixed strategies wouldn't give any greater generality
 - > A strategy profile is $\boldsymbol{\sigma} = (\sigma_1, ..., \sigma_n)$
- The games we've been considering before now are a degenerate case in which the random variables $v_1, ..., v_n$ are independent

- \succ G is an *n*-player game
- > $\mathbf{v} = (v_1, ..., v_n)$ are random variables with domains $\mathbf{D} = (D_1, ..., D_n)$
 - Joint distribution $\pi(\mathbf{d}) = \pi(d_1, \dots, d_n) = \Pr[v_1 = d_1, \dots, v_n = d_n]$
- $\triangleright \mathbf{\sigma} = (\sigma_1, ..., \sigma_n)$ is a strategy profile
 - Each strategy σ_i is a mapping from D_i to A_i
- Then the expected utility for agent *i* is $u_i(\sigma) = \sum_{\mathbf{d}} \pi(\mathbf{d}) u_i(\sigma(\mathbf{d})),$

i.e.,
$$u_i(S_1,...,S_n) = \mathop{a}_{d_1,...,d_n} \rho(d_1,...,d_n) u_i(S_1(d_1),...,S_n(d_n))$$

• (v, π, σ) is a **correlated equilibrium** if for every agent *i* and strategy σ'_i , $u_i(\sigma) \ge u_i(\sigma'_i, \sigma_{-i})$

i.e., $u_i(\sigma_1, ..., \sigma_{i-1}, \sigma_i, \sigma_{i+1}, ..., \sigma_n) \ge u_i(\sigma_1, ..., \sigma_{i-1}, \sigma_i', \sigma_{i+1}, ..., \sigma_n)$

Theorem. For every Nash equilibrium $\mathbf{s} = (s_1, ..., s_n)$, there's a corresponding correlated equilibrium $\mathbf{\sigma} = (\sigma_1, ..., \sigma_n)$

> "Corresponding" means they produce the same distribution on outcomes Basic idea of the proof: for each *i*, set up v_i and σ_i to mimic s_i

- > v_1 , ..., v_n independently distributed
- > Each v_i has domain A_i and probability distribution s_i
- > Each σ_i is the identity function (i.e., do the action that you're told to do)
- When the agents play the strategy profile σ, the distribution over outcomes is identical to that under s
- > No agent *i* can benefit by deviating from σ_i , so σ is a correlated equilibrium
- But not every correlated equilib. is equivalent to a Nash equilib.e.g.,Battle of Sexes
- Intuitively, correlated equilibrium is computable in polynomial time since it has only a single randomization over outcomes, whereas in NE this is constructed as a product of independent probabilities.

Computing CE

$$\begin{split} \sum_{a \in A \mid a_i \in a} p(a)u_i(a) &\geq \sum_{a \in A \mid a'_i \in a} p(a)u_i(a'_i, a_{-i}) \quad \forall i \in N, \, \forall a_i, a'_i \in A_i \\ p(a) &\geq 0 \qquad \qquad \forall a \in A \\ \sum_{a \in A} p(a) &= 1 \end{split}$$

- variables: p(a); constants: u_i(a)
- we could find the social-welfare maximizing CE by adding an objective function

maximize:
$$\sum_{a \in A} p(a) \sum_{i \in N} u_i(a).$$