CMSC 474, Introduction to Game Theory

11. Evolutionary Stability

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Outline

- Chapter 2 discussed two solution concepts:
  - Pareto optimality and Nash equilibrium
- Chapter 3 discusses several more:
  - Maxmin and Minmax
  - Dominant strategies
  - Correlated equilibrium
  - Trembling-hand perfect equilibrium (complicated defintion)
  - ε-Nash equilibrium
  - Evolutionarily stable strategies

Evolutionary Stability

- This concept comes from evolutionary biology
- Start with a population of some species
  - For us species are those agents playing a particular strategy *s*
- Add a small population of "invaders" species
  - For us invaders are those agents playing a different strategy *t*
  - *Assume t* **invades** *s* at level *p*, i.e., *p* is the fraction that uses *t*
  - (1–*p*) = the fraction that uses *s*
- If *s*'s fitness against the mixture of both species is higher than *t*'s, then *t*'s proportion will shrink and *s*'s will grow (thus s is ``**stable**'')
  - Fitness for species is the ability to both survive and reproduce
  - For us, fitness of a species= its expected payoff from interacting with a random member of the population, namely with species *t* with probability *p* and with species *s* with probability 1-*p*

Evolutionary Stability

- Write a payoff matrix for the two species against each other
  - Symmetric 2-player game, so we only need to look at agent 1's payoffs
- A strategy's **fitness** is its expected payoff against a randomly chosen agent
  - fitness(s) = $(1-p)a + pb$
  - fitness(t) = $(1-p)c + pd$

	s	t
s	$\mathbf{a}, a$	$\mathbf{b}, c$
t	$\mathbf{c}, b$	$\mathbf{d}, d$

- s is **evolutionarily stable against** t if there is an $\varepsilon > 0$ such that for every $p < \varepsilon$, fitness(s) > fitness(t)
  - i.e., $(1-p)a + pb \quad > \quad (1-p)c + pd$
- As $p \to 0$, $(1-p)a + pb \to a$ and $(1-p)c + pd \to c$
  - For sufficiently small p, the inequality holds if $a > c$, or if a = c and $b > d$
- Thus s is evolutionarily stable against t iff either of the following holds:
  - $a > c$
  - $a = c$ and $b > d$

Example: the Body-Size Game

- Consider two different sizes of beetles competing for food
  - When beetles of the same size compete, they get equal shares
  - When large competes with small, large gets most of the food
  - Large beetles get less fitness benefit from any given amount of food
    - Some of it is diverted into maintaining their expensive metabolism

	small	*large*
small	5, 5	1, 8
large	8, 1	3, 3

- Is a population of small beetles evolutionarily stable against large beetles?

	large	*small*
large	3, 3	8, 1
small	1, 8	5, 5

- Is a population of large beetles evolutionarily stable against small ones?

- Source:
  - http://www.cs.cornell.edu/home/kleinber/networks-book

Evolutionary Stability

- More generally, suppose *s* is a mixed strategy
- Represents a population composed of several species
- We'll talk about *s*'s evolutionary stability against all other mixed strategies
- *s* is an **evolutionarily stable strategy (ESS)** iff for ***every*** mixed strategy $t \neq s$, either of the following holds:
    - $u(s,s) > u(t,s)$
    - $u(s,s) = u(t,s)$ and $u(s,t) > u(t,t)$

    (note that $u_1 = u_2$ since the game is symmetric)
- *s* is **weakly evolutionarily stable** iff for every mixed strategy $t \neq s$, either of the following stability conditions holds:
    1. $u(s,s) > u(t,s)$
    2. $u(s,s) = u(t,s)$ and $u(s,t) \geq u(t,t)$

  - Includes cases where *s* and *t* have the same fitness
      - So the population that uses *t* neither grows nor shrinks

Example

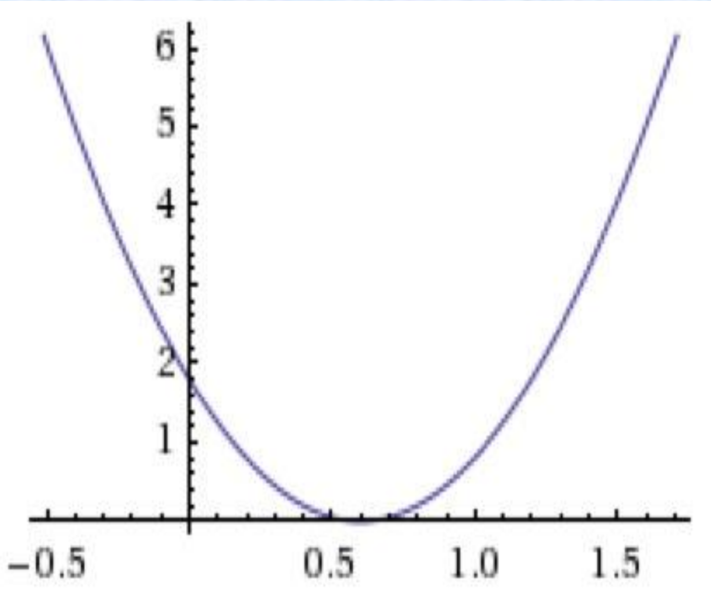
	H	D
H	–2, –2	6, 0
D	0, 6	3, 3

- **The Hawk-Dove game**
  - 2 animals contend for a piece of food
  - Each animal may be either a hawk (H) or a dove (D)
    - The prize is worth 6 to each
    - Fighting costs each 5
- When a hawk meets a dove, the hawk gets the prize without a fight: payoffs 6, 0
- When 2 doves meet, they split the prize without a fight: payoffs 3, 3
- When 2 hawks meet,
  - They fight, and each has a 50% chance of getting the prize
  - For each, the payoff is $-5 + 0.5 \cdot 6 = -2$
- Unique Nash equilibrium (s, s), where s = {(3/5, H), (2/5, D)}
  - i.e., 60% hawks, 40% doves

Example

	H	D
H	–2, –2	6, 0
D	0, 6	3, 3

- To confirm that s is also an ESS, show that for all $t \neq s$,
    - $u_1(s,s) > u_1(t,s)$ OR
    - $u_1(s, s) = u_1(t, s)$ and $u_1(s, t) > u_1(t, t)$
  - where $s = \{(3/5, \text{H}), (2/5, \text{D})\}$ and $t = \{(p, \text{H}), (1–p, \text{D})\}$
- For *every* fully-mixed strategy s, if (s,s) is a Nash equilibrium then $u_1(s,s) = u_1(t,s)$
- Next, show $u_1(s,t) > u_1(t,t)$:
  - $u_1(s,t) = (3/5)(–2p + 6(1–p)) + (2/5)(0p + 3(1–p))$
  - $u_1(t,t) = p(–2p + 6(1–p)) + (1–p)(0p + 3(1–p))$
- Let $v = u_1(s,t) - u_1(t,t)$
- Easy to solve using http://wolframalpha.com
  - Simplifies to $v = 5p^2 – 6p + 9/5$
  - Unique minimum $v = 0$ when $p = 3/5$, i.e., $t = s$
  - If $p \neq 3/5$ then $v > 0$, i.e., $u_1(s,t) > u_1(t,t)$



Evolutionary Stability and Nash Equilibria

- Recall that s is **evolutionarily stable** iff for every mixed strategy $t \neq s$, either of the following holds:
  - $u(s,s) > u(t,s)$ (1)
  - $u(s,s) = u(t,s)$ and $u(s,t) > u(t,t)$ (2)

Theorem. Let G be a symmetric 2-player game, and s be a mixed strategy. If s is an evolutionarily stable strategy, then (s, s) is a Nash equilibrium of G.

Proof. By definition, an ESS s must satisfy $u(s,s) \geq u(t,s)$, i.e., s is a best response to itself, so it must be a Nash equilibrium.

Theorem. Let G be a symmetric 2-player game, and s be a mixed strategy. If (s,s) is a strict Nash equilibrium of G, then s is evolutionarily stable.

Proof. If (s,s) is a strict Nash equilibrium, then $u(s,s) > u(t,s)$.

- This satisfies (1) above

Summary

- Maxmin and minmax strategies, and the Minimax Theorem
  - Matching Pennies, Two-Finger Morra
- dominant strategies
  - Prisoner's Dilemma, Which Side of the Road, Matching Pennies
  - Iterated elimination of dominated strategies (IESDS)
- rationalizability
  - the *p*-Beauty Contest
- correlated equilibrium
  - Battle of the Sexes
- epsilon-Nash equilibria
- evolutionarily stable strategies
  - Body-Size game, Hawk-Dove game