CMSC 474, Introduction to Game Theory

11. Evolutionary Stability

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Chapter 2 discussed two solution concepts:
  - Pareto optimality and Nash equilibrium

Chapter 3 discusses several more:
  - Maxmin and Minmax
  - Dominant strategies
  - Correlated equilibrium
  - Trembling-hand perfect equilibrium (complicated definition)
  - $\varepsilon$-Nash equilibrium
  - Evolutionarily stable strategies
Evolutionary Stability

- This concept comes from evolutionary biology
- Start with a population of some species
  - For us species are those agents playing a particular strategy $s$
- Add a small population of “invaders” species
  - For us invaders are those agents playing a different strategy $t$
  - Assume $t$ invades $s$ at level $p$, i.e., $p$ is the fraction that uses $t$
  - $(1-p) =$ the fraction that uses $s$
- If $s$’s fitness against the mixture of both species is higher than $t$’s, then $t$’s proportion will shrink and $s$’s will grow (thus $s$ is "stable")
  - Fitness for species is the ability to both survive and reproduce
  - For us, fitness of a species= its expected payoff from interacting with a random member of the population, namely with species $t$ with probability $p$ and with species $s$ with probability $1-p$
Evolutionary Stability

- Write a payoff matrix for the two species against each other
  - Symmetric 2-player game, so we only need to look at agent 1’s payoffs
- A strategy’s **fitness** is its expected payoff against a randomly chosen agent
  - fitness($s$) = $(1–p)a + pb$
  - fitness($t$) = $(1–p)c + pd$
- $s$ is **evolutionarily stable against** $t$ if there is an $\varepsilon > 0$ such that for every $p < \varepsilon$, fitness($s$) > fitness($t$)
  - i.e., $(1–p)a + pb > (1–p)c + pd$
- As $p \to 0$, $(1–p)a + pb \to a$ and $(1–p)c + pd \to c$
  - For sufficiently small $p$, the inequality holds if $a > c$, or if $a = c$ and $b > d$
- Thus $s$ is evolutionarily stable against $t$ iff either of the following holds:
  - $a > c$
  - $a = c$ and $b > d
Example: the Body-Size Game

- Consider two different sizes of beetles competing for food
  - When beetles of the same size compete, they get equal shares
  - When large competes with small, large gets most of the food
  - Large beetles get less fitness benefit from any given amount of food
    - Some of it is diverted into maintaining their expensive metabolism

- Is a population of small beetles evolutionarily stable against large beetles?
- Is a population of large beetles evolutionarily stable against small ones?

- Source:
Evolutionary Stability

- More generally, suppose $s$ is a mixed strategy
- Represents a population composed of several species
- We’ll talk about $s$’s evolutionary stability against all other mixed strategies
- $s$ is an evolutionarily stable strategy (ESS) iff for every mixed strategy $t \neq s$, either of the following holds:
  - $u(s,s) > u(t,s)$
  - $u(s,s) = u(t,s)$ and $u(s,t) > u(t,t)$
    (note that $u_1 = u_2$ since the game is symmetric)
- $s$ is weakly evolutionarily stable iff for every mixed strategy $t \neq s$, either of the following stability conditions holds:
  1. $u(s,s) > u(t,s)$
  2. $u(s,s) = u(t,s)$ and $u(s,t) \geq u(t,t)$
    - Includes cases where $s$ and $t$ have the same fitness
      - So the population that uses $t$ neither grows nor shrinks
Example

- The Hawk-Dove game
  - 2 animals contend for a piece of food
  - Each animal may be either a hawk (H) or a dove (D)
    - The prize is worth 6 to each
    - Fighting costs each 5
  - When a hawk meets a dove, the hawk gets the prize without a fight: payoffs 6, 0
  - When 2 doves meet, they split the prize without a fight: payoffs 3, 3
  - When 2 hawks meet,
    - They fight, and each has a 50% chance of getting the prize
    - For each, the payoff is \(-5 + 0.5 \cdot 6 = -2\)
  - Unique Nash equilibrium \((s, s)\), where \(s = \{(3/5, H), (2/5, D)\}\)
    - i.e., 60% hawks, 40% doves
To confirm that $s$ is also an ESS, show that for all $t \neq s$,
• $u_1(s,s) > u_1(t,s)$ OR
• $u_1(s,s) = u_1(t,s)$ and $u_1(s,t) > u_1(t,t)$
  where $s = \{(3/5, H), (2/5, D)\}$ and $t = \{(p, H), (1-p, D)\}$

For every fully-mixed strategy $s$, if $(s,s)$ is a Nash equilibrium then $u_1(s,s) = u_1(t,s)$

Next, show $u_1(s,t) > u_1(t,t)$:
• $u_1(s,t) = (3/5)(-2p + 6(1-p)) + (2/5)(0p + 3(1-p))$
• $u_1(t,t) = p(-2p + 6(1-p)) + (1-p)(0p + 3(1-p))$

Let $\nu = u_1(s,t) - u_1(t,t)$

Easy to solve using [http://wolframalpha.com](http://wolframalpha.com)
• Simplifies to $\nu = 5p^2 - 6p + 9/5$
• Unique minimum $\nu = 0$ when $p = 3/5$, i.e., $t = s$
• If $p \neq 3/5$ then $\nu > 0$, i.e., $u_1(s,t) > u_1(t,t)$
Evolutionary Stability and Nash Equilibria

- Recall that \( s \) is \textbf{evolutionarily stable} iff for every mixed strategy \( t \neq s \), either of the following holds:
  - \( u(s,s) > u(t,s) \) (1)
  - \( u(s,s) = u(t,s) \) and \( u(s,t) > u(t,t) \) (2)

**Theorem.** Let \( G \) be a symmetric 2-player game, and \( s \) be a mixed strategy. If \( s \) is an evolutionarily stable strategy, then \((s,s)\) is a Nash equilibrium of \( G \).

**Proof.** By definition, an ESS \( s \) must satisfy \( u(s,s) \geq u(t,s) \), i.e., \( s \) is a best response to itself, so it must be a Nash equilibrium.

**Theorem.** Let \( G \) be a symmetric 2-player game, and \( s \) be a mixed strategy. If \((s,s)\) is a strict Nash equilibrium of \( G \), then \( s \) is evolutionarily stable.

**Proof.** If \((s,s)\) is a strict Nash equilibrium, then \( u(s,s) > u(t,s) \).
  - This satisfies (1) above
Summary

- Maxmin and minmax strategies, and the Minimax Theorem
  - Matching Pennies, Two-Finger Morra
- dominant strategies
  - Prisoner’s Dilemma, Which Side of the Road, Matching Pennies
  - Iterated elimination of dominated strategies (IESDS)
- rationalizability
  - the $p$-Beauty Contest
- correlated equilibrium
  - Battle of the Sexes
- epsilon-Nash equilibria
- evolutionarily stable strategies
  - Body-Size game, Hawk-Dove game