CMSC 474, Introduction to Game Theory

15. Imperfect-Information Games

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Imperfect-Information Games

- So far, we've assumed that players in an extensive-form game always know what node they're at
  - Know all prior choices
    - Both theirs and the others'
  - Thus "perfect information" games
- But sometimes players
  - Don't know all the actions the others took or
  - Don't recall all their past actions
- Sequencing lets us capture some of this ignorance:
  - An earlier choice is made without knowledge of a later choice
- But it doesn't let us represent the case where two agents make choices at the same time, in mutual ignorance of each other

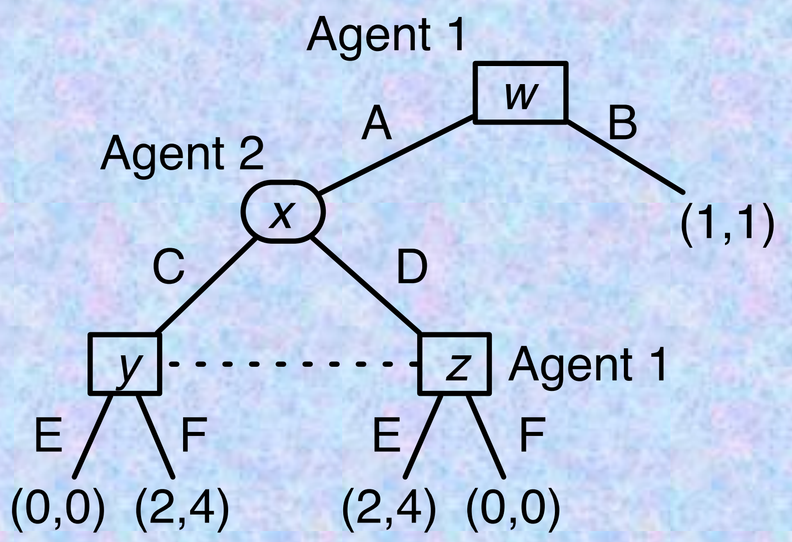
Definition

- An **imperfect-information** game is an extensive-form game in which each agent's choice nodes are partitioned into **information sets**
  - An information set = {all choice nodes an agent ***might*** be at}
    - The nodes in an information set are indistinguishable to the agent
    - So all have the same set of actions
  - Agent *i*'s information sets are $I_{i1}, \ldots, I_{im}$ for some *m*, where
    - $I_{i1} \cup \ldots \cup I_{im}$ = {all nodes where it's agent *i*'s move}
    - $I_{ij} \cap I_{ik} = \emptyset$ for all $j \neq k$
    - For all nodes $x,y \in I_{ij}$,
      - {all available actions at *x*} = {all available actions at *y*}

- A perfect-information game is a special case in which each I_{ij} contains just one node

Example

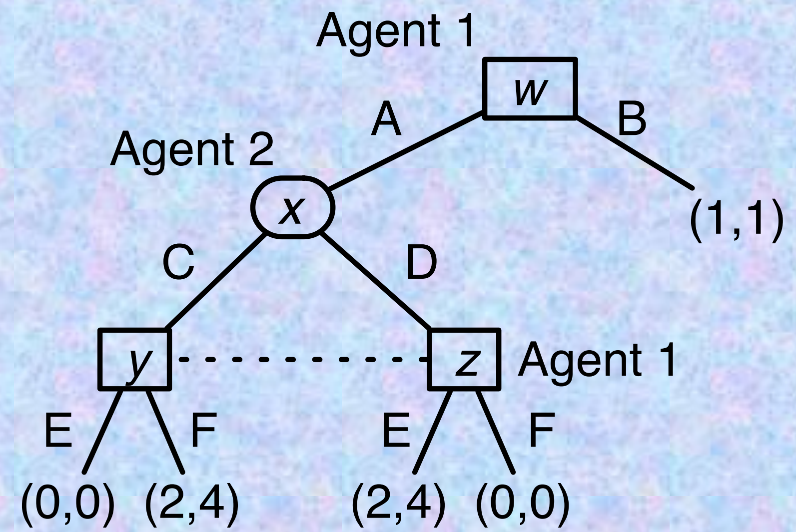
- Below, agent 1 has two information sets:
  - $I_{11} = \{w\}$
  - $I_{12} = \{y, z\}$
  - In I_{12}, agent 1 doesn't know whether agent 2's move was C or D
- Agent 2 has just one information set:
  - $I_{21} = \{x\}$



Strategies

- A **pure strategy** for agent i is a function s_i that selects an available action at each of i's information sets
  - $s_i(I)$ = agent i's action in information set I
- Thus {all pure strategies for i} is the Cartesian product
  - {actions available in I_{i1}} × … × {actions available in I_{im}}
- Agent 1's pure strategies:

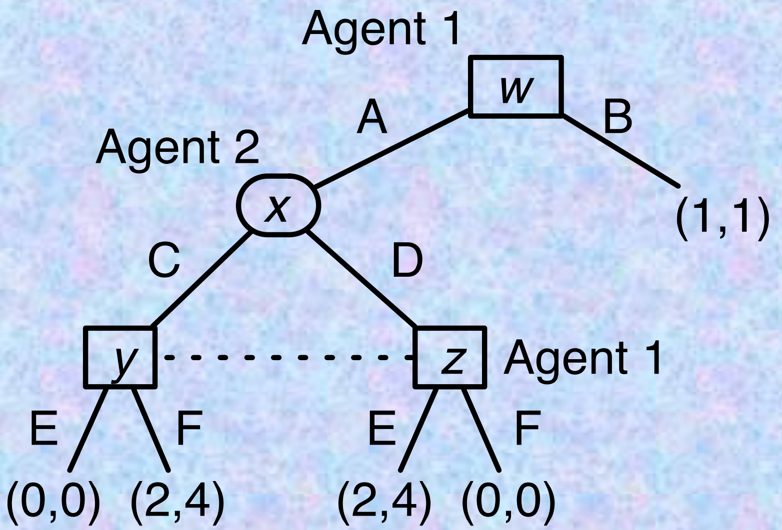
  {A,B} × {E, F} = {(A, E), (A, F), (B, E), (B, F)}
- Agent 2's pure strategies: {C, D}



Extensive Form → Normal Form

- Any extensive-form imperfect-information game can be transformed into an equivalent normal-form game
- Same strategies and same payoffs
  - Thus same Nash equilibria, same Pareto optimal strategy profiles, etc.
- Just like we did it for perfect-information games
  - Create an *n*-dimensional payoff matrix in which the *i*'th dimension corresponds to agent *i*'s pure strategies

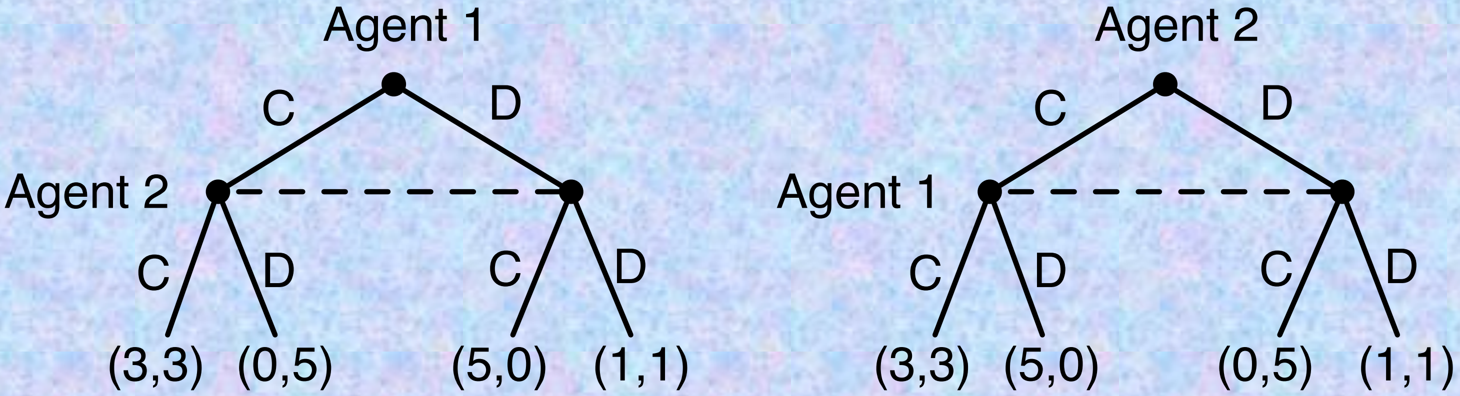
	C	D
(A,E)	0, 0	2, 4
(A,F)	2, 4	0, 0
(B,E)	1, 1	1, 1
(B,F)	1, 1	1, 1



Normal Form → Extensive Form

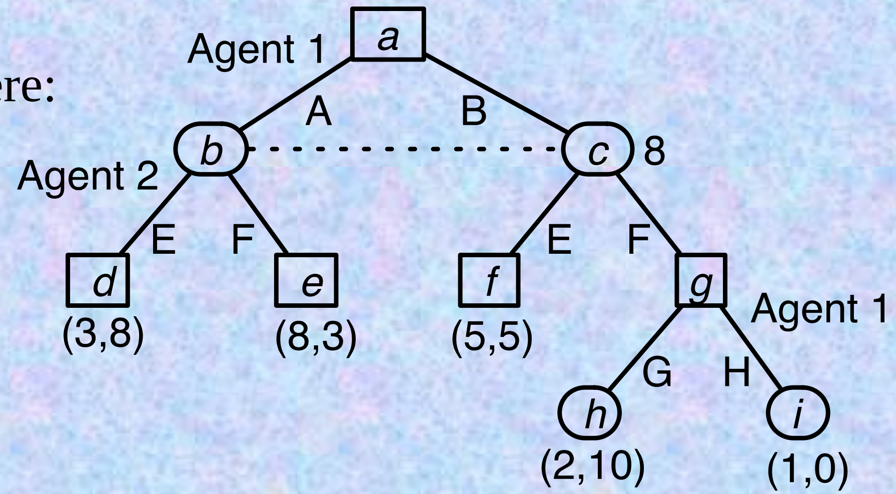
- Any normal-form game can be transformed into an equivalent extensive-form imperfect-information game
  - n-level game tree in which each agent has exactly one information set
- Same strategies and same payoffs → same Nash equilibria, same Pareto optimal strategy profiles, etc.
- Example:
  - Two imperfect-information extensive-form games that are equivalent to the Prisoner's Dilemma:

	C	*D*
C	3, 3	0, 5
D	5, 0	1, 1



Behavioral Strategies

- In imperfect-information extensive-form games, we can define a new class of strategies called **behavioral strategies**
  - An agent's (probabilistic) choice at each node is independent of his/her choices at other nodes
- Consider the imperfect-info game shown here:



- A behavioral strategy for Agent 1:
  - At node *a*, {(0.5, A), (0.5, B)}
  - At node *g*, {(0.3, G), (0.7, H)}
- Is there an equivalent mixed strategy?
  - What do we mean by "equivalent"?

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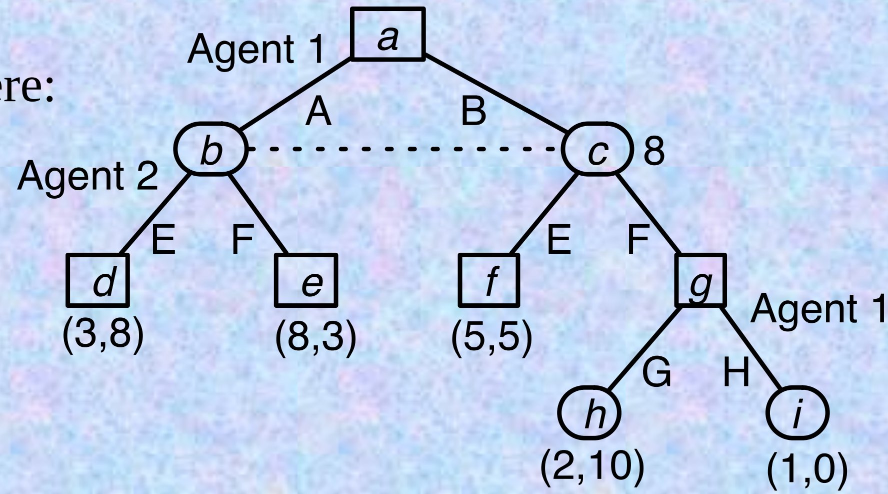

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  - At node *a*, {(0.5, A), (0.5, B)}
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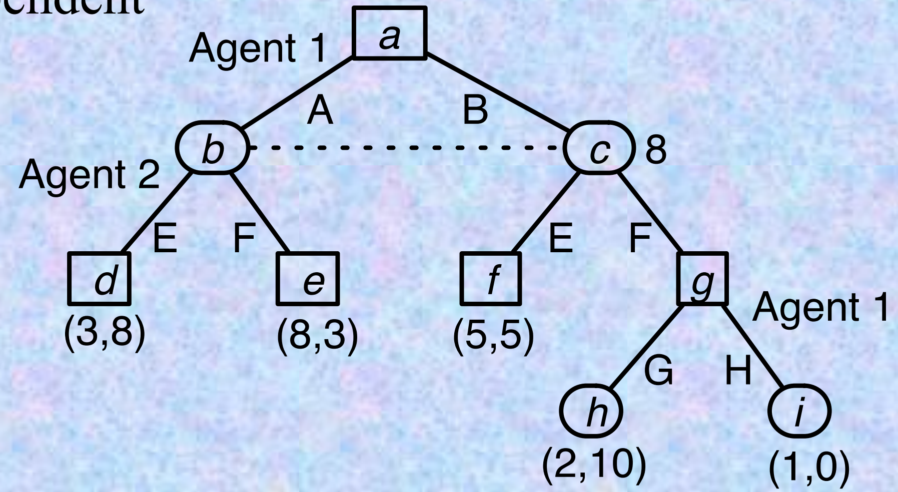
- Is there an equivalent mixed strategy?
  - What do we mean by "equivalent"?
  - Two strategies s_i and s_i' are equivalent if for every fixed strategy profile $\mathbf{s}_{-i}$ of the remaining agents, s_i and s_i' give us the same probabilities on outcomes
- An equivalent mixed strategy:
  - {(0.15, (A, G)); (0.35, (A, H)); (0.15, (B, G)); (0.35, (B, H))}

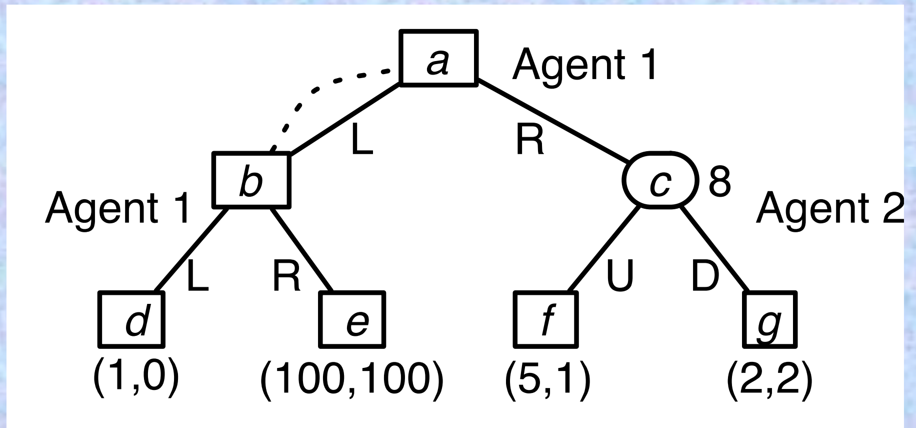
Behavioral vs. Mixed Strategies

- Consider the following mixed strategy:
  - {(0.6, (A, G)), (0.4, (B, H))}
- The choices at the two nodes aren't independent
  - Choose A at *a* ⇔ choose G at *g*
  - Choose B at *a* ⇔ choose H at *g*
- Thus not always easy to find an equivalent behavioral strategy.

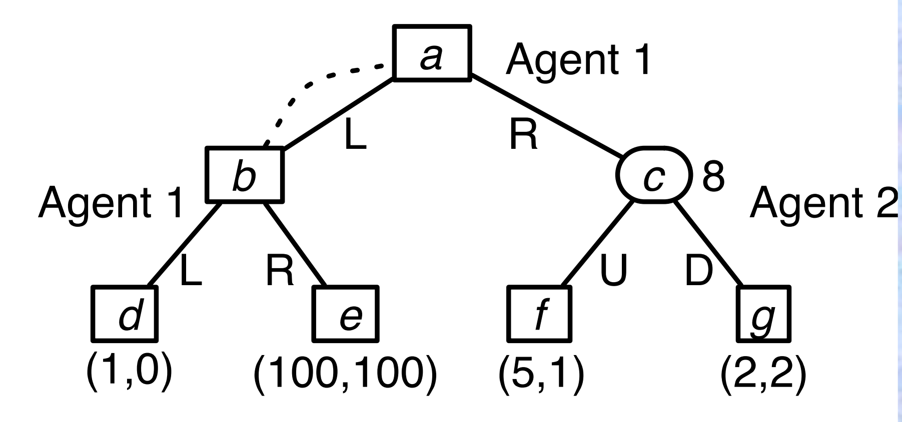


Behavioral vs. Mixed Strategies

- In some games, there are
  - mixed strategies that have no equivalent behavioral strategy
  - behavioral strategies that have no equivalent mixed strategy
- Thus mixed and behavioral strategies can produce different sets of equilibria
- Consider the game shown here:
  - At both *a* and *b*, agent 1's information set is {*a*, *b*}
  - How can this ever happen?



Behavioral vs. Mixed Strategies



- Mixed strategy {(p, L), ($1–p$, R)}: agent 1 chooses L or R randomly, but commits to it
  - Choose L → the game will end at *d*
  - Choose R → the game will end at *f* or *g*
  - The game will **never** end at node *e*
- Behavioral strategy {(q, L), ($1–q$, R)}: every time agent 1 is in {*a*, *b*}, agent 1 re-makes the choice
  - Pr[game ends at *e*] = $q(1–q)$
  - Pr[game ends at *e*] > 0, except when $q = 0$ or $q = 1$
- Only two cases in which there are equivalent mixed and behavioral strategies
  - If $p = q = 0$, then both strategies are the pure strategy L
  - If $p = q = 1$, then both strategies are the pure strategy R
- In all other cases, the mixed and behavioral strategies produce different probability distributions over the outcomes