CMSC 474, Introduction to Game Theory

16. Behavioral vs. Mixed Strategies

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Behavioral Strategies

- In imperfect-information extensive-form games, we can define a new class of strategies called **behavioral strategies**
  - An agent’s (probabilistic) choice at each node is independent of his/her choices at other nodes
- Consider the imperfect-info game shown here:
- A behavioral strategy for Agent 1:
  - At node $a$, $\{(0.5, A), (0.5, B)\}$
  - At node $g$, $\{(0.3, G), (0.7, H)\}$
- Is there an equivalent mixed strategy?
  - What do we mean by “equivalent”??
Behavioral Strategies

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- Is there an equivalent mixed strategy?
  - What do we mean by “equivalent”?
    - Two strategies $s_i$ and $s_i'$ are equivalent if for every fixed strategy profile $s_{-i}$ of the remaining agents, $s_i$ and $s_i'$ give us the same probabilities on outcomes
- An equivalent mixed strategy:
  - $\{(0.15, (A, G)); (0.35, (A, H)); (0.15, (B, G)); (0.35, (B, H))\}$
Behavioral vs. Mixed Strategies

- Consider the following mixed strategy:
  - \(\{(0.6, (A, G)), (0.4, (B, H))\}\)

- The choices at the two nodes aren’t independent
  - Choose A at \(a\) \(\Leftrightarrow\) choose G at \(g\)
  - Choose B at \(a\) \(\Leftrightarrow\) choose H at \(g\)

- Is there an equivalent behavioral strategy?
Behavioral vs. Mixed Strategies

- Consider the following mixed strategy:
  - \( \{(0.6, (A, G)), (0.4, (B, H))\} \)

- The choices at the two nodes aren’t independent
  - Choose A at \( a \) if choose G at \( g \)
  - Choose B at \( a \) if choose H at \( g \)

- Thus not always easy to find an equivalent behavioral strategy.
Behavioral vs. Mixed Strategies

- In some games, there are
  - mixed strategies that have no equivalent behavioral strategy
  - behavioral strategies that have no equivalent mixed strategy

- Thus mixed and behavioral strategies can produce different sets of equilibria

- Consider the game shown here:
  - At both $a$ and $b$, agent 1’s information set is $\{a, b\}$
  - How can this ever happen?
Behavioral vs. Mixed Strategies

- Mixed strategy \{ (p, L), (1-p, R) \}: agent 1 chooses L or R randomly, but commits to it
  - Choose L → the game will end at d
  - Choose R → the game will end at f or g
  - The game will never end at node e
- Behavioral strategy \{ (q, L), (1-q, R) \}: every time agent 1 is in \{ a, b \}, agent 1 re-makes the choice
  - \Pr[\text{game ends at } e] = q(1-q)
  - \Pr[\text{game ends at } e] > 0, except when \( q = 0 \) or \( q = 1 \)
- Only two cases in which there are equivalent mixed and behavioral strategies
  - If \( p = q = 0 \), then both strategies are the pure strategy L
  - If \( p = q = 1 \), then both strategies are the pure strategy R
- In all other cases, the mixed and behavioral strategies produce different probability distributions over the outcomes
Nash Equilibrium in Mixed Strategies

Nash equilibrium in mixed strategies:

- If agent 1 uses a mixed strategy, the game will never end at node $e$.
- Thus:
  - For agent 1, $R$ is strictly dominant.
  - For agent 2, $D$ is strictly dominant.
- So $(R,D)$ is the unique Nash equilibrium.
Nash Equilibrium in Behavioral Strategies

Nash equilibrium in behavioral strategies:

- For Agent 2, D is strictly dominant
- Find 1’s best response among behavioral strategies
  - Suppose 1 uses the behavioral strategy \{(q, L), (1 - q, R)\}
  - Then agent 1’s expected payoff is
    - \[ u_1 = q^2 + 100q(1 - q) + 2(1 - q) = -99q^2 + 98q + 2 \]
    - To find the maximum value of \( u_1 \), set \( \frac{du_1}{dq} = 0 \)
      - \(-198q + 98 + 0 = 0\), so \( q = 49/99 \)
- So (R,D) is not an equilibrium
  - The equilibrium is \(\{(49/99, L), (50/99, R)\}, D\)
Why This Happened

- The reason the strategies weren’t equivalent was because agent 1 could be in the same information set more than once
  - With a mixed-strategy, 1 made the same move both times
  - With a behavioral strategy, 1 could make a different move each time
- There are games in which this can never happen
  - Games of **perfect recall**
Games of Perfect Recall

- In an imperfect-information game $G$, agent $i$ has **perfect recall** if $i$ never forgets anything he/she knew earlier
  - In particular, $i$ remembers all his/her own moves
- $G$ is a **game of perfect recall** if every agent in $G$ has perfect recall

**Theorem:** For every history in a game of perfect recall, no agent can be in the same information set more than once

**Proof:** Let $h$ be any history for $G$. Suppose that
- At one point in $h$, $i$’s information set is $I$
- At another point later in $h$, $i$’s information set is $J$
- Then $i$ must have made at least one move in between
- If $i$ remembers all his/her moves, then
  - At $J$, $i$ remembers a longer sequence of moves than at $I$
  - Thus $I$ and $J$ are different information sets
Games of Perfect Recall

- **Theorem** (Kuhn, 1953). In a game of perfect recall, for every mixed strategy $s_i$ there is an equivalent behavioral strategy $s'_i$, and vice versa.

- In a game of perfect recall, the set of Nash equilibria doesn’t change if we consider behavioral strategies instead of mixed strategies.
Sequential Equilibrium

- For perfect-information games, subgame-perfect equilibria were useful
  - Avoided non-credible threats; could be computed more easily
- Is there something similar for imperfect-info games?

- In a subgame-perfect equilibrium, each agent’s strategy must be a best response in every subgame
  - We can’t use that definition in imperfect-information games
  - No longer have a well-defined notion of a subgame
  - Rather, at each info set, a “subforest” or a collection of subgames

- Could we require each player’s strategy to be a best response in each of the subgames in the forest?
  - Won’t work correctly …
Example

- 2’s information set is \{c, d\}
- No strategy is a best response at both c and d
- But if 1 is rational, then 1 will never choose C
- So if rationality is common knowledge
  - Then 2 only needs a best response at node d
Example

- 2’s information set is \{c,d\}
- No strategy is a best response at both c and d
- If 1 is rational, then 1 will never choose \text{L}
- Let 1’s mixed strategy be \{(p, C), (1-p, R)\}, and 2’s mixed strategy be \{(q, U), (1-q, D)\}
- Can show there is one Nash equilibrium, at \(p = q = \frac{1}{2}\)
  - But \(q = \frac{1}{2}\) is not a best response at either c or d
Sequential Equilibrium

- This leads to a complicated solution concept called **sequential equilibrium**
  - A little like a trembling-hand perfect equilibrium (which was already complicated), but with additional complications to deal with the tree structure

**Definition 5.3.1 (Sequential equilibrium).** A strategy profile \( S \) is a sequential equilibrium of an extensive-form game \( G \) if there exist probability distributions \( \mu(h) \) for each information set \( h \) in \( G \), such that the following two conditions hold:

1. \( (S, \mu) = \lim_{n \to \infty} (S^n, \mu^n) \) for some sequence \((S^1, \mu^1), (S^2, \mu^2), \ldots\), where \( S^n \) is fully mixed, and \( \mu^n \) is consistent with \( S^n \) (in fact, since \( S^n \) is fully mixed, \( \mu^n \) is uniquely determined by \( S^n \)); and

2. For any information set \( h \) belonging to agent \( i \), and any alternative strategy \( S'_i \) of \( i \), we have \( S | h, \mu(h)) \geq u_i((S', S_{-i}) | h, \mu(h)) \).

- Every finite game of perfect recall has a sequential equilibrium
- Every subgame-perfect equilibrium is a sequential equilibrium, but not vice versa
- We won’t discuss it further
Summary

- Topics covered:
  - information sets
  - behavioral vs. mixed strategies
  - games of perfect recall
    - equivalence between behavioral and mixed strategies in such games
    - Sequential equilibrium instead of subgame-perfect equilibrium