CMSC 474, Introduction to Game Theory

20. Shapley Values

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Shapley Values

- Recall that a pre-imputation is a payoff division that is both feasible and efficient

- **Theorem.** Given a coalitional game (N,v), there's a unique pre-imputation $\varphi(N,v)$ that satisfies the Symmetry, Dummy player, and Additivity axioms. For each player i, i's share of $\varphi(N,v)$ is

$$\varphi_i(N,v) = \frac{1}{|N|!} \sum_{S \subseteq N - \{i\}} |S|!\ (|N| - |S| - 1)!\ (v(S \cup \{i\}) - v(S))$$

- $\varphi_i(N,v)$ is called i's **Shapley value**
  - Lloyd Shapley introduced it in 1953

- It captures agent i's **average marginal contribution**
  - The average contribution that i makes to the coalition, averaged over every possible sequence in which the grand coalition can be built up from the empty coalition

Shapley Values

- Suppose agents join the grand coalition one by one, all sequences equally likely
- Let S = {agents that joined before i} and T = {agents that joined after i}
  - i's marginal contribution is $v(S \cup \{i\}) - v(S)$
    - independent of how S is ordered, independent of how T is ordered
  - Pr[S, then i, then T]

    = (# of sequences that include S then i then T) / (total # of sequences)

    = $|S|!\,|T|!\,/\,|N|!$
- Let $\varphi_{i,S}$ = Pr[S, then i, then T] × i's marginal contribution when it joins
- Then $\varphi_{i,S} = \dfrac{|S|!\,(|N|-|S|-1)!}{|N|!}\left(v(S \cup \{i\}) - v(S)\right)$
- Let $\varphi_i(N,v)$ = expected contribution over all possible sequences
- Then $\varphi_i(N,v) = \sum_{S \subseteq N-\{i\}} \varphi_{i,S} = \dfrac{1}{|N|!} \sum_{S \subseteq N-\{i\}} |S|!\,(|N|-|S|-1)!\,\left(v(S \cup \{i\}) - v(S)\right)$

Example

- The voting game again
  - Parties *A*, *B*, *C*, and *D* have 45, 25, 15, and 15 representatives
  - A simple majority (51 votes) is required to pass the $100M bill
- How much money is it fair for each party to demand?
  - Calculate the Shapley values of the game
- Every coalition with ≥ 51 members has value 1; other coalitions have value 0

- Recall what it means for two agents *i* and *j* to be interchangeable:
  - for every *S* that contains neither *i* nor *j* , $v(S \cup \{i\}) = v(S \cup \{j\})$
- *B* and *C* are interchangeable
  - Each adds 0 to $\varnothing$, 1 to $\{A\}$, 0 to $\{D\}$, and 0 to $\{A,D\}$
- Similarly, *B* and *D* are interchangeable, and so are *C* and *D*
- So the fairness axiom says that *B*, *C*, and *D* should each get the same amount

- Recall that

$$\varphi_{i,S} = \frac{|S|!\,(|N| - |S| - 1)!\,(v(S \cup \{i\}) - v(S))}{|N|!}$$

$$\varphi_i(N, v) = \sum_{S \subseteq N - \{i\}} \varphi_{i,S} = \frac{1}{|N|!} \sum_{S \subseteq N - \{i\}} |S|!\,(|N| - |S| - 1)!\,(v(S \cup \{i\}) - v(S))$$

- In the example, it will be useful to let $\varphi'_{i,S}$ be the term inside the summation
  - Hence $\varphi'_{i,S} = |N|!\ \varphi_{i,S}$

- Let's compute $\varphi_A(N, v)$
- $N = |\{A,B,C,D\}| = 4$, so $\varphi'_{A,S} = |S|!\,(3 - |S|)!\,(v(S \cup A) - v(S))$
- S may be any of the following:
  - $\varnothing$, {B}, {C}, {D}, {B,C}, {B,D}, {C,D}
- We need to sum over all of them:

$$\varphi_A(N, v) = \frac{1}{4!}\left(\varphi'_{A,\varnothing} + \varphi'_{A,\{B\}} + \varphi'_{A,\{C\}} + \varphi'_{A,\{D\}} + \varphi'_{A,\{B,C\}} + \varphi'_{A,\{B,D\}} + \varphi'_{A,\{C,D\}} + \varphi'_{A,\{B,C,D\}}\right)$$

$$\varphi'_{A,S} = |S|!\,(3-|S|)!\,(v(S \cup A) - v(S))$$

A has 45 members
B has 25 members
C has 15 members
D has 15 members

$S = \emptyset$	→	$v(\{A\}) - v(\emptyset) = 0 - 0 = 0$	→	$\varphi'_{A,\emptyset} = 0!\ 3!\ 0 = 0$
$S = \{B\}$	→	$v(\{A,B\}) - v(\{B\}) = 1 - 0 = 1$	→	$\varphi'_{A,\{B\}} = 1!\ 2!\ 1 = 2$
$S = \{C\}$	→	same		
$S = \{D\}$	→	same		
$S = \{B,C\}$	→	$v(\{A,B,C\}) - v(\{B,C\}) = 1 - 0 = 1$	→	$\varphi'_{A,\{B,C\}} = 2!\ 1!\ 1 = 2$
$S = \{B,D\}$	→	same		
$S = \{C,D\}$	→	same		
$S = \{B,C,D\}$	→	$v(\{A,B,C,D\}) - v(\{B,C,D\}) = 1 - 1 = 0$	→	$\varphi'_{A,\{B,C,D\}} = 3!\ 0!\ 0 = 0$

$$\varphi_A(N,v) = \frac{1}{4!}\left(\varphi'_{A,\emptyset} + \varphi'_{A,\{B\}} + \varphi'_{A,\{C\}} + \varphi'_{A,\{D\}} + \varphi'_{A,\{B,C\}} + \varphi'_{A,\{B,D\}} + \varphi'_{A,\{C,D\}} + \varphi'_{A,\{B,C,D\}}\right)$$

$$= \frac{1}{24}(0+2+2+2+2+2+2+0) = 12/24 = 1/2$$

$$\varphi_i(N,v) = \frac{1}{|N|!} \sum_{S \subseteq N - \{i\}} |S|!\ (|N| - |S| - 1)!\ (v(S \cup \{i\}) - v(S))$$

- Similarly, $\varphi_B = \varphi_C = \varphi_D = 1/6$
  - The text calculates it using Shapley's formula
- Here's another way to get it:
  - If *A* gets ½, then the other ½ will be divided among *B*, *C*, and *D*
  - They are interchangeable, so a fair division will give them equal amounts: 1/6 each

- So distribute the money as follows:
  - *A* gets (1/2) \$100M = \$50M
  - *B*, *C*, *D* each get (1/6) \$100M = $\$16\tfrac{2}{3}$M

Stability of the Grand Coalition

- Agents have incentive to form the grand coalition iff there aren't any smaller coalitions in which they could get higher payoffs
- Sometimes a subset of the agents may prefer a smaller coalition
- Recall the Shapley values for our voting example:
    - *A* gets \$50M; *B, C, D* each get \$ $16\frac{2}{3}$M
  - *A* on its own can't do better
  - But {*A, B*} have incentive to defect and divide the \$100M
    - e.g., \$75M for *A* and \$25M for *B*

- What payment divisions would make the agents want to join the grand coalition?

The Core

- The **core** of a coalitional game includes every payoff vector **x** that gives every sub-coalition S at least as much in the grand coalition as S could get by itself
  - All feasible payoff vectors **x** = ($x_1, \ldots, x_n$) such that for every $S \subseteq N$,

$$\sum_{i \in S} x_i \geq v(S)$$

- For every payoff vector **x** in the core, no S has any incentive to **deviate** from the grand coalition
  - i.e., form their own coalition, excluding the others

- It follows immediately that if **x** is in the core then **x** is efficient
  - Why?

Analogy to Nash Equilibria

- The core is an analog of the set of all Nash equilibria in a noncooperative game
  - There, no agent can do better by deviating from the equilibrium
- But the core is stricter
  - No set of agents can do better by deviating from the grand coalition
- Analogous to the set of **strong** Nash equilibria
  - Equilibria in which no coalition of agents can do better by deviating
- Unlike the set of Nash equilibria, the core may sometimes be empty
  - In some cases, no matter what the payoff vector is, some agent or group of agents has incentive to deviate

Example of an Empty Core

- Consider the voting example again:
  - Shapley values are \$50M to *A*, and \$16.33M each to *B*, *C*, *D*
- The minimal coalitions that achieve 51 votes are
  - {A,B}, {A,C}, {A,D}, {B,C,D}
- If the sum of the payoffs to B, C, and D is < \$100M, this set of agents has incentive to deviate from the grand coalition
  - Thus if **x** is in the core, **x** must allocate \$100M to {B, C, D}
  - But if B, C, and D get the entire \$100M, then A (getting \$0) has incentive to join with whichever of B, C, and D got the least
    - e.g., form a coalition {A,B} without the others
  - So if **x** allocates the entire \$100M to {B,C,D} then **x** cannot be in the core
- So the core is empty

Simple Games

- There are several situations in which the core is either guaranteed to exist, or guaranteed not to exist
  - The first one involves simple games
- Recall: G is **simple** for every coalition S, either $v(S) = 1$ or $v(S) = 0$
- Player i is a **veto player** if $v(N - \{i\}) = 0$
- **Theorem**. In a simple game, the core is empty iff there is no veto player
- Example: previous slide

Simple Games

- **Theorem**. In a simple game in which there are veto players, the core is {all payoff vectors in which non-veto players get 0}

- **Example**: consider a modified version of the voting game
    - An 80% majority is required to pass the bill
- Recall that *A*, *B*, *C*, and *D* have 45, 25, 15, and 15 representatives
    - The minimal winning coalitions are {A, B, C} and {A, B, D}
    - All winning coalitions must include both A and B
    - So A and B are veto players
        - The core includes all distributions of the $100M among A and B
        - Neither A nor B can do better by deviating

Non-Additive Constant-Sum Games

- Recall:
  - G is constant-sum if for all S, $v(S) + v(N - S) = v(N)$
  - G is additive if $v(S \cup T) = v(S) + v(T)$ whenever S and T are disjoint
- **Theorem**. Every non-additive constant-sum game has an empty core
- **Example:** consider a constant-sum game G with 3 players a, b, c
  - Suppose v(a) = 1, v(b) = 1, v(c) = 1, v({a,b,c})=4
  - Then v(a) + v({b,c}) = v({a,b})+v(c) = v({a,c}) + v(b) = 4
  - Thus v({b,c}) = 4 – 1 = 3 ≠ v(b) + v(c)
  - So G is not additive
- Consider **x** = (1.333, 1.333, 1.333)
  - v({a,b}) = 3, so if {a,b} deviate, they can allocate (1.5,1.5)
- To keep {a,b} from deviating, suppose we use **x** = (1.5, 1.5, 1)
  - v({a,c}) = 3, so if {a,c} deviate, they can allocate (1.667, 1.333)

Convex Games

- Recall:
  - G is **convex** if for all $S,T \subseteq N$, $v(S \cup T) \geq v(S) + v(T) - v(S \cap T)$
- **Theorem**. Every convex game has a nonempty core
- **Theorem**. In every convex game, the Shapley value is in the core

Modified Parliament Example

- 100 representatives from four political parties:
  - A (45 reps.), B (25 reps.), C (15 reps.), D (15 reps.)
- Any coalition of parties can approve a spending bill worth \$1K times the number of representatives in the coalition:

$$v(S) = \sum_{i \in S} \$1000 \times \text{size}(i)$$

$v(A) = \$45K$, $v(B) = \$25K$, $v(C) = \$15K$, $v(D) = \$15K$,

$v(\{A,B\}) = \$70K$, $v(\{A,C\}) = \$60K$, $v(\{A,D\}) = \$60K$,

$v(\{B,C\}) = \$40K$, $v(\{B,D\}) = \$40K$, $v(\{C,D\}) = \$30K$,

$v(\{A,B,C\}) = \$100K$

- Is the game convex?

Modified Parliament Example

- Let *S* be the grand coalition
  - What is each party's Shapley value in *S*?

- Each party's Shapley value is the average value it adds to *S*, averaged over all 24 of the possible sequences in which *S* might be formed:

A, B, C, D; *A, B, D, C;* *A, C, B, D;* *A, C, D, B;* *etc*

- In every sequence, every party adds exactly \$1K times its size
- Thus every party's Shapley value is \$1K times its size:
  - φ_A = \$45K, φ_B = \$25K, φ_C = \$15K, φ_D = \$15K

Modified Parliament Example

- Suppose we distribute $v(S)$ by giving each party its Shapley value
- Does any party or group of parties have an incentive to leave and form a smaller coalition T?
  - $v(T)$ = \$1K times the number of representatives in T
    = the sum of the Shapley values of the parties in T
  - If each party in T gets its Shapley value, it does no better in T than in S
  - If some party in T gets more than its Shapley value, then another party in T will get less than its Shapley value
- No case in which every party in T does better in T than in S
- No case in which all of the parties in T will have an incentive to leave S and join T

- Thus the Shapley value is in the core