# Network Bargaining Games and Cooperative Game Theory

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# Bargaining

# Bargaining

- Common wisdom has it that the whole is more than the sum of the parts.

- Two cooperative agents are often capable of generating a surplus that neither could achieve alone.
  - Trade creates value
  - Music studio, Music band - sell an album
  - Publishing house, author - print and sell a book
  - Job position
  - Partnership formation

# Example

- Bargaining over a division of a cake

- Take-it-or-leave-it rule
  - I offer you a piece.
  - If you accept, we trade.
  - If you reject, no one eats.

- What is the equilibrium?
  - Power to the proposer.

# Example

- Bargaining over a division of a cake

- Take-it-or-counteroffer rule
  - I offer you a piece.
  - If you accept, we trade.
  - If you reject, you may counteroffer (and $\delta$ of the cake remains, the rest melt)
- What is the equilibrium?

# Bargaining

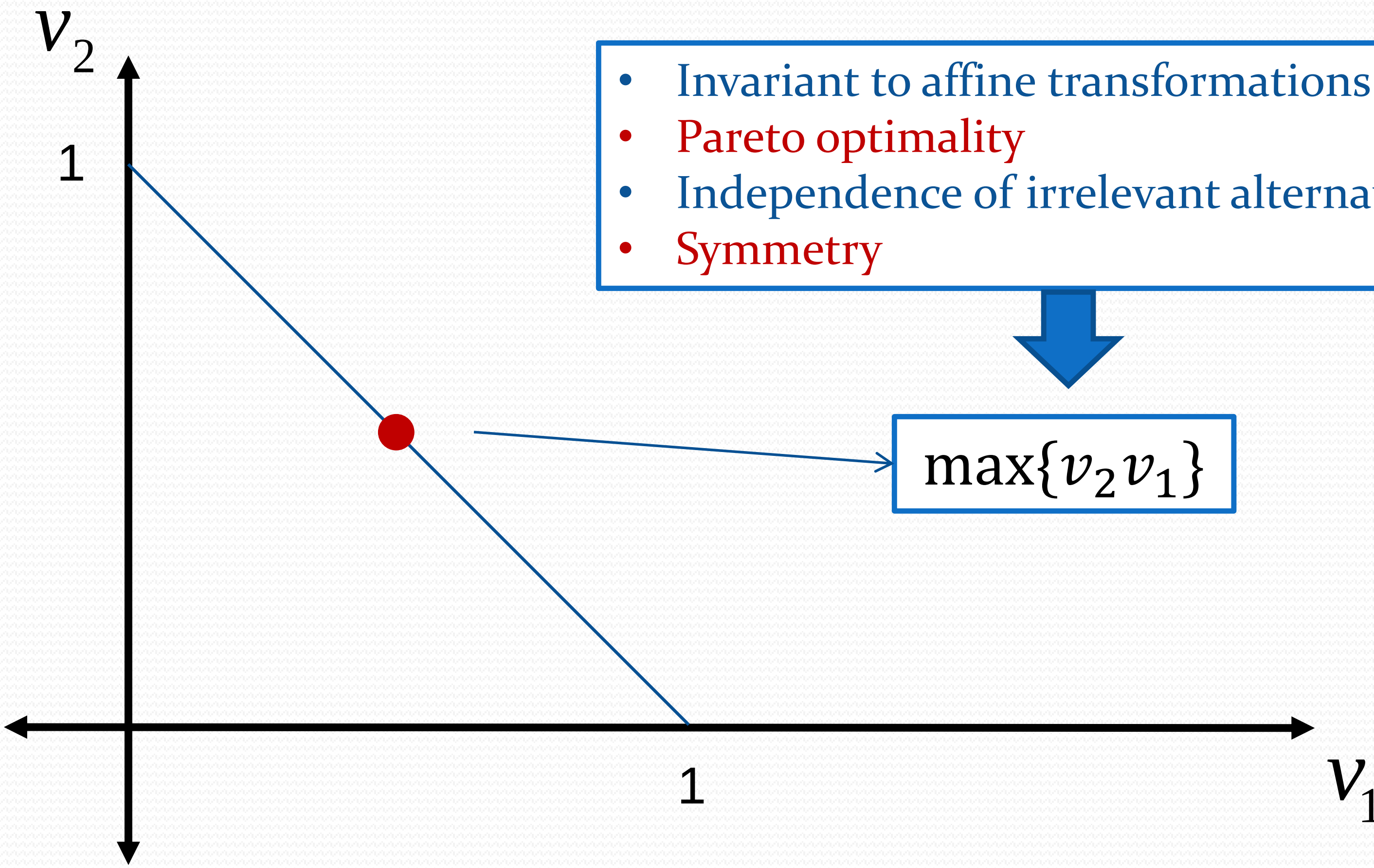
What would be the outcome?

What is the right solution?

# Nash Bargaining Solution

- Invariant to affine transformations
- Pareto optimality
- Independence of irrelevant alternatives
- Symmetry

$$\max\{v_2 v_1\}$$



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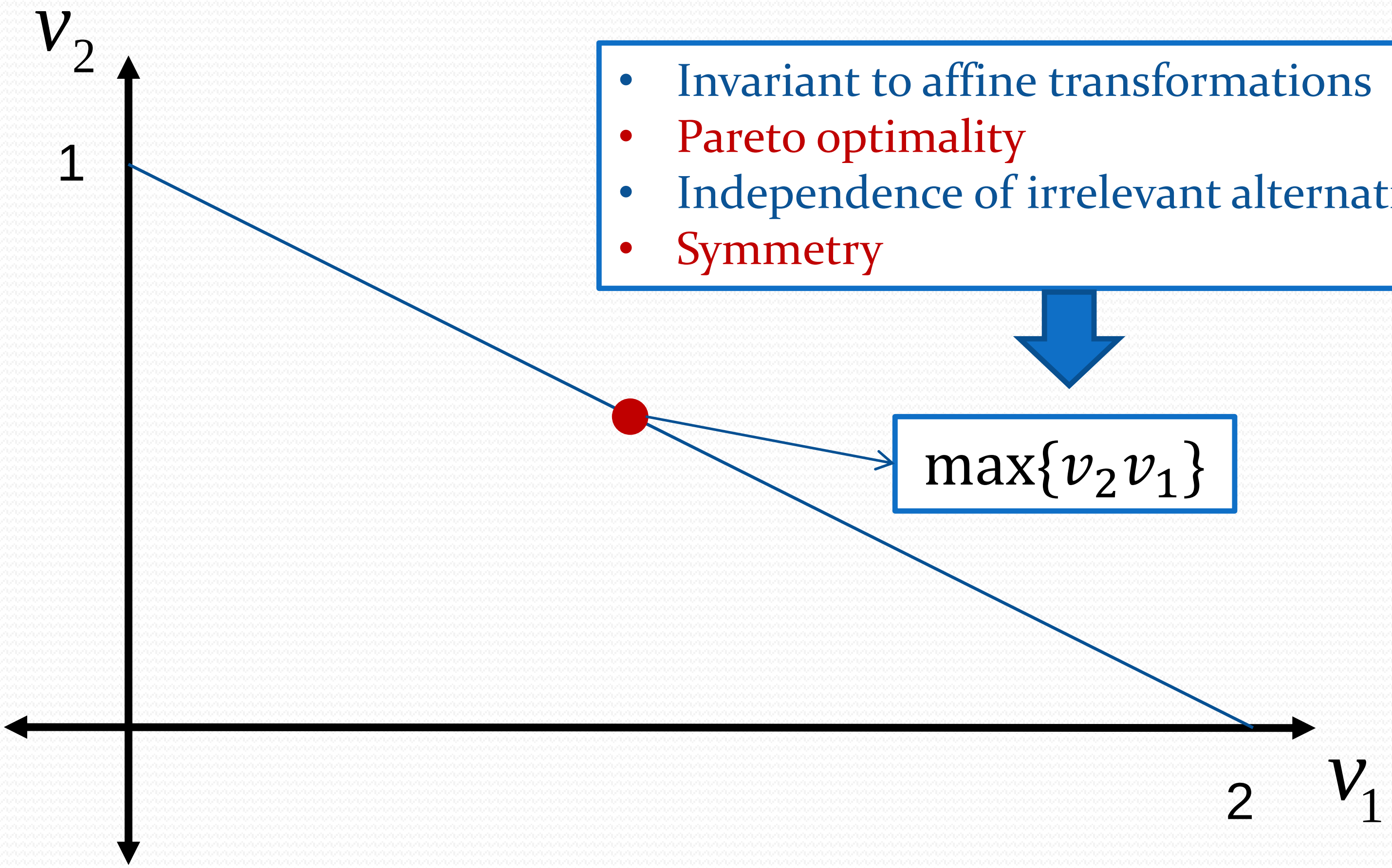
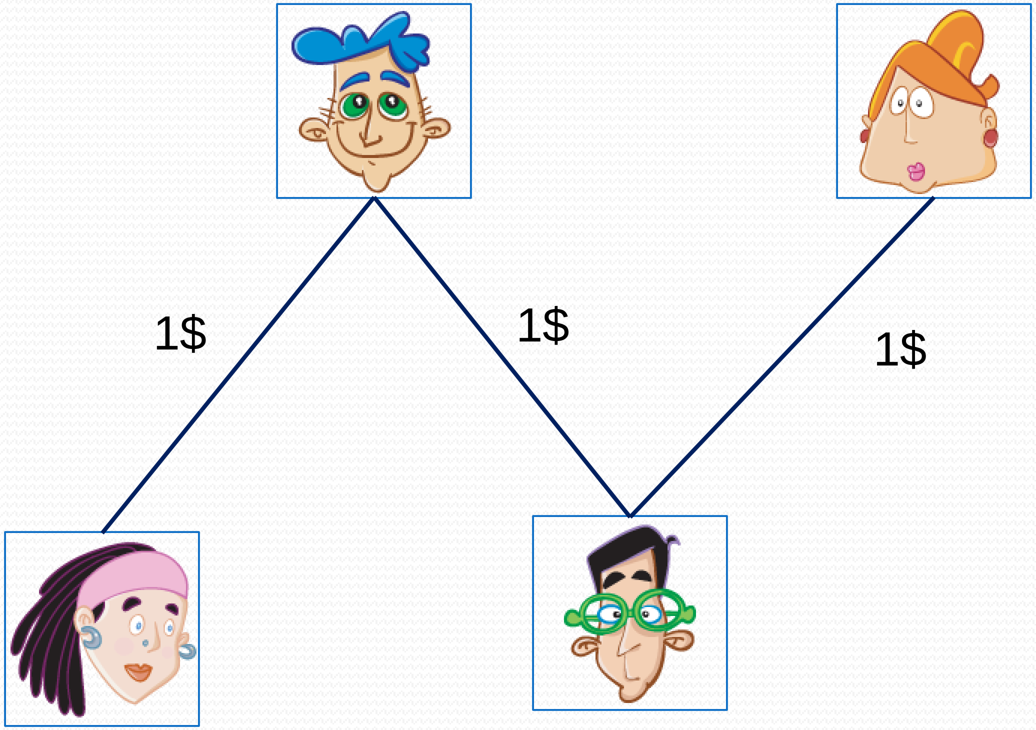
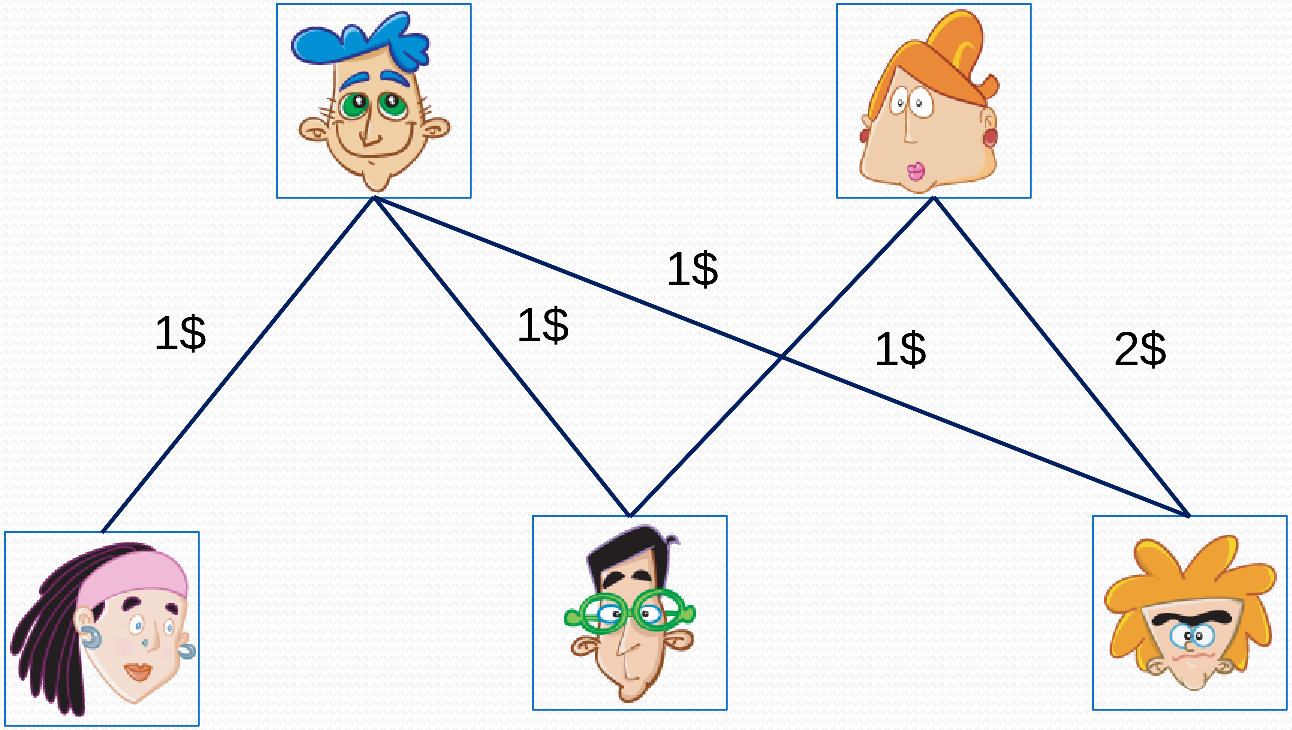


# Bargaining Game



# Bargaining Game



# Bargaining Game

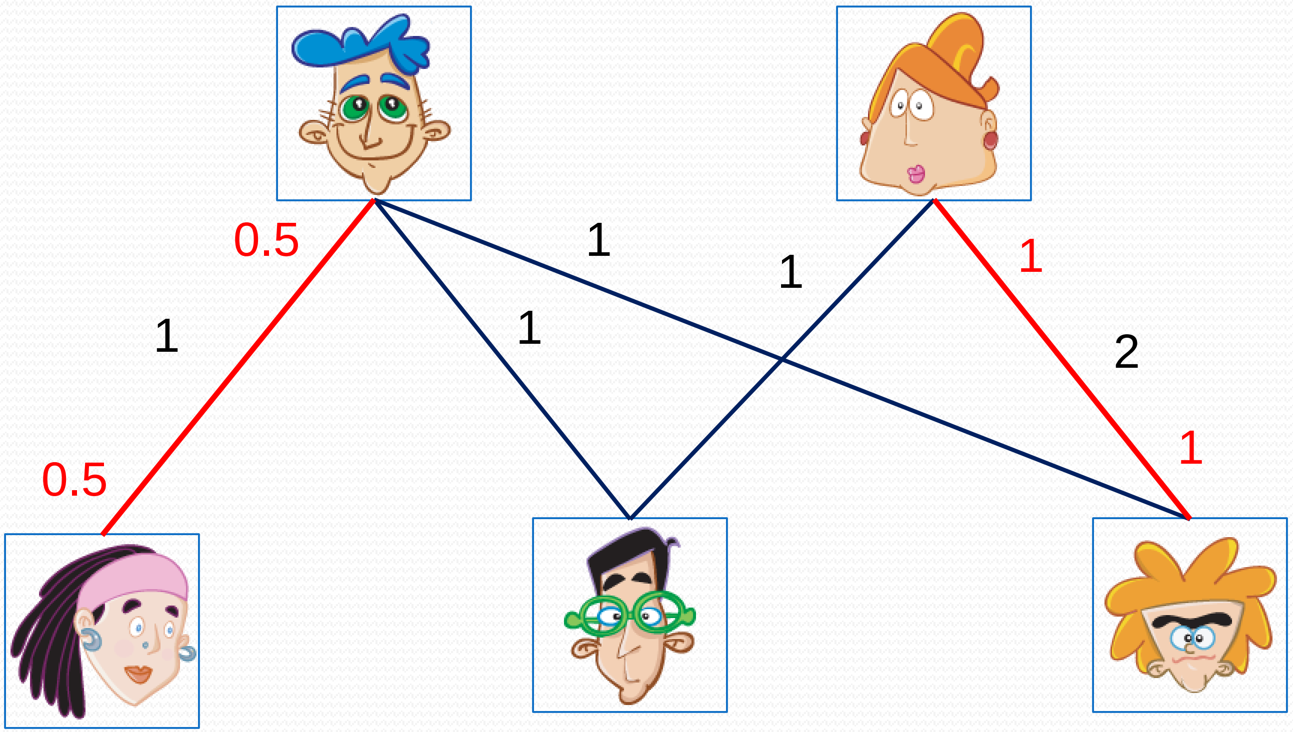
- They are $n$ agents in the market.
- Each agent may participate in at most one contract.
- For each pair of agents $i$ *and* $j$ we are given weight $w_{i,j}$
  - Representing the surplus of a contract between $i$ and $j$

Our main task is to predict the outcome of a network bargaining game.

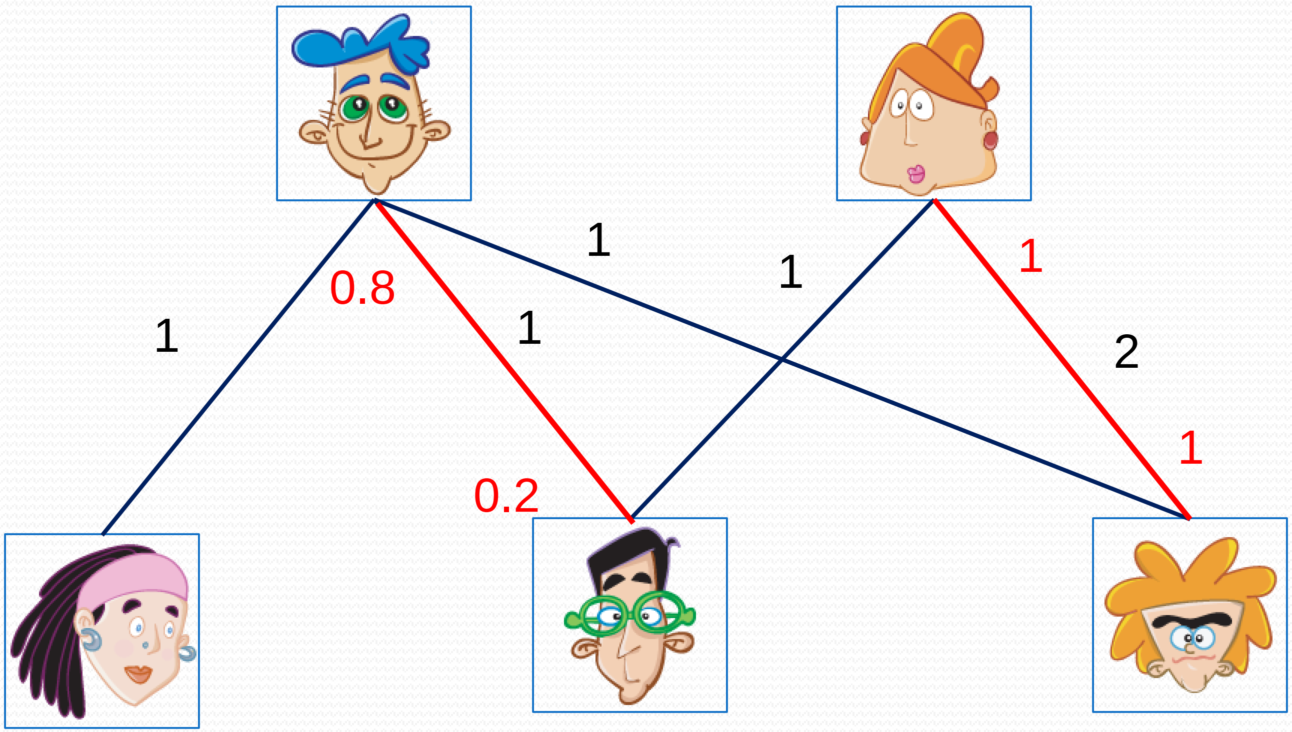
# Bargaining Solution

- We call a set of contracts $M$ feasible if:
  - Each agent $i$ is in at most $c_i$ contracts.
- A *solution* $(\{z_{i,j}\}, M)$ *of a bargaining game is:*
  - A set of feasible contracts $M$.
  - For each $(i, j)$ in $M$: $z_{i,j} + z_{j,i} = w_{i,j}$
  - $z_{i,j}$ is the amount of money $i$ earns from the contract with $j$
  - $x_i$ is the aggregate earning of agent $i$.
  - $\{x_i\}$ is the outcome of the game.

# Bargaining Solution - Example



# Bargaining Solution - Example



# Bargaining Solution

- The set of solution is quite large.
- Define a subset of solution as a result of the bargaining process.

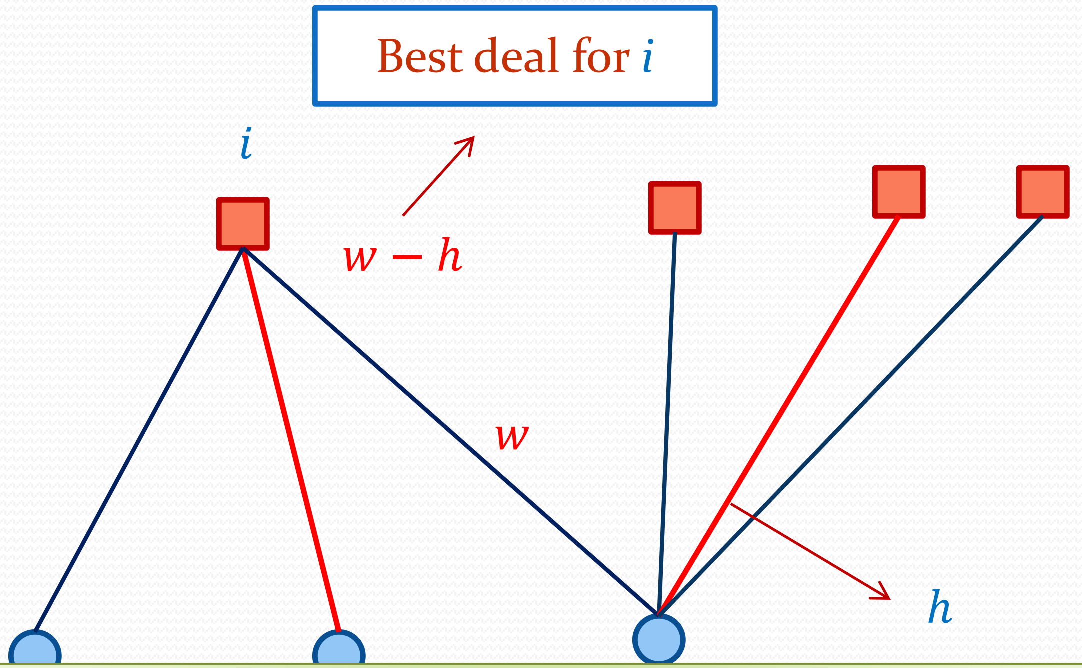
Nash bargaining solution

Cooperative game theory

# Goal

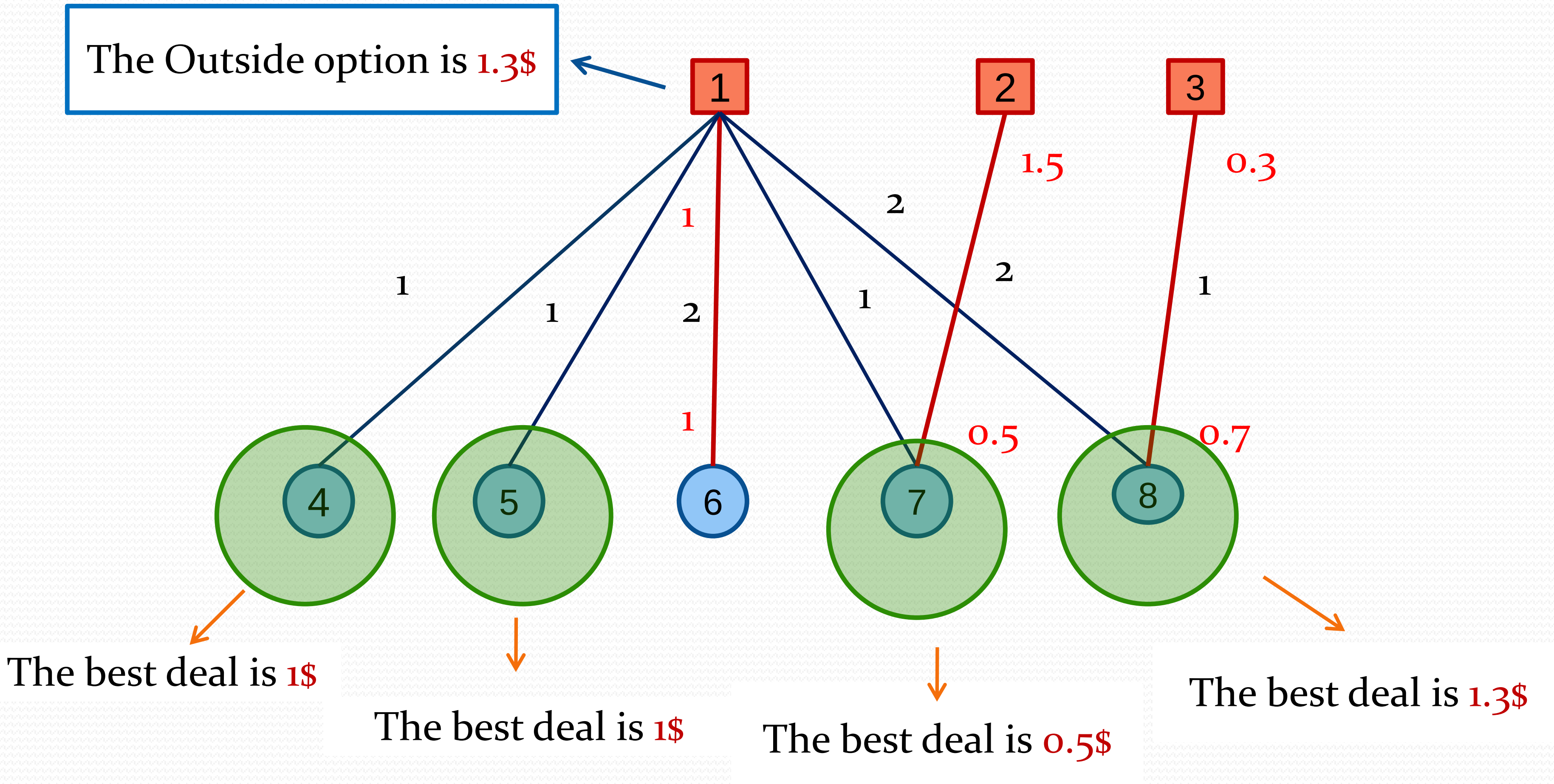
- Nash bargaining solution.
  - Stable
  - Balanced
- Cooperative game theory solutions.
  - Core
  - Kernel
- Connection between these two views.

# Outside Option




- The outside option of an agent $i$ is the best deal she could make with someone outside the contracting set $M$.

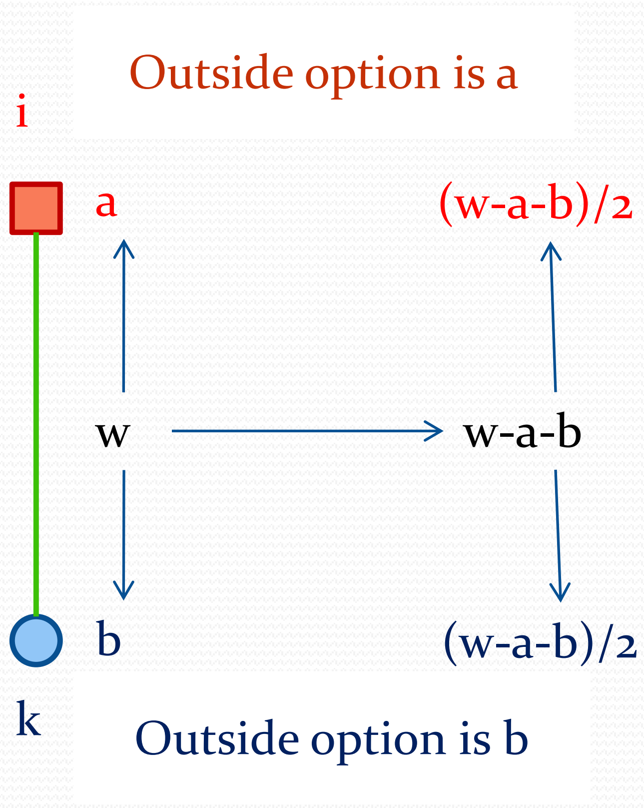
# Outside Option



# Stable and Balanced Solutions

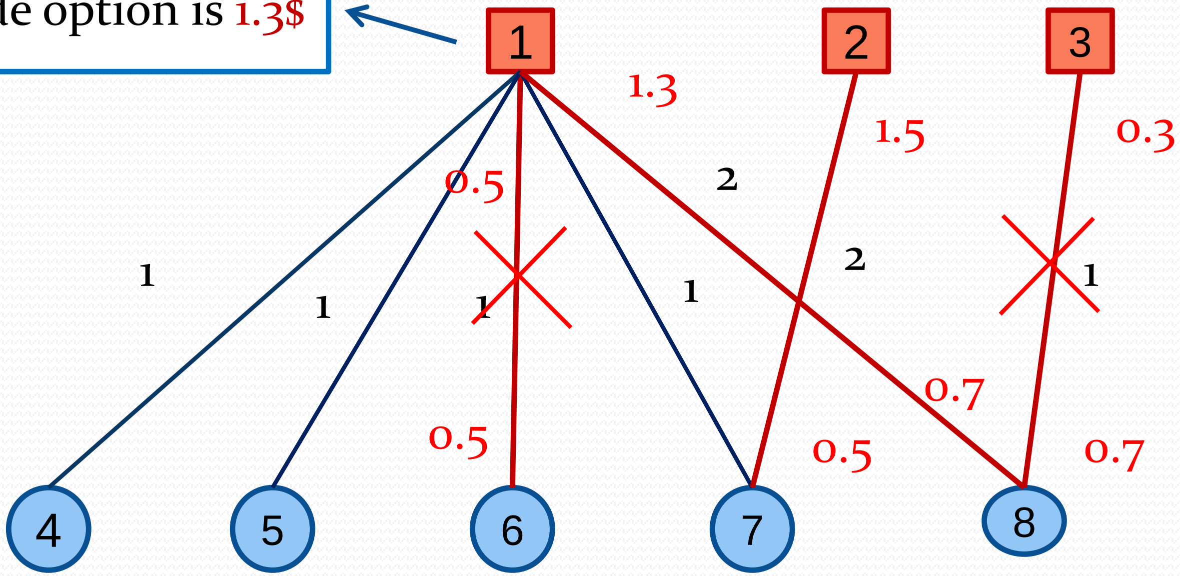
- *A* solution is stable if no agent has better outside option.
- Nash additionally argued that agents tend to split surplus equally.
- A solution is balanced if agents split the net surplus equally.
  - Each agent gets its outside option in a contract.
  - Then divide the money on the table equally.

# Balanced solution

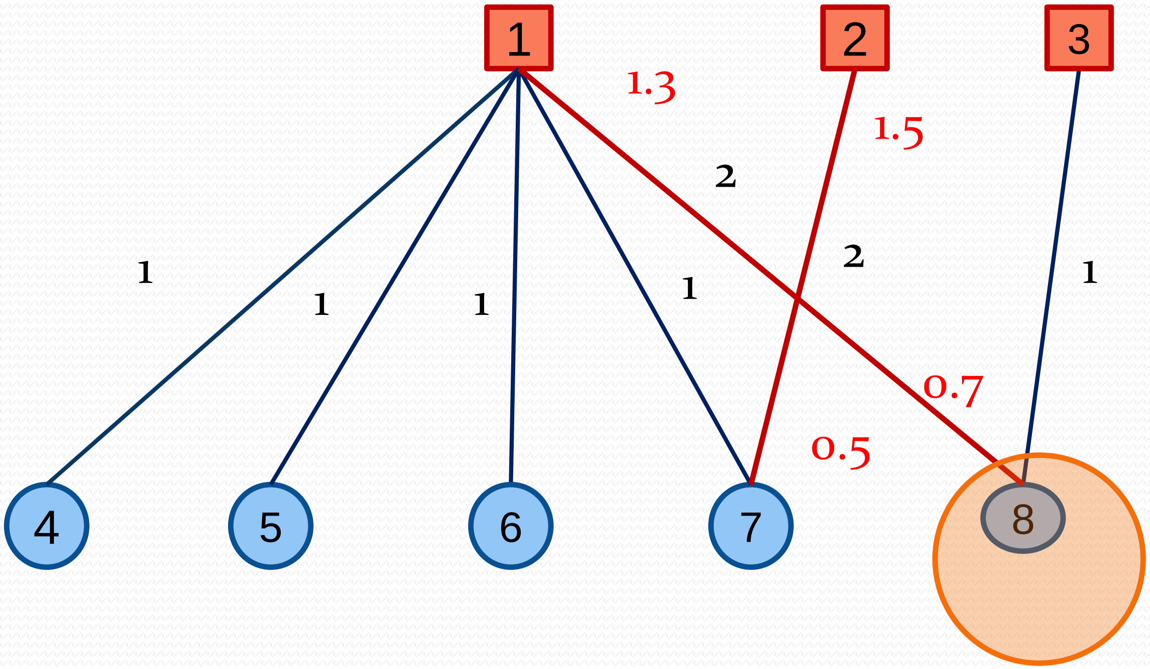


# Stable Solution

The Outside option is 1.3$

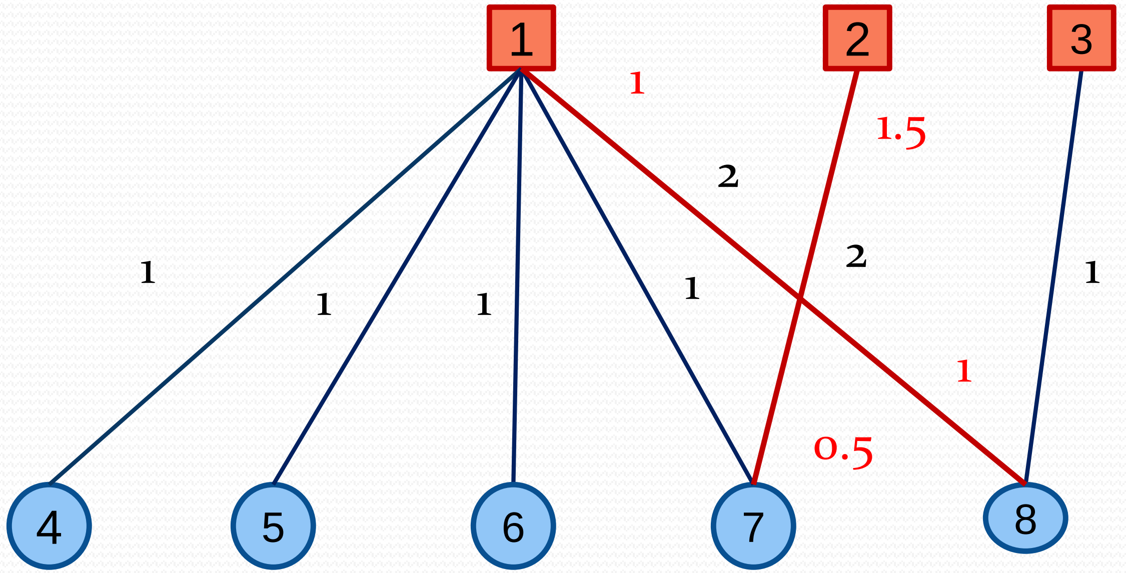


# Stable Solution




The Outside option is 1 $

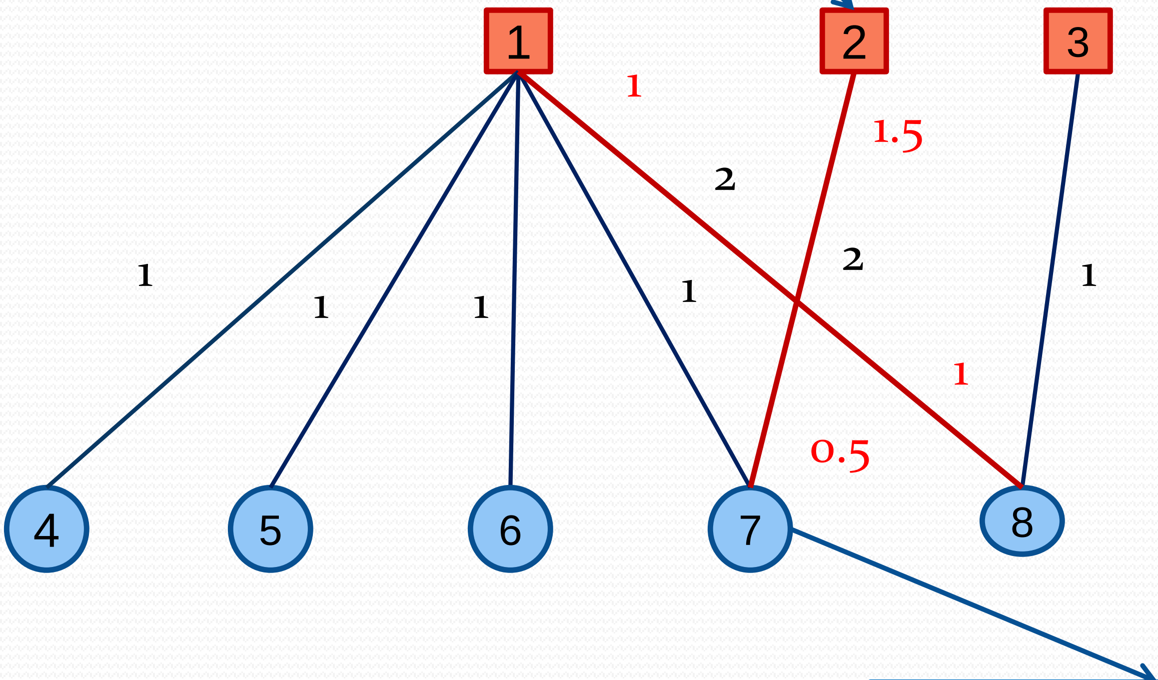
# Stable Solution




Stable Solution

# Balanced Solution

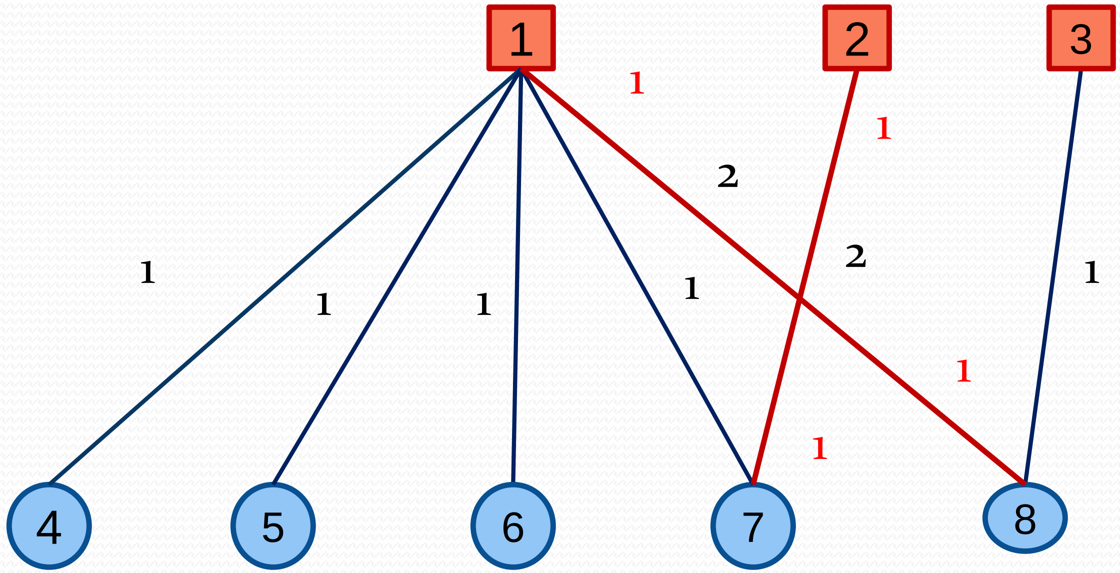
The outside option is 0$




It is not Balanced

The outside option is 0$

# Balanced Solution




Balanced Solution

# Cooperative game theory

- A cooperative game is defined by a set of agents $N$.
- A value function $v: 2^N \rightarrow R^+ \cup \{0\}$
  - The value of a set of agents represents the surplus they can achieve.
- The goal is to define an outcome of the game $\{x_i\}$

$v(S) =$ Maximum value of $\sum_{(i,j)\in M} w_{i,j}$ over all feasible contract $M$

# Core

- An outcome $\{x_i\}$ is in the core if and only if:
  - Each set of agents should earn in total at least as much as they can achieve alone: $\sum_{i \in S} x_i \geq v(S)$
  - Total surplus of all agents is exactly divided among the agents: $\sum_{i \in N} x_i = v(N)$

# Prekernel

- The power of $i$ over $j$ is the maximum amount $i$ can earn without cooperation with $j$.

$$s_{ij}(x) = \max\left\{\nu(S) - \sum_{k \in S} x_k : S \subseteq N, S \ni i, S \not\ni j\right\}$$

Prekernel: power of $i$ over $j$ = power of $j$ over $i$

# Characterizing Stable Solutions

**Primal**

**Maximize** $\sum_{ij} w_{ij} x_{ij}$

**Subject to** $\sum_j x_{ij} \leq 1, \quad \forall i$

$x_{ij} \geq 0, \forall i, j$

**Dual**

**Minimize** $\sum_i u_i$

**Subject to** $u_i + u_j \geq w_{ij}, \forall i, j$

$u_i \geq 0, \forall i$

A stable solution ≈ a pair of optimum solutions of the above linear programs

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**Stable to LP**

- given $(\{z_{ij}\}, M)$
- $x_{ij} = 1$ iff $(i, j) \in M$
- $u_i = z_{ij}$ iff $(i, j) \in M$

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**Dual**

**Minimize** $\sum_i u_i$

**Subject to** $u_i + u_j \geq w_{ij}, \forall i, j$

$u_i \geq 0, \forall i$

**LP to Stable**

- given $(\{x_{ij}\}, \{u_i\})$
- $(i, j) \in M$ iff $x_{ij} = 1$
- $z_{ij} = u_i$ for all $x_{ij} = 1$

# Core = Stable

**Stable ⊆ Core**

- We use the characterization of stable solutions
- Consider $\left(\{x_{ij}\},\{u_i\}\right)$
- Define $x_i = u_i$
- We should prove:
  - $\sum_{i\in \mathrm{N}} x_i = v(N)$
  - $\sum_{i\in R} x_i \geq v(R)$

# Core = Stable

**Core ⊆ Stable**

- Assume $(\{x_i\})$ is in the core.
- Consider an optimal set of contracts $M$
- Set $z_{ij} = x_i$ and $z_{ji} = x_j$ for all $(i, j) \in M$
  - $\sum_i x_i = v(N) =$ maximum matching
- Set $u_i = x_i$
  - $(\{u_i\})$ is a feasible solution for the dual.

# Core ∩ Kernel = Balanced

- Assume $(\{x_i\})$ is in the core ∩ kernel.
- Construct $\left(\{z_{ij}\}, M\right)$ based on the previous approach.
- Define $\hat{s}_{ij} = \alpha_i - z_{ij}$
- Prove $s_{ij} = \hat{s}_{ij}$