CMSC 474, Introduction to Game Theory

Important Normal-Form Games

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Common-payoff Games

- **Common-payoff game:**
  - For every action profile, all agents have the same payoff
  - Also called a **pure coordination** game or a **team game**
    - Need to coordinate on an action that is maximally beneficial to all

- **Which side of the road?**
  - 2 people driving toward each other in a country with no traffic rules
  - Each driver independently decides whether to stay on the left or the right
  - Need to coordinate your action with the action of the other driver

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<th>Right</th>
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<tbody>
<tr>
<td><strong>Left</strong></td>
<td>1, 1</td>
<td>0, 0</td>
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<tr>
<td><strong>Right</strong></td>
<td>0, 0</td>
<td>1, 1</td>
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A Brief Digression

- **Mechanism design**: set up the rules of the game, to give each agent an incentive to choose a desired outcome
- E.g., the law says what side of the road to drive on
  - Sweden on September 3, 1967:
Zero-sum Games

- These games are purely competitive

- **Constant-sum** game:
  - For every action profile, the sum of the payoffs is the same, i.e.,
  - there is a constant $c$ such for every action profile $a = (a_1, \ldots, a_n)$,
    - $u_1(a) + \ldots + u_n(a) = c$

- Any constant-sum game can be transformed into an equivalent game in which the sum of the payoffs is always 0
  - Positive affine transformation: subtract $c/n$ from every payoff

- Thus constant-sum games are usually called **zero-sum** games
Examples

**Matching Pennies**
- Two agents, each has a penny
- Each independently chooses to display Heads or Tails
  - If same, agent 1 gets both pennies
  - Otherwise agent 2 gets both pennies

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<th>Heads</th>
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<tr>
<td>Heads</td>
<td>1, –1</td>
<td>–1, 1</td>
</tr>
<tr>
<td>Tails</td>
<td>–1, 1</td>
<td>1, –1</td>
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**Penalty kicks in soccer**
- A kicker and a goalie
- Kicker can kick left or right
- Goalie can jump to left or right
- Kicker scores if he/she kicks to one side and goalie jumps to the other
Another Example: Rock-Paper-Scissors

- **Two players.** Each simultaneously picks an action: *Rock, Paper, or Scissors.*

- **The rewards:**
  
  - Rock beats Scissors
  - Scissors beats Paper
  - Paper beats Rock

- **The matrices:**

  $$R_1 = \begin{pmatrix}
  R & P & S \\
  R & 0 & -1 & 1 \\
  P & 1 & 0 & -1 \\
  S & -1 & 1 & 0 \\
  \end{pmatrix} \quad \quad R_2 = \begin{pmatrix}
  R & P & S \\
  R & 0 & 1 & -1 \\
  P & -1 & 0 & 1 \\
  S & 1 & -1 & 0 \\
  \end{pmatrix}$$
A game is **nonconstant-sum** (usually called **nonzero-sum**) if there are action profiles \( a \) and \( b \) such that

\[
  u_1(a) + \ldots + u_n(a) \neq u_1(b) + \ldots + u_n(b)
\]

- e.g., the Prisoner’s Dilemma

**Battle of the Sexes**

- Two agents need to coordinate their actions, but they have different preferences
- Original scenario:
  - husband prefers football, wife prefers opera
- Another scenario:
  - Two nations must act together to deal with an international crisis, and they prefer different solutions
In a symmetric game, every agent has the same actions and payoffs.

If we change which agent is which, the payoff matrix will stay the same.

For a 2x2 symmetric game, it doesn’t matter whether agent 1 is the row player or the column player.

The payoff matrix looks like this:

In the payoff matrix of a symmetric game, we only need to display $u_1$.

If you want to know another agent’s payoff, just interchange the agent with agent 1.
Strategies in Normal-Form Games

- **Pure strategy**: select a single action and play it
  - Each row or column of a payoff matrix represents both an action and a pure strategy
- **Mixed strategy**: randomize over the set of available actions according to some probability distribution
  - \( s_i(a_j) \) = probability that action \( a_j \) will be played in mixed strategy \( s_i \)
- The **support** of \( s_i = \{ \text{actions that have probability} > 0 \text{ in} \ s_i \} \)
- A pure strategy is a special case of a mixed strategy
  - support consists of a single action
- A strategy \( s_i \) is **fully mixed** if its support is \( A_i \)
  - i.e., nonzero probability for every action available to agent \( i \)
- **Strategy profile**: an \( n \)-tuple \( s = (s_1, \ldots, s_n) \) of strategies, one for each agent
Expected Utility

- A payoff matrix only gives payoffs for pure-strategy profiles
- Generalization to mixed strategies uses expected utility
  - First calculate probability of each outcome, given the strategy profile (involves all agents)
  - Then calculate average payoff for agent $i$, weighted by the probabilities
  - Given strategy profile $s = (s_1, \ldots, s_n)$
    - expected utility is the sum, over all action profiles, of the profile’s utility times its probability:

$$u_i(s) = \sum_{a \in A} u_i(a) \Pr[a | s]$$

i.e.,

$$u_i(s_1, \ldots, s_n) = \left( \sum_{(a_1, \ldots, a_n) \in A} u_i(a_1, \ldots, a_n) s_j(a_j) \right)_{j=1}^n$$
Let’s Play another Game

• Choose a number in the range from 0 to 100
  ➢ Write it on a piece of paper
  ➢ Also write your name (this is optional)
  ➢ Fold your paper in half, so nobody else can see your number
  ➢ Pass your paper to the front of the room

• The winner(s) will be whoever chose a number that’s closest to the average of all the numbers
  ➢ I’ll tell you the results later
  ➢ The winner(s) will get some prize
Summary of Past Three Sessions

- Basic concepts:
  - normal form, utilities/payoffs, pure strategies, mixed strategies
- How utilities relate to rational preferences (not in the book)
- Some classifications of games based on their payoffs
  - Zero-sum
    - Rock-paper-scissors, Matching Pennies
  - Non-zero-sum
    - Chocolate Dilemma, Prisoner’s Dilemma, Which Side of the Road?, Battle of the Sexes
  - Common-payoff
    - Which Side of the Road?
  - Symmetric
    - All of the above except Battle of the Sexes