CMSC 474, Introduction to Game Theory
Maxmin and Minmax Strategies

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Worst-Case Expected Utility

- For agent $i$, the **worst-case** expected utility of a strategy $s_i$ is the minimum over all possible combinations of strategies for the other agents:
  \[
  \min_{s_i} u_i(s_i, s_{-i})
  \]

- **Example: Battle of the Sexes**
  - Wife’s strategy $s_w = \{(p, \text{Opera}), (1-p, \text{Football})\}$
  - Husband’s strategy $s_h = \{(q, \text{Opera}), (1-q, \text{Football})\}$
  - $u_w(p,q) = 2pq + (1-p)(1-q) = 3pq - p - q + 1$
  - For any fixed $p$, $u_w(p,q)$ is linear in $q$
    - e.g., if $p = \frac{1}{2}$, then $u_w(\frac{1}{2},q) = \frac{1}{2} q + \frac{1}{2}$
  - $0 \leq q \leq 1$, so the min must be at $q = 0$ or $q = 1$
    - e.g., $\min_q (\frac{1}{2} q + \frac{1}{2})$ is at $q = 0$
  - $\min_q u_w(p,q) = \min (u_w(p,0), u_w(p,1)) = \min (1 - p, 2p)$

<table>
<thead>
<tr>
<th></th>
<th>Husband</th>
<th>Wife</th>
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<tbody>
<tr>
<td>Opera</td>
<td>2, 1</td>
<td>0, 0</td>
</tr>
<tr>
<td>Football</td>
<td>0, 0</td>
<td>1, 2</td>
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We can write $u_w(p,q)$ instead of $u_w(s_w, s_h)$
Maxmin Strategies

- **A maxmin strategy** for agent $i$
  - A strategy $s_i$ that makes $i$’s worst-case expected utility as high as possible:
    \[
    \arg\max_{s_i} \min_{s_i} u_i(s_i, s_i)
    \]
  - This isn’t necessarily unique
  - Often it is mixed

- Agent $i$’s **maxmin value**, or **security level**, is the maxmin strategy’s worst-case expected utility:
  \[
  \max_{s_i} \min_{s_i} u_i(s_i, s_i)
  \]

- For 2 players it simplifies to
  \[
  \max_{s_1} \min_{s_2} u_1(s_1, s_2)
  \]

Also called **maximin**
Example

- Wife’s and husband’s strategies
  - $s_w = \{(p, \text{Opera}), (1-p, \text{Football})\}$
  - $s_h = \{(q, \text{Opera}), (1-q, \text{Football})\}$

- Recall that wife’s worst-case expected utility is
  \[\min_q u_w(p,q) = \min (1-p, 2p)\]
  - Find $p$ that maximizes it

- Max is at $1-p = 2p$, i.e., $p = 1/3$
  - Wife’s maxmin value is $1-p = 2/3$
  - Wife’s maxmin strategy is
    \[\{(1/3, \text{Opera}), (2/3, \text{Football})\}\]

- Similarly,
  - Husband’s maxmin value is $2/3$
  - Husband’s maxmin strategy is
    \[\{(2/3, \text{Opera}), (1/3, \text{Football})\}\]
Question

- Why might an agent $i$ want to use a maxmin strategy?
Why might an agent $i$ want to use a maxmin strategy?

- Useful if $i$ is cautious (wants to maximize his/her worst-case utility) and doesn’t have any information about the other agents
  - whether they are rational
  - what their payoffs are
  - whether they draw their action choices from known distributions
- Useful if $i$ has reason to believe that the other agents’ objective is to minimize $i$’s expected utility
  - e.g., 2-player zero-sum games (we discuss this later in his session)

Solution concept: **maxmin strategy profile**
- all players use their maxmin strategies
Minmax Strategies (in 2-Player Games)

- **Minmax strategy** and **minmax value**
  - Duals of their maxmin counterparts
- Suppose agent 1 wants to punish agent 2, regardless of how it affects agent 1’s own payoff
- Agent 1’s **minmax strategy** against agent 2
  - A strategy $s_1$ that minimizes the expected utility of 2’s best response to $s_1$
    \[
    \arg\min_{s_1} \max_{s_2} u_2(s_1, s_2)
    \]
- Agent 2’s **minmax value** is 2’s maximum expected utility if agent 1 plays his/her minmax strategy:
  \[
  \min_{s_1} \max_{s_2} u_2(s_1, s_2)
  \]
- **Minmax strategy profile**: both players use their minmax strategies

Also called **minimax**
Example

- Wife’s and husband’s strategies
  - \( s_w = \{(p, \text{Opera}), (1-p, \text{Football})\} \)
  - \( s_h = \{(q, \text{Opera}), (1-q, \text{Football})\} \)

- \( u_h(p,q) = pq + 2(1-p)(1-q) = 3pq - 2p - 2q + 2 \)

- Given wife’s strategy \( p \), husband’s expected utility is linear in \( q \)
  - e.g., if \( p = \frac{1}{2} \), then \( u_h(\frac{1}{2},q) = -\frac{1}{2} q + 1 \)

- Max is at \( q = 0 \) or \( q = 1 \)
  - \( \max_q u_h(p,q) = (2-2p, p) \)

- Find \( p \) that minimizes this

- Min is at \(-2p + 2 = p \) \( \Rightarrow p = \frac{2}{3} \)

- Husband/s minmax value is \( \frac{2}{3} \)

- Wife’s minmax strategy is \( \{(\frac{2}{3}, \text{Opera}), (\frac{1}{3}, \text{Football})\}\)
Minmax Strategies in $n$-Agent Games

- In $n$-agent games ($n > 2$), agent $i$ usually can’t minimize agent $j$’s payoff by acting unilaterally.

- But suppose all the agents “gang up” on agent $j$.
  - Let $s^*_{-j}$ be a mixed-strategy profile that minimizes $j$’s maximum payoff, i.e.,
    \[
    s^*_{-j} = \arg \min_{s_{-j}} \left( \max_{s_j} u_j(s_j, s_{-j}) \right)
    \]
  - For every agent $i \neq j$, a **minmax strategy for** $i$ is $i$’s component of $s^*_{-j}$.

- **Agent $j$’s minmax value** is $j$’s maximum payoff against $s^*_{-j}$.
  - \[
    \max_{s_j} u_j(s_j, s^*_{-j}) = \min_{s_j} \max_{s_j} u_j(s_j, s_{-j})
    \]

- We have equality since we just replaced $s^*_{-j}$ by its value above.
Minimax Theorem (von Neumann, 1928)

- **Theorem.** Let $G$ be any finite two-player zero-sum game. For each player $i$,
  - $i$’s expected utility in any Nash equilibrium
    - $= i$’s maxmin value
    - $= i$’s minmax value
  - In other words, for every Nash equilibrium $(s_1^*, s_2^*)$,
    \[
    u_1(s_1^*, s_2^*) = \min_{s_1} \max_{s_2} u_1(s_1, s_2) = \max_{s_2} \min_{s_1} u_1(s_1, s_2) = -u_2(s_1^*, s_2^*)
    \]

- **Corollary.** For two-player zero-sum games: \{Nash equilibria\} = \{maxmin strategy profiles\} = \{minmax strategy profiles\}

- Note that this is **not necessary true** for non-zero-sum games as we saw for Battle of Sexes in previous slides

- Terminology: the value (or minmax value) of $G$ is agent 1’s minmax value.
Maximin and Minimax via LP

- Let \( u_2 = u_1 = u \) and let mixed strategies \( s_1 = x = (x_1, \ldots, x_k) \) and \( s_2 = y = (y_1, \ldots, y_r) \), in which player 1 has \( k \) strategies and player 2 has \( r \) strategies.

- Then \( u(x, y) = \sum_i \sum_j x_i y_j u_{i,j} = \sum_j y_j \sum_i x_i u_{i,j} \)

- We want to find \( x^* \) which optimizes \( v^1 = \max_x \min_y u(x,y) \)

- Since player 2 is doing his best response (in \( \min_y u(x,y) \)) he sets \( y_j > 0 \) only if \( \sum_i x_i u_{i,j} \) is minimized.

Thus \( v^1 = \sum_j \sum_i x_i y_j u_{i,j} = (\sum_j y_j) \min_j \sum_i x_i u_{i,j} = \min_j \sum_i x_i u_{i,j} \leq \sum_i x_i u_{i,j} \) for any \( j \)

We have the following LP to find \( v^1 \) and the first player strategy \( x^* \)

\[
\begin{align*}
\max & \quad v^1 \\
\text{such that} & \quad v^1 \leq \sum_i x_i u_{i,j} \text{ for all } j \\
\sum_i x_i & = 1 \\
x_i & \geq 0
\end{align*}
\]
Maximin and Minimax via LP

- Similarly by writing an LP for minimax value $v^2 = \min_y \max_x u(x, y)$, we can obtain the second player strategy
  $$\min v^2$$
  such that $v^2 \geq \sum_j y_j u_{i,j}$ for all $i$
  $$\sum_j y_j = 1$$
  $$y_j \geq 0$$

- Note that due to Minimax Theorem $v^1 = v^2$
  ($v^1 \leq v^2$ is trivial just by definitions). Also $(s_1, s_2) = (x, y)$ is a Nash equilibrium.
Example: Matching Pennies

- Agent 1’s strategy: display heads with probability $p$
- Agent 2’s strategy: display heads with probability $q$

\[ u_1(p, q) = pq + (1 - p)(1 - q) - p(1 - q) - q(1 - p) = 1 - 2p - 2q + 4pq \]

\[ u_2(p, q) = -u_1(p, q) \]

Want to show that

- \{Nash equilibria\} = \{maxmin strategy profiles\} = \{minmax strategy profiles\} = \{(p = \frac{1}{2}, q = \frac{1}{2})\}
Example: Matching Pennies

- **Find Nash equilibria**
  
  \[
  u_1(p, q) = 1 - 2p - 2q + 4pq \\
  u_2(p, q) = -u_1(p, q)
  \]

- If \( p = q = \frac{1}{2} \), then \( u_1 = u_2 = 0 \)

- If agent 1 changes to \( p \neq \frac{1}{2} \) and agent 2 keeps \( q = \frac{1}{2} \), then
  - \( u_1(p, \frac{1}{2}) = 1 - 2p - 1 + 2p = 0 \)

- If agent 2 changes to \( q \neq \frac{1}{2} \) and agent 1 keeps \( p = \frac{1}{2} \), then
  - \( u_2(\frac{1}{2}, q) = -(1 - 2q - 1 + 2q) = 0 \)

- Thus \( p = q = \frac{1}{2} \) is a Nash equilibrium

- Are there any others?
Example: Matching Pennies

- Show there are no other Nash equilibria
  
  \[ u_1(p, q) = 1 - 2p - 2q + 4pq \]
  
  \[ u_2(p, q) = -u_1(p, q) \]

- Consider any strategy profile \((p, q)\)
  where \(p \neq \frac{1}{2}\) or \(q \neq \frac{1}{2}\) or both

  - Several different cases, depending on the exact values of \(p\) and \(q\)
  - In every one of them, either agent 1 can increase \(u_1\) by changing \(p\), or agent 2 can increase \(u_2\) by changing \(q\), or both

- So there are no other Nash equilibria
Example: Matching Pennies

- Find all maxmin strategy profiles
  \[ u_1(p, q) = 1 - 2p - 2q + 4pq \]
  \[ u_2(p, q) = -u_1(p, q) \]

- If agent 1’s strategy is \( p = \frac{1}{2} \)
  then regardless of 2’s value of \( q \),
  \[ u_1(\frac{1}{2}, q) = 1 - 2q - 1 + 2q = 0 \]

- If agent 1’s strategy is \( p > \frac{1}{2} \)
  then 2’s best response is \( q = 0 \)
  (see the diagram)
  \[ u_1(p, 0) = 1 - 2p < 0 \]

- If agent 1’s strategy is \( p < \frac{1}{2} \)
  then 2’s best response is \( q = 1 \)
  \[ u_1(p, 1) = -1 + 2p < 0 \]

- Thus 1 has one maxmin strategy: \( p = \frac{1}{2} \)

- Similarly, 2 has one maxmin strategy: \( q = \frac{1}{2} \)
Example: Matching Pennies

- **Find all minmax strategy profiles**
  
  \[ u_1(p, q) = 1 - 2p - 2q + 4pq \]
  
  \[ u_2(p, q) = - u_1(p, q) \]

- If agent 1’s strategy is \( p = \frac{1}{2} \)
  then regardless of 2’s value of \( q \),
  \[ u_2\left(\frac{1}{2}, q\right) = -(1 - 2q - 1 + 2q) = 0 \]

- If agent 1’s strategy is \( p > \frac{1}{2} \)
  then 2’s best response is \( q = 0 \)
  (see the diagram)
  \[ u_2(p, 0) = -(1 - 2p) > 0 \]

- If agent 1’s strategy is \( p < \frac{1}{2} \)
  then 2’s best response is \( q = 1 \)
  \[ u_2(p, 1) = -(1 - 2p) > 0 \]

- Thus 1 has one minmax strategy: \( p = \frac{1}{2} \)

- Similarly, 2 has one minmax strategy: \( q = \frac{1}{2} \)