CMSC 474, Introduction to Game Theory

Maxmin and Minmax Strategies

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Worst-Case Expected Utility

- For agent i, the **worst-case** expected utility of a strategy s_i is the minimum over all possible combinations of strategies for the other agents:

$$\min_{\mathbf{s}_{-i}} u_i\left(s_i, \mathbf{s}_{-i}\right)$$

Wife \ Husband	Opera	Football
Opera	2, 1	0, 0
Football	0, 0	1, 2

- **Example: Battle of the Sexes**
  - Wife's strategy $s_w = \{(p, \text{Opera}), (1 - p, \text{Football})\}$
  - Husband's strategy $s_h = \{(q, \text{Opera}), (1 - q, \text{Football})\}$
  - $u_w(p,q) = 2pq + (1 - p)(1 - q) = 3pq - p - q + 1$
  - For any fixed p, $u_w(p,q)$ is linear in q
    - e.g., if $p = ½$, then $u_w(½,q) = ½\, q + ½$
  - $0 \leq q \leq 1$, so the min must be at $q = 0$ or $q = 1$
    - e.g., $\min_q (½\, q + ½)$ is at $q = 0$
  - $\min_q u_w(p,q) = \min(u_w(p,0), u_w(p,1)) = \min(1 - p, 2p)$

We can write $u_w(p,q)$ instead of $u_w(s_w, s_h)$

Maxmin Strategies

Also called **maximin**

- A **maxmin strategy** for agent *i*
  - A strategy s_i that makes *i*'s worst-case expected utility as high as possible:
    $$\arg\max_{s_i} \min_{\mathbf{s}_{-i}} \; u_i\left(s_i, \mathbf{s}_{-i}\right)$$
  - This isn't necessarily unique
  - Often it is mixed

- Agent *i*'s **maxmin value**, or **security level**, is the maxmin strategy's worst-case expected utility:
  $$\max_{s_i} \min_{\mathbf{s}_{-i}} \; u_i\left(s_i, \mathbf{s}_{-i}\right)$$
- For 2 players it simplifies to
  $$\max_{s_1} \min_{s_2} u_1\left(s_1, s_2\right)$$

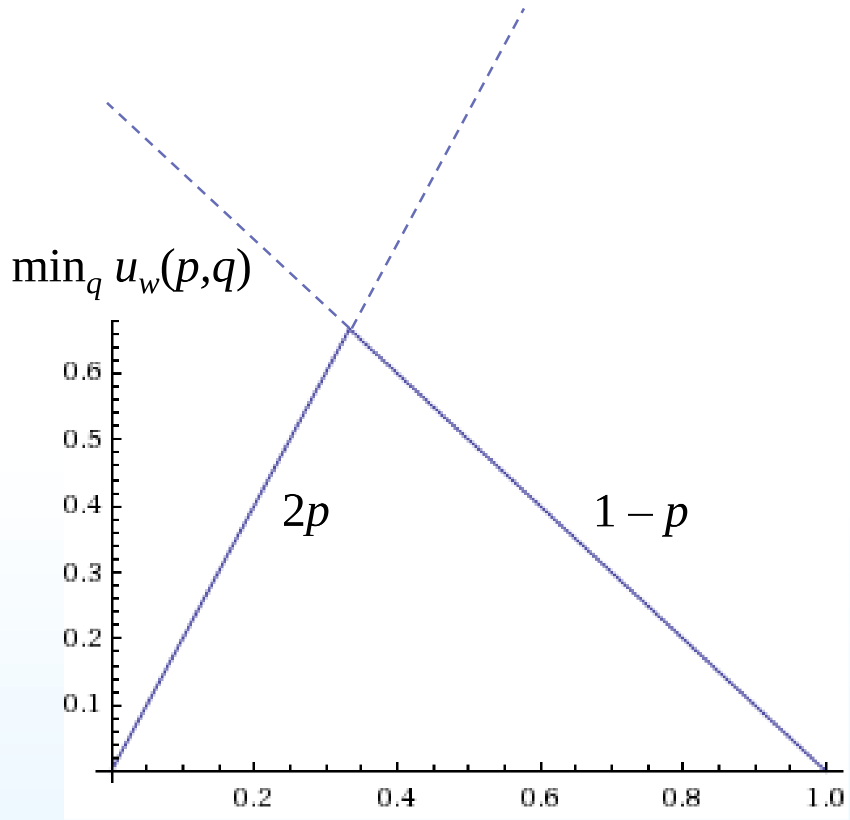
Example

- Wife's and husband's strategies
  - $s_w = \{(p, \text{Opera}), (1 - p, \text{Football})\}$
  - $s_h = \{(q, \text{Opera}), (1 - q, \text{Football})\}$
- Recall that wife's worst-case expected utility is

$$\min_q u_w(p,q) = \min\,(1 - p, 2p)$$

  - Find p that maximizes it
- Max is at $1 - p = 2p$, i.e., $p = 1/3$
  - Wife's maxmin value is $1 - p = 2/3$
  - Wife's maxmin strategy is $\{(1/3, \text{Opera}), (2/3, \text{Football})\}$
- Similarly,
  - Husband's maxmin value is 2/3
  - Husband's maxmin strategy is $\{(2/3, \text{Opera}), (1/3, \text{Football})\}$

Wife \ Husband	Opera	Football
Opera	2, 1	0, 0
Football	0, 0	1, 2



Question

- Why might an agent i want to use a maxmin strategy?

Answers

- Why might an agent *i* want to use a maxmin strategy?

  - Useful if *i* is cautious (wants to maximize his/her worst-case utility) and doesn't have any information about the other agents
    - whether they are rational
    - what their payoffs are
    - whether they draw their action choices from known distributions
  - Useful if *i* has reason to believe that the other agents' objective is to minimize *i*'s expected utility
    - e.g., 2-player zero-sum games (we discuss this later in his session)

- Solution concept: **maxmin strategy profile**
  - all players use their maxmin strategies

Minmax Strategies (in 2-Player Games)

- **Minmax strategy** and **minmax value**
  - Duals of their maxmin counterparts
- Suppose agent 1 wants to punish agent 2, regardless of how it affects agent 1's own payoff
- Agent 1's **minmax strategy** against agent 2
  - A strategy s_1 that minimizes the expected utility of 2's best response to s_1

$$\arg\min_{s_1}\max_{s_2} u_2(s_1, s_2)$$

- **Agent 2's minmax value** is 2's maximum expected utility if agent 1 plays his/her minmax strategy:

$$\min_{s_1}\max_{s_2} u_2(s_1, s_2)$$

- **Minmax strategy profile**: both players use their minmax strategies

Also called **minimax**

Example

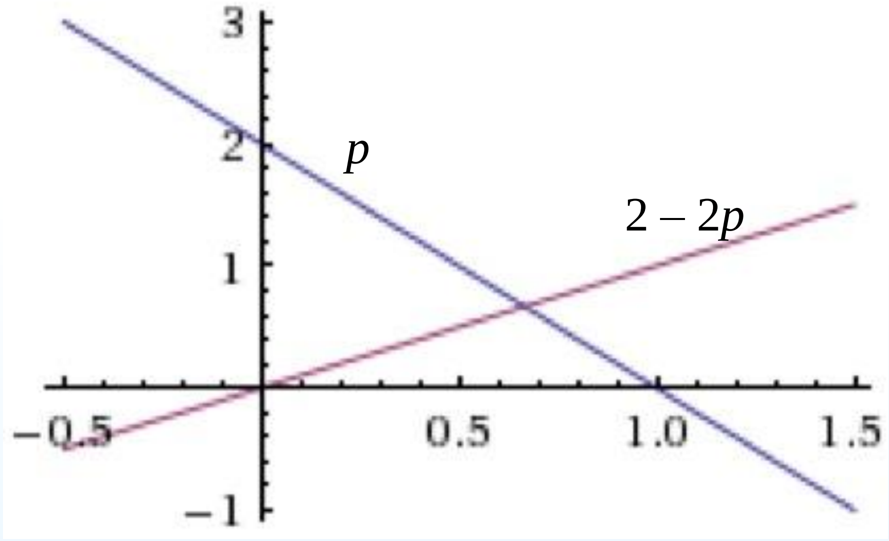
- Wife's and husband's strategies
  - s_w = {(p, Opera), (1 – p, Football)}
  - s_h = {(q, Opera), (1 – q, Football)}

Husband / Wife	Opera	Football
Opera	2, 1	0, 0
Football	0, 0	1, 2

- $u_h(p,q) = pq + 2(1-p)(1-q) = 3pq - 2p - 2q + 2$
- Given wife's strategy p, husband's expected utility is linear in q
  - e.g., if $p = ½$, then $u_h(½,q) = -½\, q + 1$
- Max is at $q = 0$ or $q = 1$

$$\max_q u_h(p,q) = (2-2p, p)$$

- Find p that minimizes this
- Min is at $-2p + 2 = p$ → $p = 2/3$
- Husband/s minmax value is 2/3
- Wife's minmax strategy is {(2/3, Opera), (1/3, Football)}



Minmax Strategies in *n*-Agent Games

- In *n*-agent games ($n > 2$), agent *i* usually can't minimize agent *j*'s payoff by acting unilaterally
- But suppose all the agents "gang up" on agent *j*
  - Let $\mathbf{s}^*_{-j}$ be a mixed-strategy profile that minimizes *j*'s maximum payoff, i.e.,

$$\mathbf{s}^*_{-j} = \arg\min_{\mathbf{s}_{-j}} \left(\max_{s_j} u_j\left(s_j, \mathbf{s}_{-j}\right) \right)$$

  - For every agent $i \neq j$, a **minmax strategy for *i*** is *i*'s component of $\mathbf{s}_{-j}$*

- **Agent *j*'s minmax value** is *j*'s maximum payoff against $\mathbf{s}_{-j}$*

$$\max_{s_j} \; u_j\left(s_j, \mathbf{s}^*_{-j}\right) = \min_{\mathbf{s}_{-j}} \max_{s_j} \; u_j\left(s_j, \mathbf{s}_{-j}\right)$$

- We have equality since we just replaced $\mathbf{s}_{-j}$* by its value above

Minimax Theorem (von Neumann, 1928)

- **Theorem.** Let G be any finite two-player zero-sum game. For each player i,
  - i's expected utility in any Nash equilibrium
    = i's maxmin value
    = i's minmax value
  - In other words, for every Nash equilibrium (s_1*, s_2*),

$$u_1(s_1^*, s_2^*) = \min_{s_1} \max_{s_2} u_1(s_1, s_2) = \max_{s_2} \min_{s_1} u_1(s_1, s_2) = -u_2(s_1^*, s_2^*)$$

- **Corollary.** For two-player zero-sum games:{Nash equilibria} = {maxmin strategy profiles}= {minmax strategy profiles}
- Note that this is **not necessary true** for **non-zero-sum** games as we saw for Battle of Sexes in previous slides
- Terminology: the **value** (or **minmax value**) of G is agent 1's minmax value

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Maximin and Minimax via LP

- Let-u_2=u_1 = u and let mixed strategies $s_1 = \mathrm{x} = (\mathrm{x}_1, \dots, \mathrm{x}_\mathrm{k})$ and $\mathrm{s}_2 = \mathrm{y} = (\mathrm{y}_1, \dots, \mathrm{y}_\mathrm{r})$, in which player 1 has k strategies and player 2 has r strategies.
- Then $\mathrm{u}(x, y) = \sum_i \sum_j x_i y_j u_{i,j}$=$\sum_j y_j \sum_i x_i u_{i,j}$
- We want to find x^* which optimizes $v^1 = \max_x \min_y u(\mathrm{x,y})$
- Since player 2 is doing his best response (in $\min_y u(\mathrm{x,y})$) he sets $y_j > 0$ only if $\sum_i x_i u_{i,j}$ is minimized.

Thus $v^1 = \sum_j \sum_i x_i y_j u_{i,j} = (\sum_j y_j) \min_j \sum_i x_i u_{i,j} = \min_\mathrm{j} \sum_i x_i u_{i,j} \leq \sum_i x_i u_{i,j}$ for any j

We have the following LP to find v^1 and the first player strategy x^*

$$\max v^1$$

$$\text{such that } v^1 \leq \sum_i x_i u_{i,j} \text{ for all j}$$

$$\sum_i x_i = 1$$

$$x_i \geq 0$$

Maximin and Minimax via LP

- Similarly by writing an LP for minimax value $v^2 = \min_y \max_x u(x,y)$, we can obtain the second player strategy

$$\min v^2$$

$$\text{such that } v^2 \geq \sum_j y_j u_{i,j} \text{ for all i}$$

$$\sum_j y_j = 1$$

$$y_j \geq 0$$

- Note that due to Minimax Theorem $v^1 = v^2$

($v^1 \leq v^2$ is trivial just by definitions). Also (s_1, s_2)= (x, y) is a Nash equilibrium.

Example: Matching Pennies

	Heads	Tails
Heads	1, –1	–1, 1
Tails	–1, 1	1, –1

- Agent 1's strategy: display heads with probability p
- Agent 2's strategy: display heads with probability q

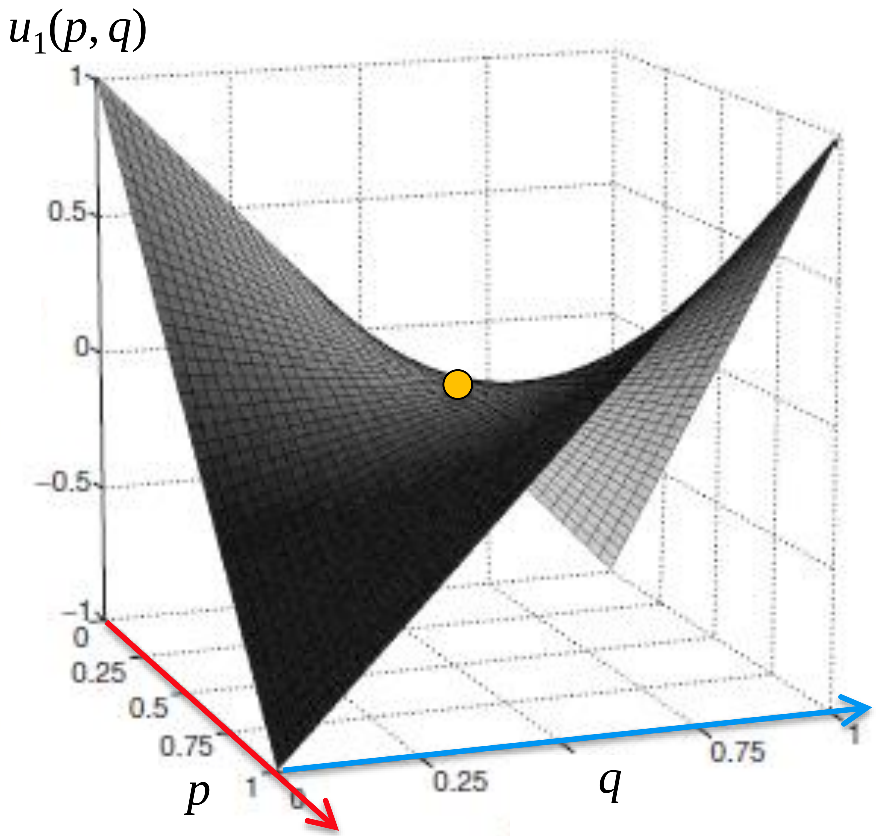
$$u_1(p, q) = p\,q + (1 - p)(1 - q) - p(1 - q) - q(1 - p)$$
$$= 1 - 2p - 2q + 4pq$$
$$u_2(p, q) = - u_1(p, q)$$

- Want to show that
  - {Nash equilibria}

    = {maxmin strategy profiles}

    = {minmax strategy profiles}

    = $\{(p = ½, q = ½)\}$



Example: Matching Pennies

	Heads	Tails
Heads	1, –1	–1, 1
Tails	–1, 1	1, –1

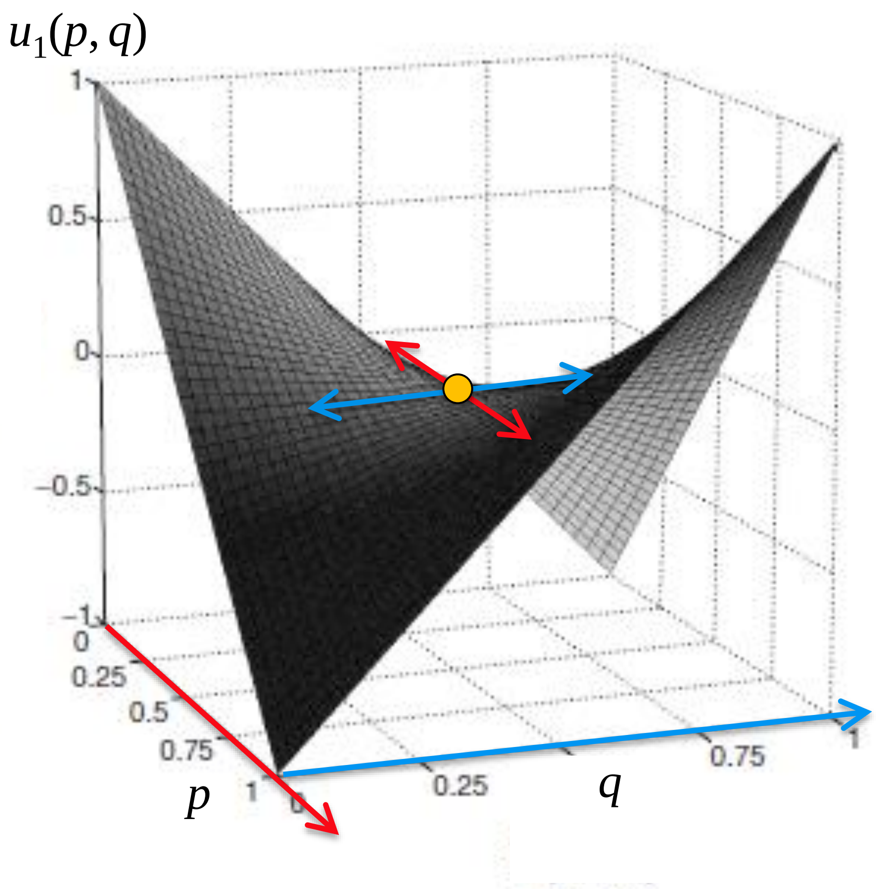
- **Find Nash equilibria**

$$u_1(p, q) = 1 - 2p - 2q + 4pq$$

$$u_2(p, q) = -\, u_1(p, q)$$

- If $p = q = ½$, then $u_1 = u_2 = 0$
- If agent 1 changes to $p \neq ½$ and agent 2 keeps $q = ½$, then
  - ➢ $u_1\,(p, ½) = 1 - 2p - 1 + 2p = 0$
- If agent 2 changes to $q \neq ½$ and agent 1 keeps $p = ½$, then
  - ➢ $u_2\,(½, q) = -\,(1 - 2q - 1 + 2q) = 0$
- Thus $p = q = ½$ is a Nash equilibrium

- Are there any others?



Example: Matching Pennies

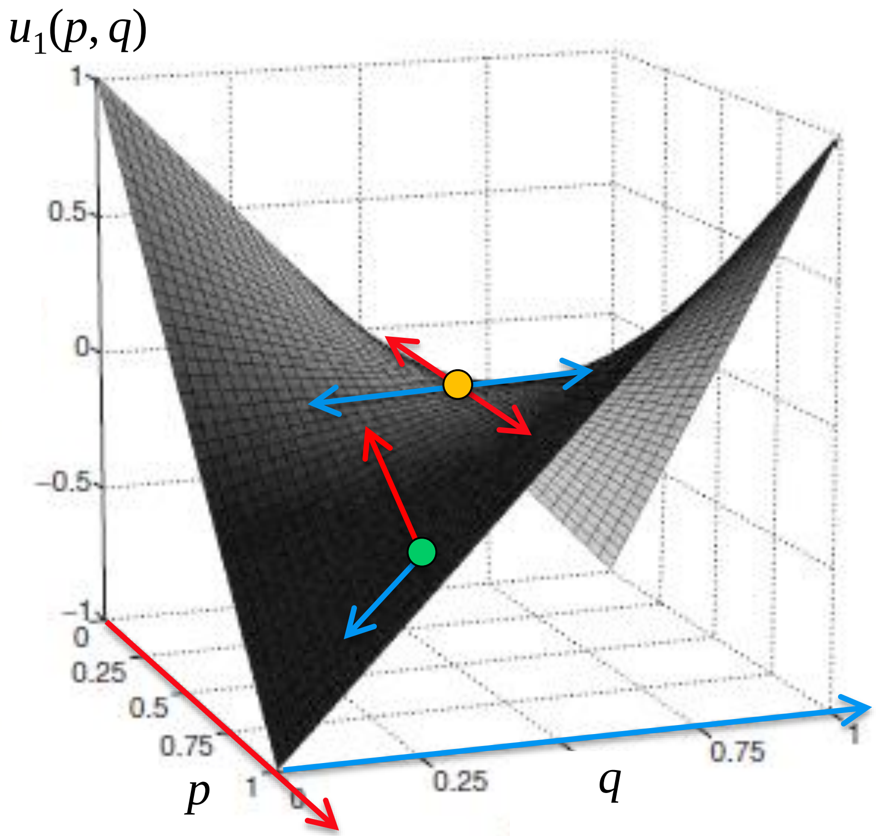
	Heads	Tails
Heads	1, –1	–1, 1
Tails	–1, 1	1, –1

- **Show there are no other Nash equilibria**

$$u_1(p, q) = 1 - 2p - 2q + 4pq$$

$$u_2(p, q) = -\,u_1(p, q)$$

- Consider any strategy profile (p, q) where $p \neq ½$ or $q \neq ½$ or both
  - Several different cases, depending on the exact values of p and q
  - In every one of them, either agent 1 can increase u_1 by changing p, or agent 2 can increase u_2 by changing q, or both
- So there are no other Nash equilibria



Example: Matching Pennies

	Heads	Tails
Heads	1, –1	–1, 1
Tails	–1, 1	1, –1

- **Find all maxmin strategy profiles**

$$u_1(p, q) = 1 - 2p - 2q + 4pq$$
$$u_2(p, q) = - u_1(p, q)$$

- If agent 1's strategy is $p = ½$ then regardless of 2's value of q, $u_1(½, q) = 1 - 2q - 1 + 2q = 0$
- If agent 1's strategy is $p > ½$ then 2's best response is $q = 0$

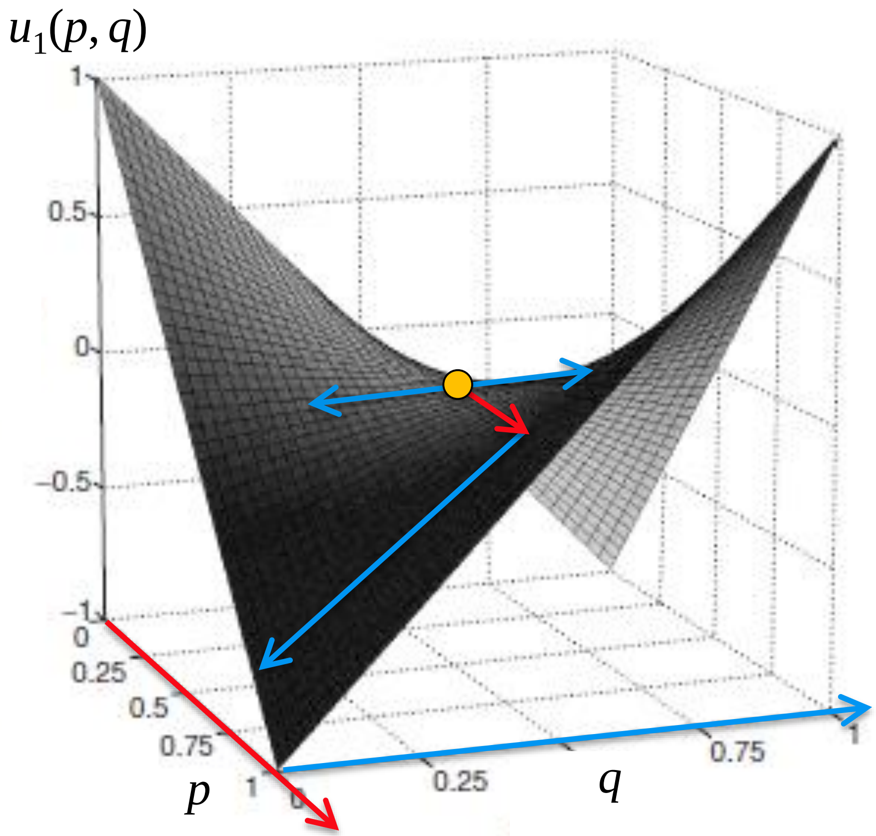
  (see the diagram)

$$u_1(p, 0) = 1 - 2p < 0$$

- If agent 1's strategy is $p < ½$ then 2's best response is $q = 1$

$$u_1(p, 1) = -1 + 2p < 0$$

- Thus 1 has one maxmin strategy: $p = ½$



- Similarly, 2 has one maxmin strategy: $q = ½$

Example: Matching Pennies

	Heads	Tails
Heads	1, –1	–1, 1
Tails	–1, 1	1, –1

- **Find all minmax strategy profiles**

$$u_1(p, q) = 1 - 2p - 2q + 4pq$$

$$u_2(p, q) = -u_1(p, q)$$

- If agent 1's strategy is $p = ½$ then regardless of 2's value of q, $u_2(½, q) = -(1 - 2q - 1 + 2q) = 0$
- If agent 1's strategy is $p > ½$ then 2's best response is $q = 0$

  (see the diagram)

$$u_2(p, 0) = -(1 - 2p) > 0$$

- If agent 1's strategy is $p < ½$ then 2's best response is $q = 1$

$$u_2(p, 1) = -(-1 + 2p) > 0$$

- Thus 1 has one minmax strategy: $p = ½$
- Similarly, 2 has one minmax strategy: $q = ½$