CMSC 474, Introduction to Game Theory

Perfect-Information Extensive Form Games

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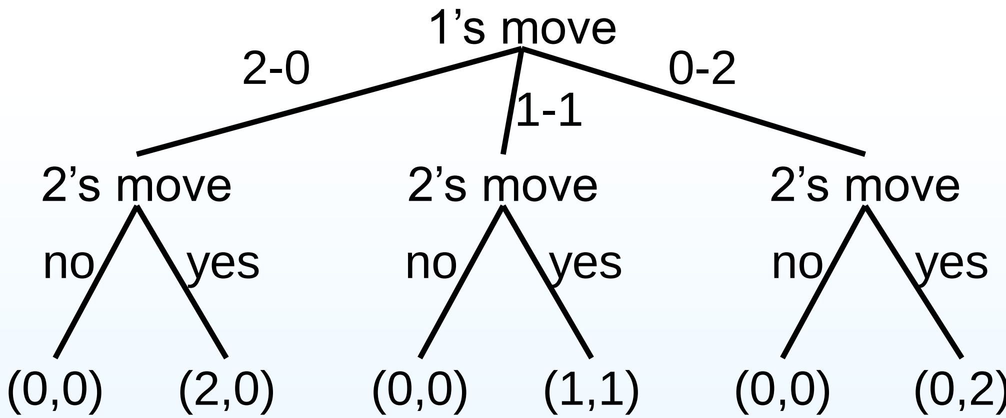
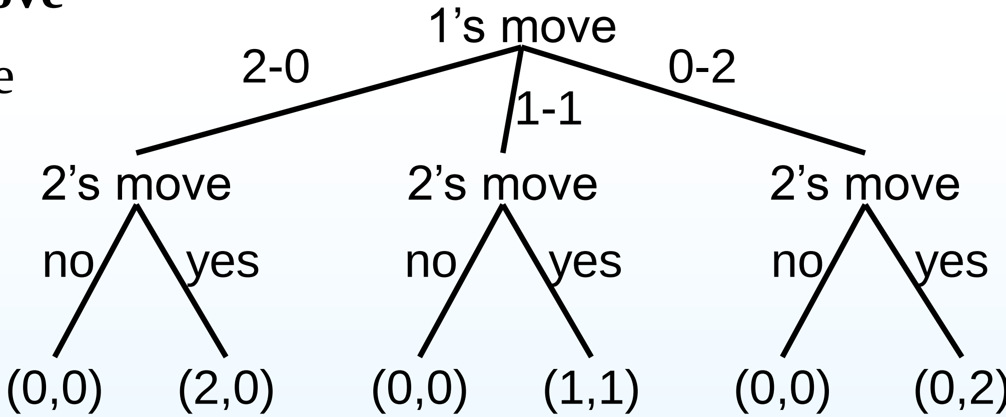
The Sharing Game

- Suppose agents 1 and 2 are two children
- Someone offers them two cookies, but only if they can agree how to share them
- Agent 1 chooses one of the following options:
  - Agent 1 gets 2 cookies, agent 2 gets 0 cookies
  - They each get 1 cookie
  - Agent 1 gets 0 cookies, agent 2 gets 2 cookies
- Agent 2 chooses to accept or reject the split:
  - Accept => they each get their cookies(s)
  - Otherwise, neither gets any



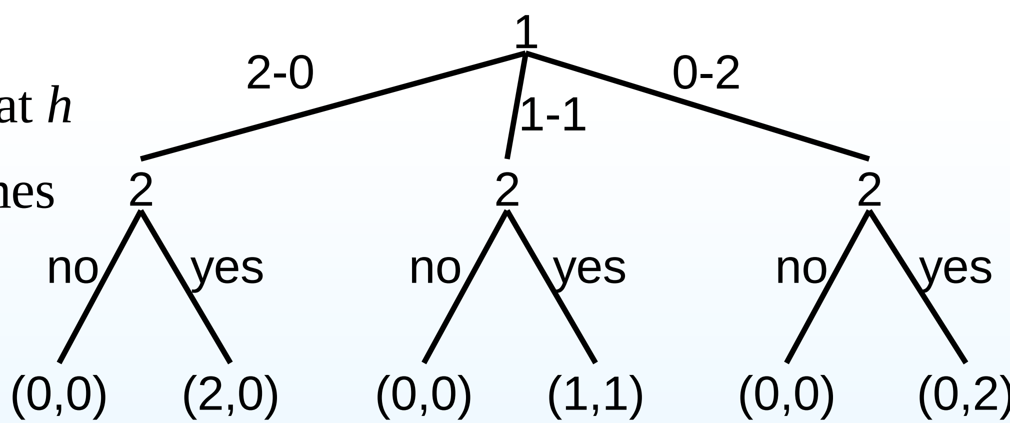
Extensive Form

- The sharing game is a game in **extensive form**
  - A game representation that makes the temporal structure explicit
  - Doesn't assume agents act simultaneously
- Extensive form can be converted to normal form
  - So previous results carry over
  - But there are additional results that depend on the temporal structure
- In a perfect-information game, the extensive form is a **game tree**:
  - **Choice** (or **nonterminal**) **node**: place where an agent chooses an action
  - **Edge**: an available **action** or **move**
  - **Terminal node**: a final outcome
  - At each terminal node h, each agent i has a utility $u_i(h)$



Notation from the Book (Section 4.1)

- H = {nonterminal nodes}
- Z = {terminal nodes}
- If h is a nonterminal node, then
  - $\rho(h)$ = the player to move at h
  - $\chi(h)$ = {all available actions at h}
  - $\sigma(h,a)$ = node produced by action a at node h
  - h's **children** or **successors** = $\{\sigma(h,a) : a \in \chi(h)\}$
- If h is a node (either terminal or nonterminal), then
  - h's **history** = the sequence of actions leading from the root to h
  - h's **descendants**
    = all nodes in the subtree rooted at h
- The book doesn't give the nodes names
  - The labels tell which agent makes the next move

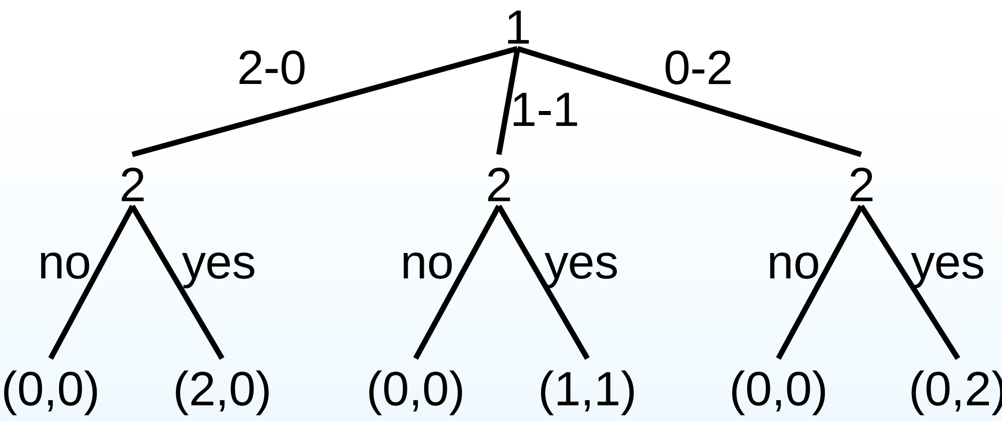


Pure Strategies

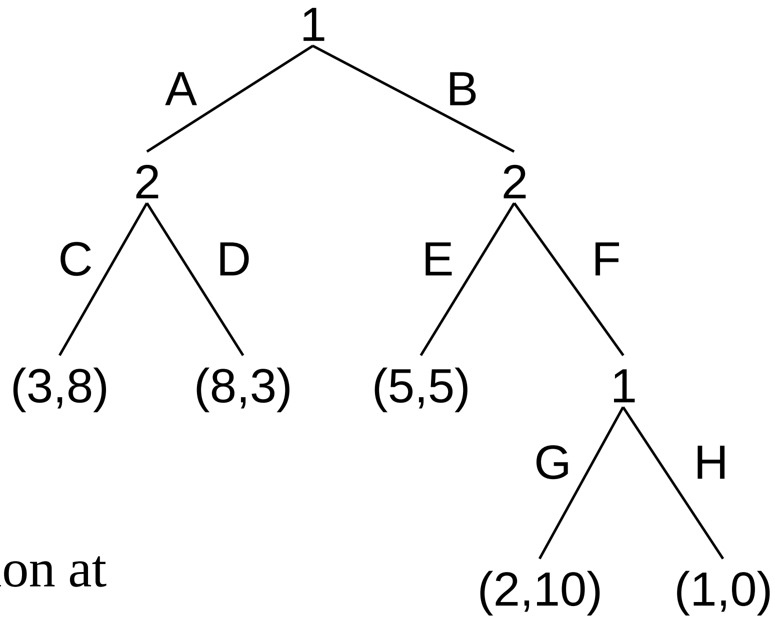
- Pure strategy for agent i in a perfect-information game:
  - Function telling what action to take at every node where it's i's choice
    - i.e., every node h at which $\rho(h) = i$
- The book specifies pure strategies as lists of actions
  - Which action at which node?
  - Either assume a canonical ordering on the nodes, or use different action names at different nodes

Sharing game:

- Agent 1 has 3 pure strategies: S_1 = {2-0, 1-1, 0-2}
- Agent 2 has 8 pure strategies:
- S_2 = {(yes, yes, yes), (yes, yes, no),
  (yes, no, yes), (yes, no, no),
  (no, yes, yes), (no, yes, no),
  (no, no, yes), (no, no, no)}

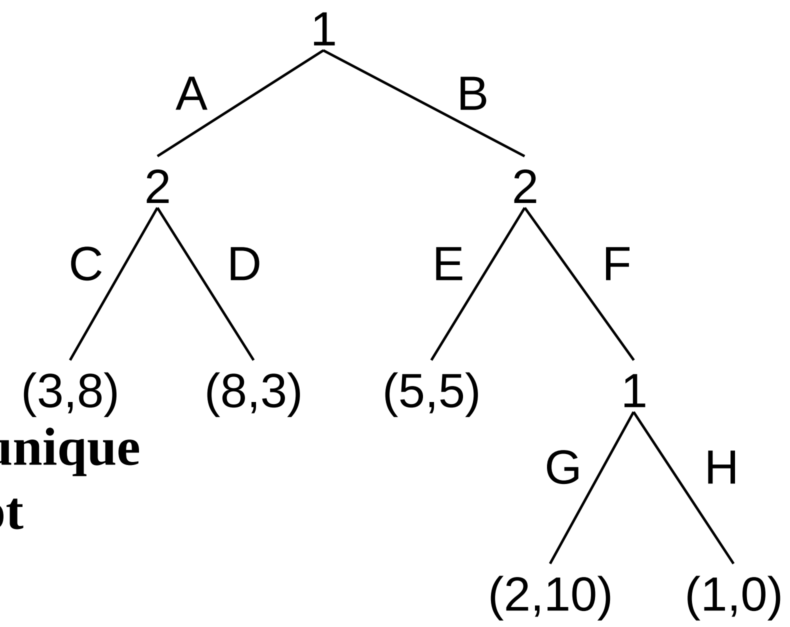


Extensive form vs. normal form



- Every game tree corresponds to an equivalent normal-form game
- The first step is to get all of the agents' pure strategies
- Each pure strategy for *i* must specify an action at every node where it's *i*'s move
- Example: the game tree shown here
  - Agent 1 has four pure strategies:
    - s_1 = {(A, G), (A, H), (B, G), (B, H)}
      - Mathematically, (A, G) and (A, H) are different strategies, even though action A makes the G-versus-H choice irrelevant
  - Agent 2 also has four pure strategies:
    - s_2 = {(C, E), (C, F), (D, E), (D, F)}

Extensive form vs. normal form

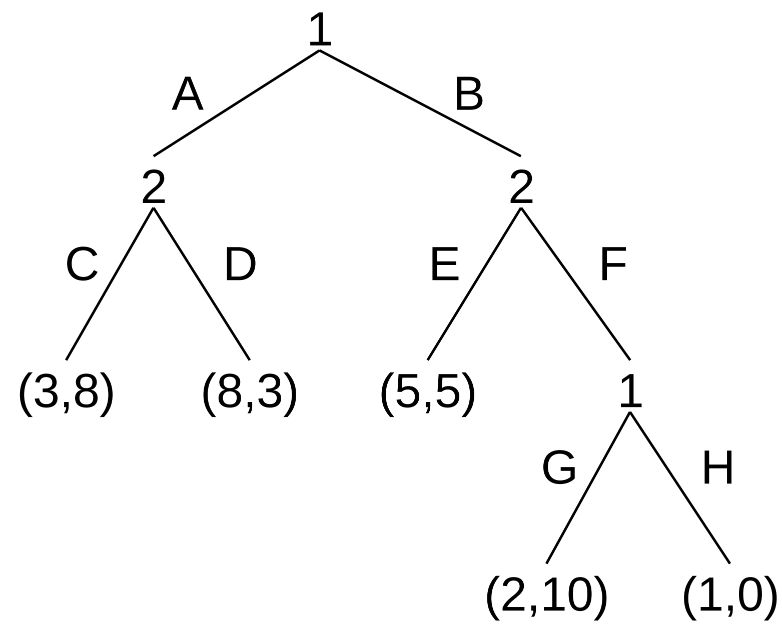


- Once we have all of the pure strategies, we can rewrite the game in normal form
- **Note that payoffs come from that of the unique leaf which will be accessible from the root**
- Converting to normal form introduces redundancy
  - 16 outcomes in the payoff matrix, versus 5 outcomes in the game tree
  - Payoff (3,8) occurs
    - once in the game tree
    - four times in the payoff matrix
- This can cause an exponential blowup

	(C,E)	(C,F)	(D,E)	(D,F)
(A,G)	3,8	3,8	8,3	8,3
(A,H)	3,8	3,8	8,3	8,3
(B,G)	5,5	2,10	5,5	2,10
(B,H)	5,5	1,0	5,5	1,0

Nash Equilibrium

- **Theorem.** Every perfect-information game in extensive form has a pure-strategy Nash equilibrium
  - This theorem has been attributed to Zermelo (1913), but there's some controversy about that
- Intuition:
  - Agents take turns, and everyone sees what's happened so far before making a move
  - So never need to introduce randomness into action selection to find an equilibrium
- In our example, there are three pure-strategy Nash equilibria



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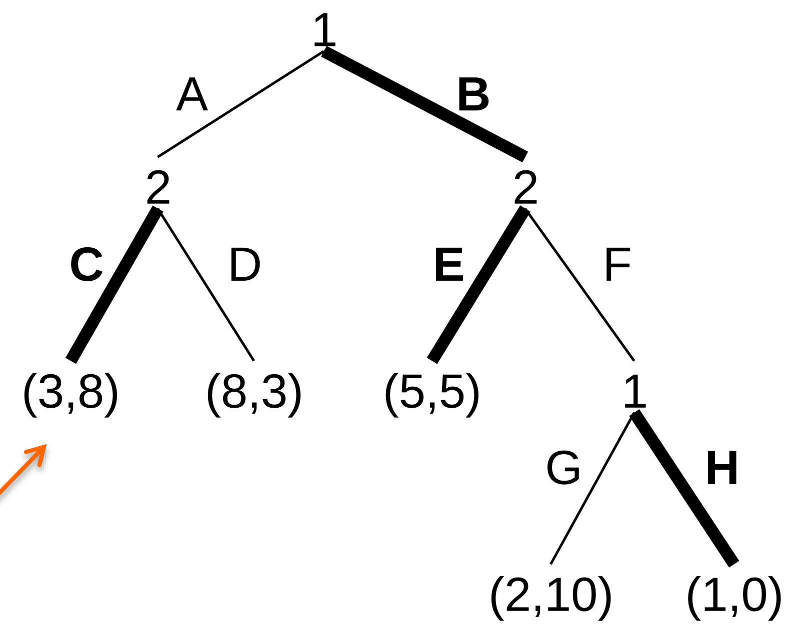
(B,H)	5,5	1,0	5,5	1,0

Nash Equilibrium

- The concept of a Nash equilibrium can be too weak for use in extensive-form games
- Recall that our example has three pure-strategy Nash equilibria:
  - {(A,G), (C,F)}
  - {(A,H), (C,F)}
  - {(B,H), (C,E)}
- Here is {(B,H), (C,E)} with the game in extensive form



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Subgame-Perfect Equilibrium

- Given a perfect-information extensive-form game G, the **subgame** of G rooted at node h is the restriction of G to the descendants of h
- Now we can define a refinement of a Nash equilibrium
- A **subgame-perfect equilibrium** (SPE) is a strategy profile **s** such that for every subgame G' of G, the restriction of **s** to G' is a Nash equilibrium of G'
  - Since G itself is is a subgame of G, every SPE is also a Nash equilibrium
- Every perfect-information extensive-form game has at least 1 SPE
  - Can prove this by induction on the height of the game tree

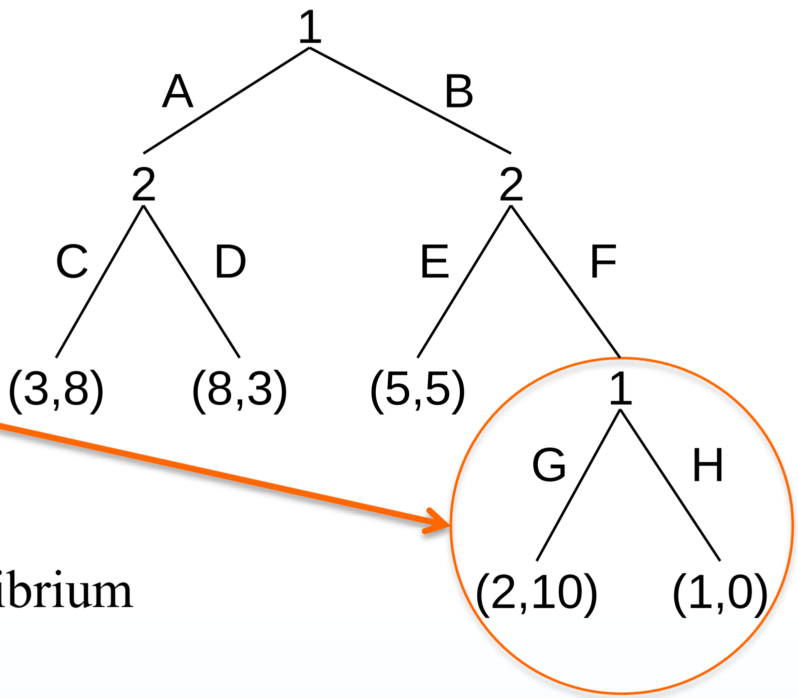
Example

- Recall that we have three Nash equilibria:

  {(*A*, *G*), (*C*, *F*)}

  {(*A*, *H*), (*C*, *F*)}

  {(*B*, *H*), (*C*, *E*)}



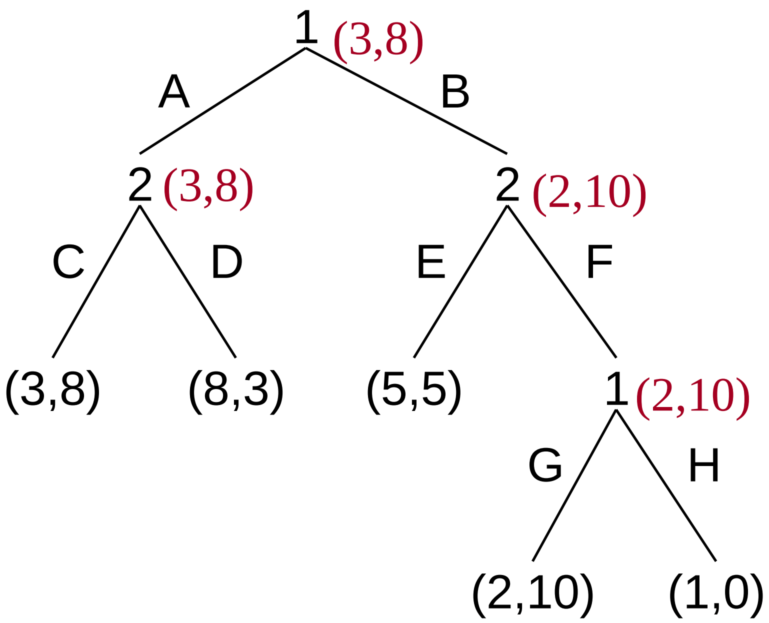
- Consider this subgame:
  - For agent 1,

    G strictly dominates *H*
  - Thus *H* can't be part of a Nash equilibrium

- This excludes {(*A*, *H*), (*C*, *F*)} and {(*B*, *H*), (*C*, *E*)}
- Just one subgame-perfect equilibrium
  - {(*A*, *G*), (*C*, *F*)}

Backward Induction

- To find subgame-perfect equilibria, we can use **backward induction**
- Identify the Nash equilibria in the bottom-most nodes
  - Assume they'll be played if the game ever reaches these nodes
- For each node *h*, recursively compute a vector $v_h = (v_{h1}, \ldots, v_{hn})$ that gives every agent's equilibrium utility
  - At each node *h*,
    - If *i* is the agent to move, then *i*'s equilibrium action is to move to a child *h'* of *h* for which *i*'s equilibrium utility $v_{h'i}$ is highest



Backward Induction

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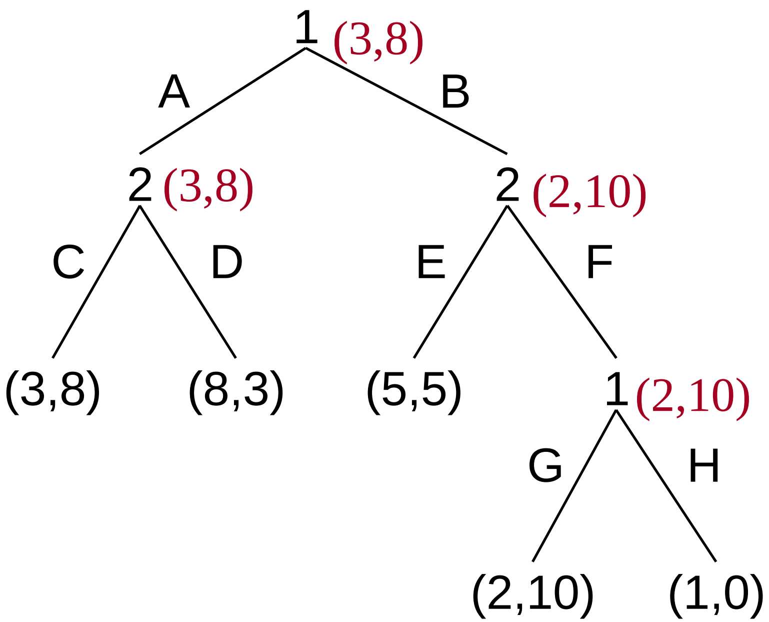


procedure backward-induction(h)

  if $h \in Z$ **then return u**(h)

  bestv = $(-\infty, \ldots, -\infty)$

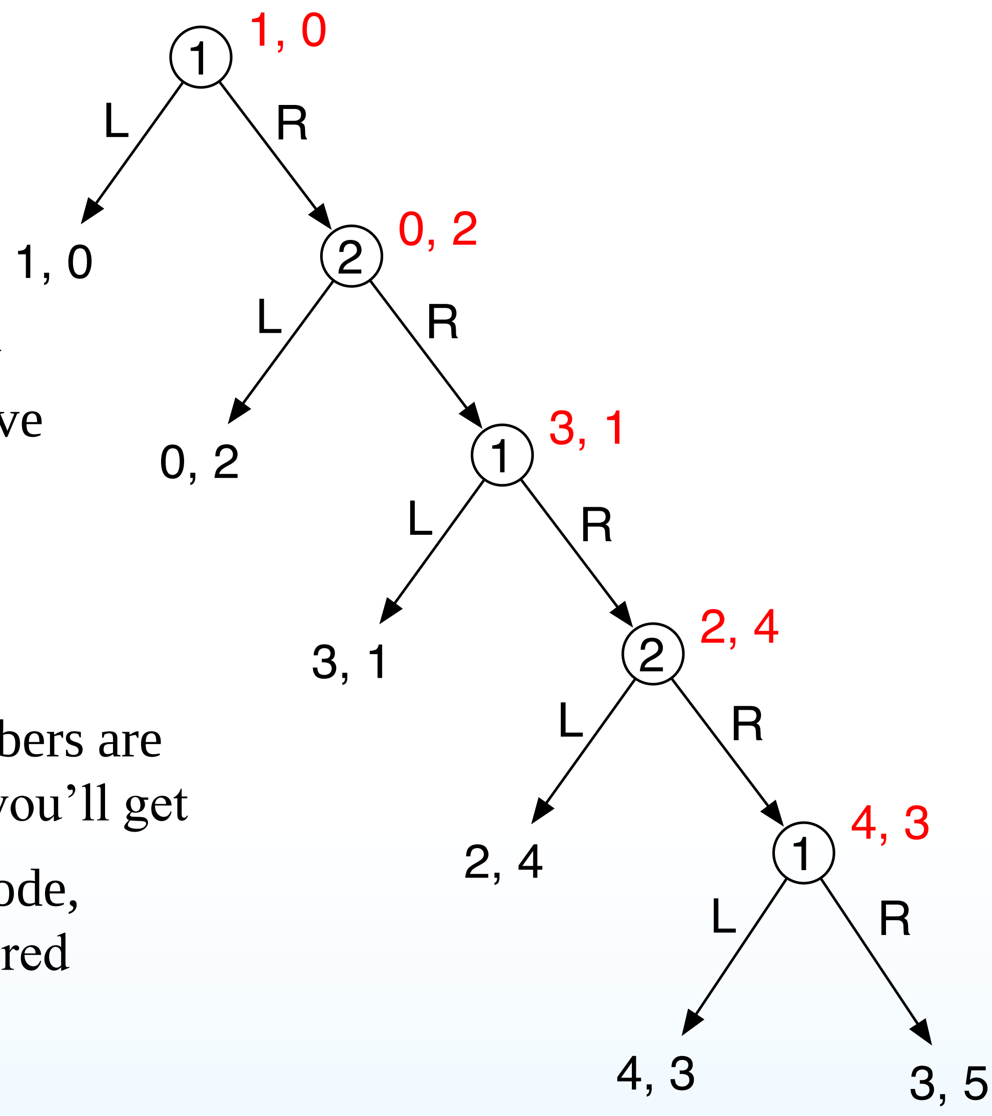
  forall $a \in \chi(h)$ **do**

    v = backward-induction($\sigma(h,a)$)

    if v[$\rho(h)$] > **bestv**[$\rho(h)$] **then bestv** = **v**

  return bestv

The Centipede Game

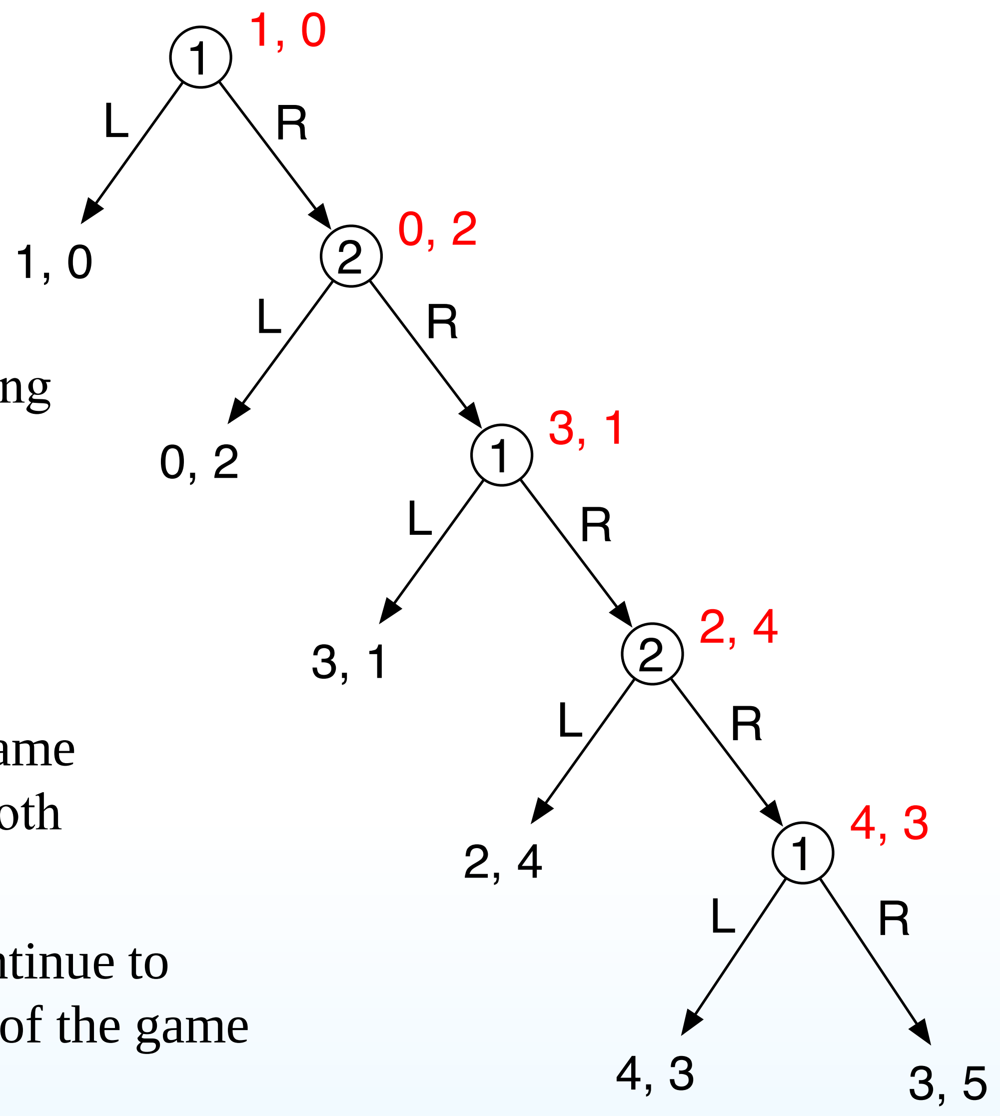


- The players move in alternation
  - Player 1 makes the first move
  - Each player can go either *Left* or *Right*
- At each terminal node, the numbers are how many pieces of chocolate you'll get
  - Next to each nonterminal node, I've put the SPE payoffs in red

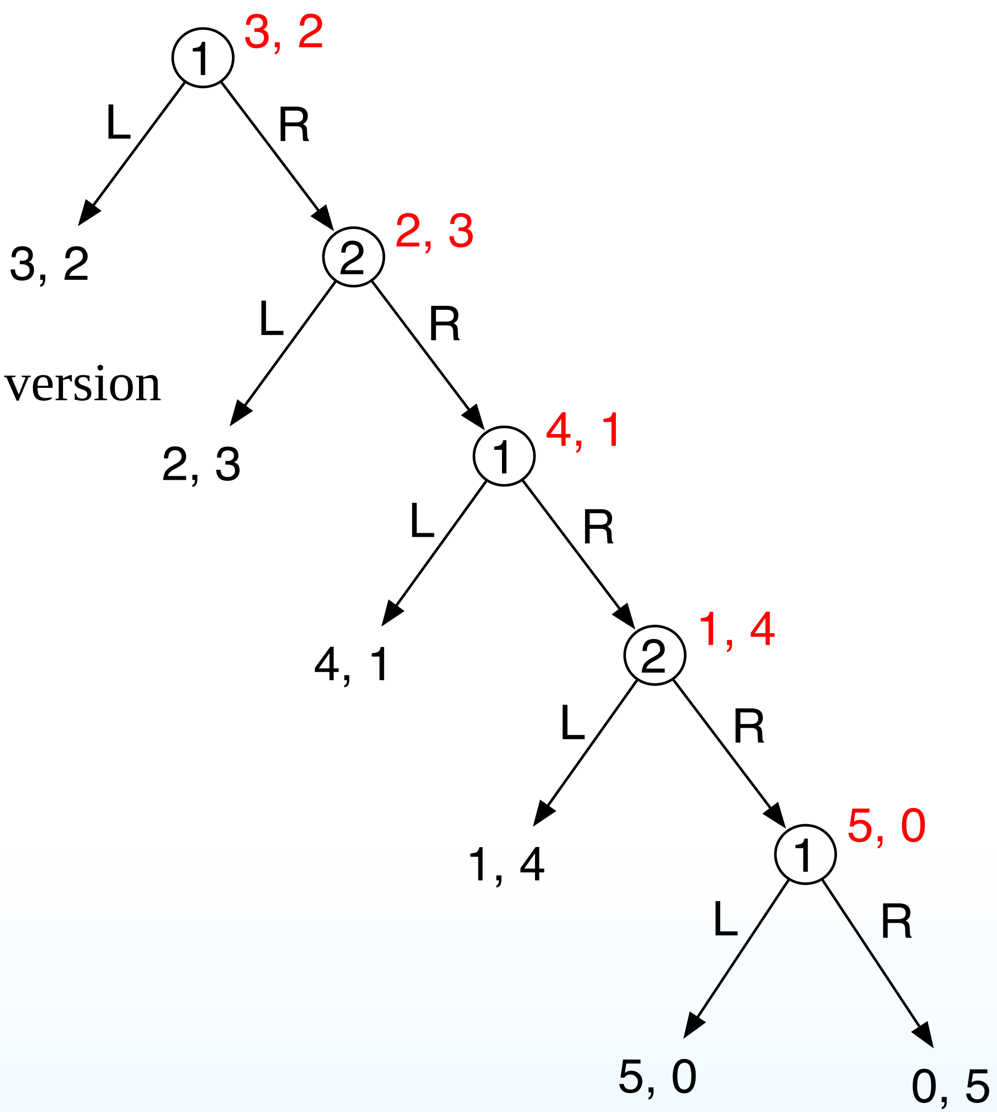
A Problem with Backward Induction

- Can extend the centipede game to any length
- The only SPE is for each agent always to move *Left*
- But this isn't intuitively appealing

- Seems unlikely that one would want to choose *Left* near the start of the game
  - ➢ If the agents continue the game for several moves, they'll both get higher payoffs
- In lab experiments, subjects continue to choose *Right* until near the end of the game

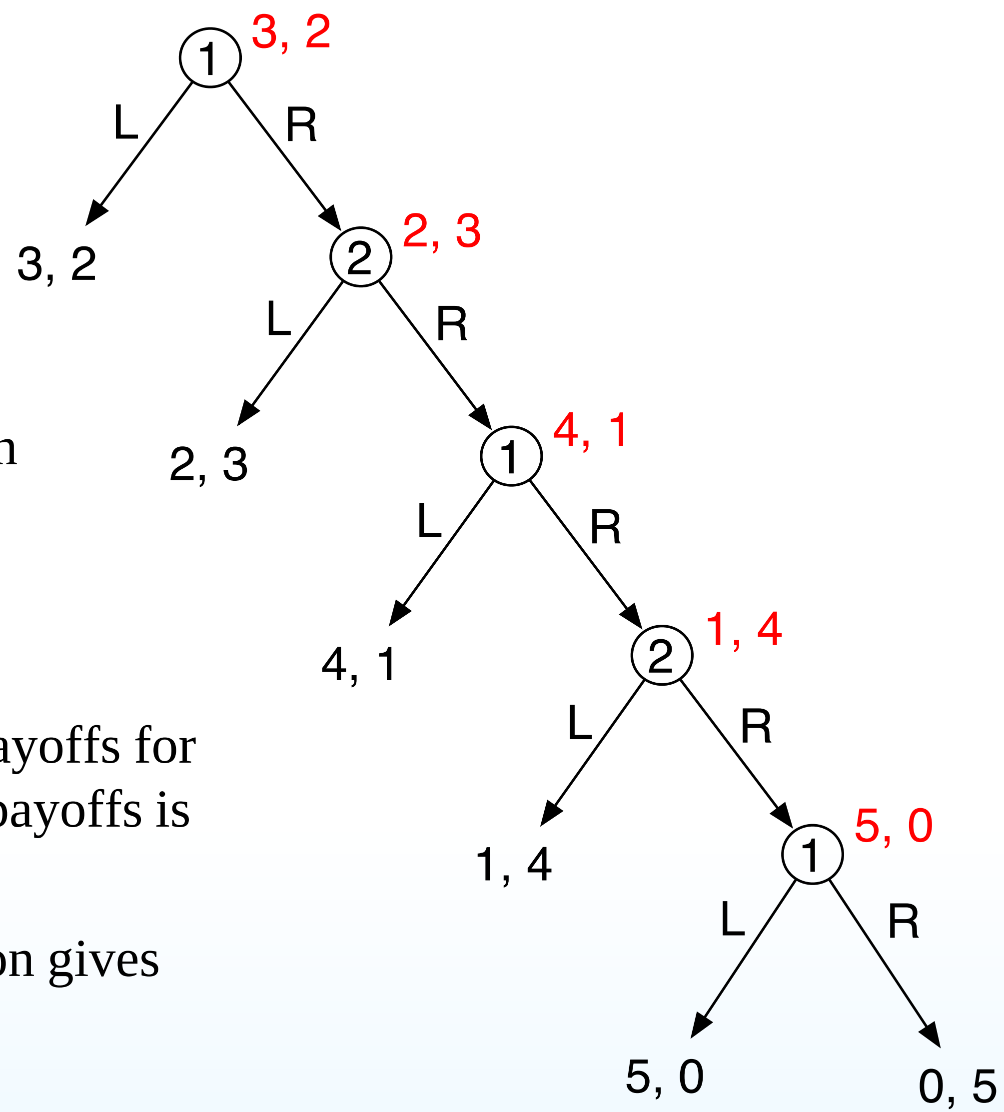


Constant-Sum Centipede Game



- Now consider a **constant-sum** version of the centipede game
- At every node, $u_2 = 5 - u_1$

Constant-Sum Centipede Game



- I need two more volunteers to play a **constant-sum** version of the centipede game
- At every node, $u_2 = 5 - u_1$
- Instead of having increasing payoffs for both players, the sum of their payoffs is always the same
- In this case, backward induction gives much more accurate results

The Minimax Algorithm

- In constant-sum games, only need to compute agent 1's SPE utility, u_1
  - $u_2 = c - u_1$
- From the Minimax Theorem,
  - at each node,

    agent 1's minmax value
    = agent 1's maxmin value
    = agent 1's SPE utility

procedure minimax(h)

if $h \in Z$ **then return** $u_1(h)$

else if $\rho(h) = 1$ **then return** $\max_{a \in \chi(h)} u_1(\sigma(h,a))$

else return $\min_{a \in \chi(h)} u_1(\sigma(h,a))$