CMSC 474, Introduction to Game Theory

18. Bayesian Games & Games of Incomplete Information

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Introduction

- All the kinds of games we’ve looked at so far have assumed that everything relevant about the game being played is common knowledge to all the players:
  - the number of players
  - the actions available to each
  - the payoff vector associated with each action vector
- True even for imperfect-information games
  - The actual moves aren’t common knowledge, but the game is

- We’ll now consider games of **incomplete** *(not imperfect)* information
  - Players are uncertain about the game being played
Example

- Consider the payoff matrix shown here
  - \( \varepsilon \) is a small positive constant; Agent 1 knows its value
  - Agent 1 doesn’t know the values of \( a, b, c, d \)
    - Thus the matrix represents a *set* of games
    - Agent 1 doesn’t know which of these games is the one being played
- Agent 1 wants a strategy that makes sense despite this lack of knowledge

- If Agent 1 thinks Agent 2 is malicious, then Agent 1 might want to play a maxmin, or “safety level,” strategy
  - minimum payoff of T is \( 1 - \varepsilon \)
  - minimum payoff of B is 1
    - So agent 1’s maxmin strategy is B
Bayesian Games

- Suppose we know the set $G$ of all possible games and we have enough information to put a probability distribution over the games in $G$
- A **Bayesian Game** is a class of games $G$ that satisfies two fundamental conditions

  - **Condition 1:**
    - The games in $G$ have the same number of agents, and the same strategy space (set of possible strategies) for each agent. The only difference is in the payoffs of the strategies.
  - This condition isn’t very restrictive
    - Other types of uncertainty can be reduced to the above, by reformulating the problem
Example

- Suppose we don’t know whether player 2 only has strategies L and R, or also an additional strategy C:

\[
\begin{array}{c|cc}
\text{Strategy} & L & R \\
\hline
U & 1, 1 & 1, 3 \\
D & 0, 5 & 1, 13 \\
\end{array}
\]

\[
\begin{array}{c|ccc}
\text{Strategy} & L & C & R \\
\hline
U & 1, 1 & 0, 2 & 1, 3 \\
D & 0, 5 & 2, 8 & 1, 13 \\
\end{array}
\]

- If player 2 doesn’t have strategy C, this is equivalent to having a strategy C that’s strictly dominated by other strategies:

\[
\begin{array}{c|ccc}
\text{Strategy} & L & C & R \\
\hline
U & 1, 1 & 0, -100 & 1, 3 \\
D & 0, 5 & 2, -100 & 1, 13 \\
\end{array}
\]

- The Nash equilibria for \( G_1' \) are the same as the Nash equilibria for \( G_1 \)
- We’ve reduced the problem to whether C’s payoffs are those of \( G_1' \) or \( G_2 \)
Bayesian Games

- **Condition 2 (common prior):**
  - The probability distribution over the games in $G$ is **common knowledge** (i.e., known to all the agents)
- So a Bayesian game defines
  - the uncertainties of agents about the game being played,
  - what each agent believes the other agents believe about the game being played
- The beliefs of the different agents are posterior probabilities
  - Combine the common prior distribution with individual “private signals” (what’s “revealed” to the individual players)
- The common-prior assumption rules out whole families of games
  - But it greatly simplifies the theory, so most work in game theory uses it
- There are some examples of games that don’t satisfy Condition 2
Definitions of Bayesian Games

- The book discusses three different ways to define Bayesian games
  - All are
    - equivalent (ignoring a few subtleties)
    - useful in some settings
    - intuitive in their own way

- The first definition (Section 7.1.1) is based on information sets

- A Bayesian game consists of
  - a set of games that differ only in their payoffs
  - a common (i.e., known to all players) prior distribution over them
  - for each agent, a partition structure (set of information sets) over the games

- Formal definition on the next page
7.1.1 Definition based on Information Sets

- **A Bayesian game** is a 4-tuple \((N, G, P, I)\) where:
  - \(N\) is a set of agents
  - \(G\) is a set of \(N\)-agent games
  - For every agent \(i\), every game in \(G\) has the same strategy space
  - \(P\) is a **common prior** over \(G\)
    - **common**: common knowledge (known to all the agents)
    - **prior**: probability before learning any additional info
  - \(I = (I_1, \ldots, I_N)\) is a tuple of partitions of \(G\), one for each agent
    - Information sets

- **Example:**

  \(G = \{\text{Matching Pennies (MP), Prisoner’s Dilemma (PD), Coordination (Crd), Battle of the Sexes (BoS)}\}\)

\[
\begin{array}{|c|c|c|}
\hline
& L & R \\
\hline
\text{MP} \ (p = 0.3) & 2, 0 & 0, 2 \\
\text{PD} \ (p = 0.1) & 2, 2 & 0, 3 \\
\text{Crd} \ (p = 0.2) & 2, 2 & 0, 0 \\
\text{BoS} \ (p = 0.4) & 2, 1 & 0, 0 \\
\hline
\end{array}
\]
Example (Continued)

- \( G = \{ \text{Matching Pennies (MP), Prisoner’s Dilemma (PD), Coordination (Crd), Battle of the Sexes (BoS)} \} \)
- Suppose the randomly chosen game is MP
- Agent 1’s information set is \( I_{1,1} \)
  - 1 knows it’s MP or PD
  - 1 can infer **posterior probabilities** for each
    
    \[
    \Pr[\text{MP} | I_{1,1}] = \frac{\Pr[\text{MP}]}{\Pr[\text{MP}] + \Pr[\text{PD}]} = \frac{0.3}{0.3 + 0.1} = \frac{3}{4}
    \]
    
    \[
    \Pr[\text{PD} | I_{1,1}] = \frac{\Pr[\text{PD}]}{\Pr[\text{MP}] + \Pr[\text{PD}]} = \frac{0.1}{0.3 + 0.1} = \frac{1}{4}
    \]

- Agent 2’s information set is \( I_{2,1} \)
  
  \[
  \Pr[\text{MP} | I_{2,1}] = \frac{\Pr[\text{MP}]}{\Pr[\text{MP}] + \Pr[\text{CrdD}]} = \frac{0.3}{0.3 + 0.2} = \frac{3}{5}
  \]
  
  \[
  \Pr[\text{Crd} | I_{2,1}] = \frac{\Pr[\text{Crd}]}{\Pr[\text{MP}] + \Pr[\text{CrdD}]} = \frac{0.2}{0.3 + 0.2} = \frac{2}{5}
  \]

\[
\begin{array}{c|cc}
I_{1,1} & \\ 
\hline
\text{MP} (p = 0.3) & L & R \\
\hline
U & 2, 0 & 0, 2 \\
\hline
D & 0, 2 & 2, 0 \\
\hline
\end{array}
\]

\[
\begin{array}{c|cc}
I_{1,2} & \\ 
\hline
\text{Crd} (p=0.2) & L & R \\
\hline
U & 2, 2 & 0, 0 \\
\hline
D & 0, 0 & 1, 1 \\
\hline
\end{array}
\]

\[
\begin{array}{c|cc}
I_{2,2} & \\ 
\hline
\text{PD} (p = 0.1) & L & R \\
\hline
U & 2, 2 & 0, 3 \\
\hline
D & 3, 0 & 1, 1 \\
\hline
\end{array}
\]

\[
\begin{array}{c|cc}
I_{2,1} & \\ 
\hline
\text{BoS} (p = 0.4) & L & R \\
\hline
U & 2, 1 & 0, 0 \\
\hline
D & 0, 0 & 1, 2 \\
\hline
\end{array}
\]
7.1.2 Extensive Form with Chance Moves

- Extensive form with Chance Moves
  - The book gives a description, but not a formal definition
- Hypothesize a special agent, Nature
- Nature has no utility function
  - At the start of the game, Nature makes a probabilistic choice according to the common prior
- The agents receive individual signals about Nature’s choice
  - Some of Nature’s choices are “revealed” to some players, others to other players
  - The players receive no other information
    - In particular, they cannot see each other’s moves
Example

- Same example as before, but translated into extensive form

  - Nature randomly chooses MP, sends signal \( I_{1,1} \) to Agent 1, sends signal \( I_{2,1} \) to Agent 2

\[
\text{Nature}
\]

\[
\begin{array}{l}
\begin{array}{c|cc}
& \text{L} & \text{R} \\
\hline
\text{U} & 2, 0 & 0, 2 \\
\text{D} & 0, 2 & 2, 0 \\
\end{array}
\end{array}
\]

\[
\begin{array}{l}
\begin{array}{c|cc}
& \text{L} & \text{R} \\
\hline
\text{U} & 2, 2 & 0, 0 \\
\text{D} & 0, 0 & 1, 1 \\
\end{array}
\end{array}
\]

\[
\begin{array}{l}
\begin{array}{c|cc}
& \text{L} & \text{R} \\
\hline
\text{U} & 2, 2 & 0, 3 \\
\text{D} & 3, 0 & 1, 1 \\
\end{array}
\end{array}
\]

\[
\begin{array}{l}
\begin{array}{c|cc}
& \text{L} & \text{R} \\
\hline
\text{U} & 2, 1 & 0, 0 \\
\text{D} & 0, 0 & 1, 2 \\
\end{array}
\end{array}
\]
Extensions

- The definition in section 7.1.2 can be extended to include the following:
  - Players sometimes get information about each other’s moves
  - Nature makes choices and sends signals throughout the game
- This allows us to model Backgammon and Bridge
Bridge

- At the start of the game, Nature makes one move
  - The deal of the cards
- Nature signals to each player what that player’s cards are
- Each player can always see the other players’ moves
  - But imperfect information, since the players can’t see each others’ hands
Backgammon

- Nature makes choices throughout the game
  - The random outcomes of the dice rolls
- Nature reveals its choices to both players
  - Both players can see the dice
- Both players always see each other’s moves of checkers
- Hence, perfect information
7.1.3 Definition Based on Epistemic Types

- **Epistemic types**
  - Recall that we can assume the only thing players are uncertain about is the game’s utility function
  - Thus we can define uncertainty directly over a game’s utility function

- **Definition 7.1.2**: a **Bayesian game** is a tuple \((N, A, \Theta, p, u)\) where:
  - \(N\) is a set of agents;
  - \(A = A_1 \times \ldots \times A_n\), where \(A_i\) is the set of actions available to player \(i\);
  - \(\Theta = \Theta_1 \times \ldots \times \Theta_n\), where \(\Theta_i\) is the type space of player \(i\);
  - \(p : \Theta \rightarrow [0, 1]\) is a common prior over types; and
  - \(u = (u_1, \ldots, u_n)\), where \(u_i : A \times \Theta \rightarrow \mathbb{R}\) is the utility function for player \(i\)

- All this is common knowledge among the players
  - And each agent knows its own type
Types

- An agent’s type consists of all the information it has that isn’t common knowledge, e.g.,
  - The agent’s actual payoff function
  - The agent’s beliefs about other agents’ payoffs,
  - The agent’s beliefs about their beliefs about his own payoff
  - Any other higher-order beliefs
Example

- Agent 1’s possible types: $\theta_{1,1}$ and $\theta_{1,2}$
- 1’s type is $\theta_{1,j} \iff$ 1’s info set is $I_{1,j}$
- Agent 2’s possible types: $\theta_{2,1}$ and $\theta_{2,2}$
- 2’s type is $\theta_{2,j} \iff$ 2’s info set is $I_{2,j}$
- Joint distribution on the types:
  \[
  \begin{align*}
  \Pr[\theta_{1,1}, \theta_{2,1}] &= 0.3; \quad \Pr[\theta_{1,1}, \theta_{2,2}] = 0.1 \\
  \Pr[\theta_{1,2}, \theta_{2,1}] &= 0.2; \quad \Pr[\theta_{1,2}, \theta_{2,2}] = 0.4
  \end{align*}
  \]
- Conditional probabilities for agent 1:
  \[
  \begin{align*}
  \Pr[\theta_{2,1} \mid \theta_{1,1}] &= 0.3/(0.3 + 0.1) = 3/4; \quad \Pr[\theta_{2,2} \mid \theta_{1,1}] = 0.1/(0.3 + 0.1) = 1/4 \\
  \Pr[\theta_{2,1} \mid \theta_{1,2}] &= 0.2/(0.2 + 0.4) = 1/3; \quad \Pr[\theta_{2,2} \mid \theta_{1,2}] = 0.4/(0.2 + 0.4) = 2/3
  \end{align*}
  \]
Example (continued)

- The players’ payoffs depend on both their types and their actions
  - The types determine what game it is
  - The actions determine the payoff within that game
Strategies

- In principle, we could use any of the three definitions of a Bayesian game
  - The book uses the 3rd one (epistemic types)
- Strategies are similar to what we had in imperfect-information games
  - A pure strategy for player $i$ maps each of $i$’s types to an action
    - what $i$ would play if $i$ had that type
  - A mixed strategy $s_i$ is a probability distribution over pure strategies
    - $s_i(a_i | \theta_j) = \Pr[i \text{ plays action } a_j | i \text{'s type is } \theta_j]$

- Three kinds of expected utility: ex post, ex interim, and ex ante
  - Depend on what we know about the players’ types
- We mainly consider ex ante in this class (which is simpler than others)
- A type profile is a vector $\theta = (\theta_1, \theta_2, \ldots, \theta_n)$ of types, one for each agent
  - $\theta_{-i} = (\theta_1, \theta_2, \ldots, \theta_{i-1}, \theta_{i+1}, \ldots, \theta_n)$
  - $\theta = (\theta_i, \theta_{-i})$
Expected Utility

- Three different kinds of expected utility, depending on what we know about the agents’ types
- If we know every agent’s type (i.e., the type profile $\theta$)
  - agent $i$’s **ex post** expected utility:
    \[
    EU_i(s, \theta) = \sum_a \Pr[a \mid s, \theta] u_i(a, \theta) = \sum_a \left( \prod_{j \in N} s_j(a_j \mid \theta_j) \right) u_i(a, \theta)
    \]
- If we only know the common prior
  - agent $i$’s **ex ante** expected utility:
    \[
    EU_i(s) = \sum_{\theta} \Pr[\theta] \cdot EU_i(s, \theta) = \sum_{\theta_i} \Pr[\theta_i] \cdot EU_i(s, \theta_i)
    \]
- If we know the type $\theta_i$ of one agent $i$, but not the other agents’ types
  - $i$’s **ex interim** expected utility:
    \[
    EU_i(s, \theta_i) = \sum_{\theta_{-i}} \Pr[\theta_{-i} \mid \theta_i] \cdot EU_i(s, (\theta_i, \theta_{-i}))
    \]
Bayes-Nash Equilibria

- Given a strategy profile $s_{-i}$, a **best response** for agent $i$ is a strategy $s_i$ such that

$$s_i \in \operatorname{arg\, max}(EU_i(s'_i, s_{-i}))$$

- Above, the set notation is because more than one strategy may produce the same expected utility.

- A **Bayes-Nash** equilibrium is a strategy profile $s$ such that for every $s_i$ in $s$, $s_i$ is a best response to $s_{-i}$.
  
  - Just like the definition of a Nash equilibrium, except that we’re using Bayesian-game strategies.
Computing Bayes-Nash Equilibria

- The idea is to construct a payoff matrix for the entire Bayesian game, and find equilibria on that matrix.

- First, write each of the pure strategies as a list of actions, one for each type.

  - **Agent 1’s pure strategies:**
    - UU: U if type $\theta_{1,1}$, U if type $\theta_{1,2}$
    - UD: U if type $\theta_{1,1}$, D if type $\theta_{1,2}$
    - DU: D if type $\theta_{1,1}$, U if type $\theta_{1,2}$
    - DD: D if type $\theta_{1,1}$, D if type $\theta_{1,2}$

  - **Agent 2’s pure strategies:**
    - LL: L if type $\theta_{2,1}$, L if type $\theta_{2,2}$
    - LR: L if type $\theta_{2,1}$, R if type $\theta_{2,2}$
    - RL: R if type $\theta_{2,1}$, L if type $\theta_{2,2}$
    - RR: R if type $\theta_{2,1}$, R if type $\theta_{2,2}$
Next, compute the *ex ante* expected utility for each pure-strategy profile

- e.g., (note that $\theta$, UU, and LL determine dots)

$$EU_2(UU, LL) = \sum_\theta \Pr[\theta]u_2(\ldots, \theta)$$

$$= \Pr[\theta_{1,1}, \theta_{2,1}]u_2(U, L, \theta_{1,1}, \theta_{2,1})$$

$$+ \Pr[\theta_{1,1}, \theta_{2,2}]u_2(U, L, \theta_{1,1}, \theta_{2,2})$$

$$+ \Pr[\theta_{1,2}, \theta_{2,1}]u_2(U, L, \theta_{1,2}, \theta_{2,1})$$

$$+ \Pr[\theta_{1,2}, \theta_{2,2}]u_2(U, L, \theta_{1,2}, \theta_{2,2})$$

$$= 0.3(0) + 0.1(2) + 0.2(2) + 0.4(1)$$

$$= 1$$
Computing Bayes-Nash Equilibria (continued)

- Put all of the *ex ante* expected utilities into a payoff matrix
  - e.g., $EU_2(UU,LL) = 1$
- Now we can compute best responses and Nash equilibria

![Payoff Matrix](image)

- **MP** ($p = 0.3$)
  - $U$: 2, 0, 0, 2
  - $D$: 0, 2, 2, 0
- **PD** ($p = 0.1$)
  - $U$: 2, 2, 0, 3
  - $D$: 3, 0, 1, 1
- **Crd** ($p = 0.2$)
  - $U$: 2, 2, 0, 0
  - $D$: 0, 0, 1, 1
- **BoS** ($p = 0.4$)
  - $U$: 2, 1, 0, 0
  - $D$: 0, 0, 1, 2
Summary

- Incomplete information vs. imperfect information
- Incomplete information vs. uncertainty about payoffs
- Bayesian games (three different definitions)
  - Changing uncertainty about games into uncertainty about payoffs
  - *Ex ante, ex interim, and ex post* utilities
  - Bayes-Nash equilibria
- Bayesian-game interpretations of Bridge and Backgammon
- Base-Nash instead of Nash