CMSC 474, Introduction to Game Theory

Coalition Game Theory

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Coalitional Games with Transferable Utility

- Given a set of agents, a coalitional game defines how well each group (or **coalition**) of agents can do for itself—its payoff
  - Not concerned with
    - how the agents make individual choices within a coalition,
    - how they coordinate, or
    - any other such detail
- **Transferable utility** assumption: the payoffs to a coalition may be freely redistributed among its members
  - Satisfied whenever there is a universal **currency** that is used for exchange in the system
  - Implies that each coalition can be assigned a single value as its payoff

Coalitional Games with Transferable Utility

- A **coalitional game with transferable utility** is a pair $G = (N,v)$, where
  - N = {1, 2, …, *n*} is a finite set of players
  - **(nu)** $v : 2^N \rightarrow \Re$ associates with each coalition $S \subseteq N$ a real-valued payoff $v(S)$, that the coalition members can distribute among themselves
- *v* is the **characteristic function**
  - We assume $v(\varnothing) = 0$
- A coalition's payoff is also called its **worth**
- Coalitional game theory is normally used to answer two questions:

  (1) Which coalition will form?

  (2) How should that coalition divide its payoff among its members?
- The answer to (1) is often "the grand coalition" (all of the agents)
  - But this answer can depend on making the right choice about (2)

Example: A Voting Game

- Consider a parliament that contains 100 representatives from four political parties:
  - A (45 reps.), B (25 reps.), C (15 reps.), D (15 reps.)
- They're going to vote on whether to pass a $100 million spending bill| (and how much of it should be controlled by each party)
- Need a majority ($\geq$ 51 votes) to pass legislation
  - If the bill doesn't pass, then every party gets 0
- More generally, a voting game would include
  - a set of agents N
  - a set of *winning* coalitions $W \subseteq 2^N$
    - In the example, all coalitions that have enough votes to pass the bill
  - $v(S) = 1$ for each coalition $S \in W$
    - Or equivalently, we could use $v(S)$ = $100 million
  - $v(S) = 0$ for each coalition $S \notin W$

Superadditive Games

- A coalitional game $G = (N,v)$ is **superadditive** if the union of two disjoint coalitions is worth at least the sum of its members' worths
  - for all $S, T \subseteq N$, if $S \cap T = \varnothing$, then $v(S \cup T) \geq v(S) + v(T)$
- The voting-game example is superadditive
  - If $S \cap T = \varnothing$, $v(S) = 0$, and $v(T) = 0$, then $v(S \cup T) \geq 0$
  - If $S \cap T = \varnothing$ and $v(S) = 1$, then $v(T) = 0$ and $v(S \cup T) = 1$
  - Hence $v(S \cup T) \geq v(S) + v(T)$
- If G is superadditive, the grand coalition always has the highest possible payoff
  - For any $S \neq N$, $v(N) \geq v(S) + v(N-S) \geq v(S)$
- $G = (N,v)$ is **additive** (or **inessential**) if
  - For $S, T \subseteq N$ and $S \cap T = \varnothing$, then $v(S \cup T) = v(S) + v(T)$

Constant-Sum Games

- G is **constant-sum** if the worth of the grand coalition equals the sum of the worths of any two coalitions that partition N
  - $v(S) + v(N - S) = v(N)$, for every $S \subseteq N$

- Every additive game is constant-sum
  - additive => $v(S) + v(N - S) = v(S \cup (N - S)) = v(N)$
- But not every constant-sum game is additive
  - Example is a good exercise

Convex Games

- G is **convex (supermodular)** if for all $S,T \subseteq N$,
    - $v(S \cup T) + v(S \cap T) \geq v(S) + v(T)$
- It can be shown the above definition is equivalent to for all i in N and for all $S \subseteq T \subseteq N\text{-}\{i\}$,
    - $v(\mathrm{T} \cup \{i\})\text{-} v(\mathrm{T}) \geq v(S \cup \{i\}) - v(S)$
    - Prove it as an exercise
- Recall the definition of a superadditive game:
    - for all $S,T \subseteq N$, if $S \cap T = \varnothing$, then $v\ (S \cup T) \geq v\ (S\ ) + v\ (T\ )$
- It follows immediately that every super-additive game is a convex game

Simple Coalitional Games

- A game $G = (N, v)$ is **simple** for every coalition S,
    - either $v(S) = 1$ (i.e., S **wins**) or $v(S) = 0$ (i.e., S **loses**)
  - Used to model voting situations (e.g., the example earlier)
- Often add a requirement that if S wins, all supersets of S would also win:
    - if $v(S) = 1$, then for all $T \supseteq S$, $v(T) = 1$

- This doesn't quite imply superadditivity
  - Consider a voting game G in which 50% of the votes is sufficient to pass a bill
  - Two coalitions S and T, each is exactly 50% N
  - $v(S) = 1$ and $v(T) = 1$
  - But $v(S \cup T) \neq 2$

Proper-Simple Games

- G is a **proper simple game** if it is both simple and constant-sum
  - If S is a winning coalition, then $N - S$ is a losing coalition
    - $v(S) + v(N - S) = 1$, so if $v(S) = 1$ then $v(N - S) = 0$

- Relations among the classes of games:

$$\{\text{Additive games}\} \subseteq \{\text{Super-additive games}\} \subseteq \{\text{Convex games}\}$$

$$\{\text{Additive games}\} \subseteq \{\text{Constant-sum game}\}$$

$$\{\text{Proper-simple games}\} \subseteq \{\text{Constant-sum games}\}$$

$$\{\text{Proper-simple games}\} \subseteq \{\text{Simple game}\}$$

Analyzing Coalitional Games

- Main question in coalitional game theory
  - How to divide the payoff to the grand coalition?
- Why focus on the grand coalition?
  - Many widely studied games are super-additive
    - Expect the grand coalition to form because it has the highest payoff
  - Agents may be required to join
    - E.g., public projects often legally bound to include all participants
- Given a coalitional game $G = (N, v)$, where $N = \{1, \ldots, n\}$
  - We'll want to look at the agents' shares in the grand coalition's payoff
    - The book writes this as **(Psi)** $\psi(N,v) = \mathbf{x} = (x_1, \ldots, x_n)$, where $\psi_i(N,v) = x_i$ is the agent's payoff
  - We won't use the ψ notation much
    - Can be useful for talking about several different coalitional games at once, but we usually won't be doing that

Terminology

- **Feasible payoff set**

  = {all payoff profiles that don't distribute more than the worth of the grand coalition}

  = $\{(x_1, \ldots, x_n) \mid x_1 + x_2 + \ldots + x_n\} \leq v(N)$

- **Pre-imputation set**

  P = {feasible payoff profiles that are **efficient**, i.e., distribute the entire worth of the grand coalition}

  = $\{(x_1, \ldots, x_n) \mid x_1 + x_2 + \ldots + x_n\} = v(N)$

- **Imputation set**

  C = {payoffs in P in which each agent gets at least what he/she would get by going alone (i.e., forming a singleton coalition)}

  = $\{(x_1, \ldots, x_n) \in \mathsf{P} : \forall i \in N, x_i \geq v(\{i\})\}$

im•pute: verb [trans.] represent as being done, caused, or possessed by someone; attribute *: the crimes* ***imputed to*** *Richard.*

Fairness, Symmetry

- What is a **fair** division of the payoffs?
  - Three axioms describing fairness
    - *Symmetry*, *dummy player*, and *additivity* axioms

- Definition: agents i and j are **interchangeable** if they always contribute the same amount to every coalition of the other agents
  - i.e., for every S that contains neither i nor j, $v(S \cup \{i\}) = v(S \cup \{j\})$

- **Symmetry axiom**: in a fair division of the payoffs, interchangeable agents should receive the same payments, i.e.,
  - if i and j are interchangeable and $(x_1, \ldots, x_n)$ is the payoff profile, then $x_i = x_j$

Dummy Players

- Agent i is a **dummy player** if i's contributes to any coalition is exactly the amount i can achieve alone
  - i.e., for all S s.t. $i \notin S$, $v(S \cup \{i\}) = v(S) + v(\{i\})$

- **Dummy player axiom**: in a fair distribution of payoffs, dummy players should receive payment equal to the amount they achieve on their own
  - i.e., if i is a dummy player and $(x_1, \ldots, x_n)$ is the payoff profile, then $x_i = v(\{i\})$

Additivity

- Let $G_1 = (N, v_1)$ and $G_2 = (N, v_2)$ be two coalitional games with the same agents
- Consider the combined game $G = (N, v_1 + v_2)$, where
  - $(v_1 + v_2)(S) = v_1(S) + v_2(S)$

- **Additivity axiom**: in a fair distribution of payoffs for G, the agents should get the sum of what they would get in the two separate games
  - i.e., for each player i, $\psi_i(N, v_1 + v_2) = \psi_i(N, v_1) + \psi_i(N, v_2)$