CMSC 474, Introduction to Game Theory

More Auctions

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First-Price Sealed-Bid Auctions

- Examples:
  - construction contracts (lowest bidder)
  - real estate
  - art treasures

- Typical rules
  - Bidders write their bids for the object and their names on slips of paper and deliver them to the auctioneer
  - The auctioneer opens the bid and finds the highest bidder
  - The highest bidder gets the object being sold, for a price equal to his/her own bid
  - Winner’s profit = BV – price paid
  - Everyone else’s profit = 0
First-Price Sealed-Bid (continued)

- Suppose that
  - There are $n$ bidders
  - Each bidder has a private valuation, $v_i$, which is private information
  - But a probability distribution for $v_i$ is common knowledge
    - Let’s say $v_i$ is uniformly distributed over [0, 100]
  - Let $B_i$ denote the bid of player $i$
  - Let $\pi_i$ denote the profit of player $i$

- What is the Nash equilibrium bidding strategy for the players?
  - Need to find the optimal bidding strategies

- First we’ll look at the case where $n = 2$
Finding the optimal bidding strategies

- Let $B_i$ be agent $i$’s bid, and $\pi_i$ be agent $i$’s profit
- If $B_i \geq v_i$, then $\pi_i \leq 0$
  - So, assuming rationality, $B_i < v_i$
- Thus
  - $\pi_i = 0$ if $B_i \neq \max_j \{B_j\}$
  - $\pi_i = v_i - B_i$ if $B_i = \max_j \{B_j\}$

- How much below $v_i$ should your bid be?
- The smaller $B_i$ is,
  - the less likely that $i$ will win the object
  - the more profit $i$ will make if $i$ wins the object
First-Price Sealed-Bid (continued)

- **Case** \( n = 2 \)
  - Suppose your BV is \( v \) and your bid is \( B \)
  - Let \( x \) be the other bidder’s BV and \( \alpha x \) be his/her bid, where \( 0 < \alpha < 1 \)
    - You don’t know the values of \( x \) and \( \alpha \)
  - Your expected profit is
    - \( E(\pi) = P(\text{your bid is higher}) \cdot (v - B) + P(\text{your bid is lower}) \cdot 0 \)

- If \( x \) is uniformly distributed over \([0, 100]\), then the probability distribution function (pdf) is \( f(x) = 1/100, 0 \leq x \leq 100 \)
  - \( P(\text{your bid is higher}) = P(\alpha x < B) = P(x < B/\alpha) = \int_{0}^{B/\alpha} (1/100) \, dx = B/100\alpha \)
  - so \( E(\pi) = B(v - B)/100\alpha \)

- If you want to maximize your expected profit (hence your valuation of money is risk-neutral), then your maximum bid is
  - \( \max_{B} B(v - B)/100\alpha = \max_{B} B(v - B) = \max_{B} Bv - B^2 \)
    - maximum occurs when \( v - 2B = 0 \) \( \Rightarrow \) \( B = v/2 \)

- So, bid \( 1/2 \) of what the item is worth to you!
First-Price Sealed-Bid (continued)

- With $n$ bidders, if your bid is $B$, then
  - $P(\text{your bid is the highest}) = \left(\frac{B}{100\alpha}\right)^{n-1}$

- Assuming risk neutrality, you choose your bid to be
  - $\max_B B^{n-1}(v-B) = v(n-1)/n$

- As $n$ increases, $B \rightarrow v$
  - I.e., increased competition drives bids close to the valuations
Dutch Auctions

- **Examples**
  - flowers in the Netherlands, fish market in England and Israel, tobacco market in Canada

- **Typical rules**
  - Auctioneer starts with a high price
  - Auctioneer lowers the price gradually, until some buyer shouts “Mine!”
  - The first buyer to shout “Mine!” gets the object at the price the auctioneer just called
  - Winner’s profit $= BV - \text{price}$
  - Everyone else’s profit $= 0$

- **Dutch auctions are game-theoretically equivalent to first-price, sealed-bid auctions**
  - The object goes to the highest bidder at the highest price
  - A bidder must choose a bid without knowing the bids of any other bidders
  - The optimal bidding strategies are the same
Sealed-Bid, Second-Price Auctions

- **Background:** Vickrey (1961)
- **Used for**
  - stamp collectors’ auctions
  - US Treasury’s long-term bonds
  - Airwaves auction in New Zealand
  - eBay and Amazon
- **Typical rules**
  - Bidders write their bids for the object and their names on slips of paper and deliver them to the auctioneer
  - The auctioneer opens the bid and finds the highest bidder
  - The highest bidder gets the object being sold, for a price equal to the *second highest* bid
- **Winner’s profit** = BV – price
- **Everyone else’s profit** = 0
Sealed-Bid, Second-Price (continued)

- Equilibrium bidding strategy:
  - It is a weakly dominant strategy to bid your true value: This property is also called truthfulness or strategyproofness of an auction.
- To show this, need to show that overbidding or underbidding cannot increase your profit and might decrease it.
- Let $V$ be your valuation of the object, and $X$ be the highest bid made by anyone else.
- Let $s_V$ be the strategy of bidding $V$, and $\pi_V$ be your profit when using $s_V$.
- Let $s_B$ be a strategy that bids some $B \neq V$, and $\pi_B$ be your profit when using $s_B$.
- There are $3! = 6$ possible numeric orderings of $B$, $V$, and $X$:
  - Case 1, $X > B > V$: You don’t get the commodity either way, so $\pi_B = \pi_V = 0$.
  - Case 2, $B > X > V$: $\pi_B = V - X < 0$, but $\pi_V = 0$.
  - Case 3, $B > V > X$: you pay $X$ rather than your bid, so $\pi_B = \pi_V = V - X > 0$.
  - Case 4, $X < B < V$: you pay $X$ rather than your bid, so $\pi_B = \pi_V = V - X > 0$.
  - Case 5, $B < X < V$: $\pi_B = 0$, but $\pi_V = V - X > 0$.
  - Case 6, $B < V < X$: You don’t get the commodity either way, so $\pi_B = \pi_V = 0$. 

Sealed-Bid, Second-Price (continued)

- Sealed-bid, 2nd-price auctions are nearly equivalent to English auctions
  - The object goes to the highest bidder
  - Price is close to the second highest BV
Summary

- Auctions and their equilibria
  - English
  - Dutch
  - Sealed bid, first price
  - Sealed bid, second price (Vickrey)