CMSC 474, Introduction to Game Theory

More Auctions

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First-Price Sealed-Bid Auctions

- Examples:
  - construction contracts (lowest bidder)
  - real estate
  - art treasures
- Typical rules
  - Bidders write their bids for the object and their names on slips of paper and deliver them to the auctioneer
  - The auctioneer opens the bid and finds the highest bidder
  - The highest bidder gets the object being sold, for a price equal to his/her own bid
  - Winner's profit = BV– price paid
  - Everyone else's profit = 0

First-Price Sealed-Bid (continued)

- Suppose that
  - There are n bidders
  - Each bidder has a private valuation, v_i, which is private information
  - But a probability distribution for v_i is common knowledge
    - Let's say v_i is uniformly distributed over [0, 100]
  - Let B_i denote the bid of player i
  - Let π_i denote the profit of player i

- What is the Nash equilibrium bidding strategy for the players?
  - Need to find the optimal bidding strategies

- First we'll look at the case where $n = 2$

First-Price Sealed-Bid (continued)

- Finding the optimal bidding strategies
  - Let B_i be agent i's bid, and π_i be agent i's profit
  - If $B_i \geq v_i$, then $\pi_i \leq 0$
    - So, assuming rationality, $B_i < v_i$
  - Thus
    - $\pi_i = 0$ if $B_i \neq \max_j \{B_j\}$
    - $\pi_i = v_i - B_i$ if $B_i = \max_j \{B_j\}$
  - How much below v_i should your bid be?
  - The smaller B_i is,
    - the less likely that i will win the object
    - the more profit i will make if i wins the object

First-Price Sealed-Bid (continued)

- Case $n = 2$
  - Suppose your BV is v and your bid is B
  - Let x be the other bidder's BV
    and αx be his/her bid, where $0 < \alpha < 1$
    - You don't know the values of x and α
  - Your expected profit is
    - $E(\pi) = P(\text{your bid is higher}) \cdot (v - B) + P(\text{your bid is lower}) \cdot 0$
- If x is uniformly distributed over [0, 100], then the probability distribution function (pdf) is $f(x) = 1/100,\ 0 \le x \le 100$
  - $P(\text{your bid is higher}) = P(\alpha x < B) = P(x < B/\alpha) = \int_0^{B/\alpha} (1/100)\, dx = B/100\alpha$
  - so $E(\pi) = B(v - B)/100\alpha$
- If you want to maximize your expected profit (hence your valuation of money is risk-neutral), then your maximum bid is
    - $\max_B B(v-B)/100\alpha = \max_B B(v-B) = \max_B Bv - B^2$
    - maximum occurs when $v - 2B = 0 \;\Rightarrow\; B = v/2$
- So, bid ½ of what the item is worth to you!

First-Price Sealed-Bid (continued)

- With n bidders, if your bid is B, then
  - P(your bid is the highest) = $(B/100\alpha)^{n-1}$

- Assuming risk neutrality, you choose your bid to be
  - $\max_B B^{n-1}(v-B) = v(n-1)/n$
- As n increases, $B \rightarrow v$
  - I.e., increased competition drives bids close to the valuations

Dutch Auctions

- Examples
  - flowers in the Netherlands, fish market in England and Israel, tobacco market in Canada
- Typical rules
  - Auctioneer starts with a high price
  - Auctioneer lowers the price gradually, until some buyer shouts “Mine!”
  - The first buyer to shout “Mine!” gets the object at the price the auctioneer just called
  - Winner’s profit = BV – price
  - Everyone else’s profit = 0
- Dutch auctions are game-theoretically equivalent to first-price, sealed-bid auctions
  - The object goes to the highest bidder at the highest price
  - A bidder must choose a bid without knowing the bids of any other bidders
  - The optimal bidding strategies are the same

Sealed-Bid, Second-Price Auctions

- Background: Vickrey (1961)
- Used for
  - stamp collectors' auctions
  - US Treasury's long-term bonds
  - Airwaves auction in New Zealand
  - eBay and Amazon
- Typical rules
  - Bidders write their bids for the object and their names on slips of paper and deliver them to the auctioneer
  - The auctioneer opens the bid and finds the highest bidder
  - The highest bidder gets the object being sold, for a price equal to the *second highest* bid
- Winner's profit = BV – price
- Everyone else's profit = 0

Sealed-Bid, Second-Price (continued)

- Equilibrium bidding strategy:
  - It is a weakly dominant strategy to bid your true value: This property is also called **truthfulness** or **strategyproofness** of an auction.
- To show this, need to show that overbidding or underbidding cannot increase your profit and might decrease it.
- Let V be your valuation of the object, and X be the highest bid made by anyone else
- Let s_V be the strategy of bidding V, and π_V be your profit when using s_V
- Let s_B be a strategy that bids some $B \neq V$, and π_B be your profit when using s_B
- There are 3! = 6 possible numeric orderings of B, V, and X:
  - Case 1, $X > B > V$: You don't get the commodity either way, so $\pi_B = \pi_V = 0$.
  - Case 2, $B > X > V$: $\pi_B = V - X < 0$, but $\pi_V = 0$
  - Case 3, $B > V > X$: you pay X rather than your bid, so $\pi_B = \pi_V = V - X > 0$
  - Case 4, $X < B < V$: you pay X rather than your bid, so $\pi_B = \pi_V = V - X > 0$
  - Case 5, $B < X < V$: $\pi_B = 0$, but $\pi_V = V - X > 0$
  - Case 6, $B < V < X$: You don't get the commodity either way, so $\pi_B = \pi_V = 0$

Sealed-Bid, Second-Price (continued)

- Sealed-bid, 2nd-price auctions are nearly equivalent to English auctions
  - The object goes to the highest bidder
  - Price is close to the second highest BV

Summary

- Auctions and their equilibria
  - English
  - Dutch
  - Sealed bid, first price
  - Sealed bid, second price (Vickrey)