CMSC 474, Introduction to Game Theory

Linear Programming

Mohammad T. Hajiaghayi
University of Maryland

Linear Programming

- minimize or maximize a linear objective
- subject to linear equalities and inequalities

Example. Max is in a pie eating contest that lasts 1 hour. Each torte that he eats takes 2 minutes. Each apple pie that he eats takes 3 minutes. He receives 4 points for each torte and 5 points for each pie. What should Max eat so as to get the most points?

Step 1. Determine the decision variables

- Let x be the number of tortes eaten by Max.
- Let y be the number of pies eaten by Max.

Max's linear program

Step 2. Determine the *objective function*

Step 3. Determine the *constraints*

Maximize $z = 4x + 5y$ (objective function)

subject to $2x + 3y \leq 60$ (constraint)

$x \geq 0$; $y \geq 0$ (non-negativity constraints)

A ***feasible solution*** satisfies all of the constraints.

$x = 10, y = 10$ is feasible; $x = 10, y = 15$ is ***infeasible***.

An ***optimal solution*** is the best feasible solution.

The optimal solution is $x = 30, y = 0$.

Terminology

- ***Decision variables:*** e.g., x and y.
  - In general, these are quantities you can control to improve your objective which should completely describe the set of decisions to be made.
- ***Constraints:*** e.g., $2x + 3y \leq 24$, $x \geq 0$, $y \geq 0$
  - Limitations on the values of the decision variables.
- ***Objective Function.*** e.g., $4x + 5y$
  - Value measure used to rank alternatives
  - Seek to maximize or minimize this objective
  - examples: maximize NPV, minimize cost

Linear Programs

- A *linear function* is a function of the form:

$$f(x_1, x_2, \ldots, x_n) = c_1x_1 + c_2x_2 + \ldots + c_nx_n$$
$$= \sum_{i=1 \text{ to } n} c_ix_i$$

e.g., $3x_1 + 4x_2 - 3x_4$.

- A mathematical program is a *linear program (LP)* if the objective is a linear function and the constraints are linear equalities or inequalities.

e.g., $3x_1 + 4x_2 - 3x_4 \geq 7$

$x_1 - 2x_5 = 7$

- Typically, an LP has non-negativity constraints.

An integer program is a linear program plus constraints that some or all of the variables are integer valued.

- Maximize $3x_1 + 4x_2 - 3x_3$

$$
\begin{aligned}
3x_1 + 2x_2 - x_3 &\geq 17 \\
3x_2 - x_3 &= 14 \\
x_1 \geq 0,\ x_2 \geq 0,\ x_3 &\geq 0 \text{ and}
\end{aligned}
$$

x_1 , x_2, x_3 are all integers

Complexity of LP & IP

1. Simplex Method runs in exponential time.
2. Ellepsoid Method runs in $O(n^6)$ time.
3. Interior Point Method runs in $O(n^{3.5})$ time.
4. Unfortunately Integer Programming (IP) is NP - complete.