①

* Why do we study game theory? Algorithmic Game Theory: incentive-aware algorithms
  - we have selfish agents or self-interested agents, which optimize their own objective functions.
  - Goal of Mechanism Design: encourage selfish agents to act socially by designing rewarding rules such that when agents optimize their own objective, a social objective is met.

Note: Game Theory is NOT a tool
It is a concept.

* How do we study these systems?
  - First model the system (usually a network) as a game
  - We analyze equilibrium points and then compare the social value of equi. points to global optimum
* An equilibrium point or just equil. is a state in which no person involved in the game wants any change. more precisely an equib. is simply a state of the world where economic forces are balanced and in the absence of external influences the (equib) values of economic variables will not change.

- Two Important class of Equilibrium: Nash equib and Market equib. We introduce both of them in this session (or maybe the next) and give some important results for them. (first Nash and then Market)

change the order

- Important Factors:
  - Existence of equib as a subject of study ← economics
  - Performance of the output (Approximation Factor) ← both cs & economics
  - Convergence (Running time) ← Computer Science
    OR do we converge at all?

Some important game theory concepts (some of them will be discussed in this course)

- Lack of coordination in networks and equilibrium concepts
- Price of Anarchy (or stability) in Games such as load balancing game, Selfish Routing games, Congestion games, Market sharing games, network creation games, Network formation games, etc.
- Coordination Mechanisms to obtain better price of Anarchy
- Convergence and best response dynamics and their outcomes e.g. in sink equilibri
- Market Equib. and applications e.g. in wireless networks
- Interdomain routing and stable paths problems, Gao-Rexford Conditions
- Auctions, VCG, truthfulness, Sponsored Search Auctions, Online Auctions
- Cost sharing, privacy and complexity (hardness)

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Nash equib. is a solution concept (a condition which identifies the equib) of a game involving two or more players, in which no player has anything to gain by changing only his or her own strategy unilaterally. In other words, if each player has chosen a strategy and no player can benefit by changing his or her strategy while the other players keep theirs unchanged, then the current set of strategy choices and the corresponding payoffs (profits, rewards) constitute a Nash Equib. It is named after John Nash. In this case, each player either knows the strategies of other players or can derive these.

lets first start with a two-player game (to be more formal)

Some examples first:

Example ①: prisoners' dilemma: two prisoners are on a trial for a crime and each one faces a choice of confessing to the crime or remaining silent. If they both remain silent, the authorities will not be able to prove charges against them and they will both serve a short prison term, say 2 years, for minor offenses. If only one confesses, he will get a reduced 1 year prison and he will be used as a witness against the other who will get 5 years. Finally if they both confesses, they both will get a small break of cooperating with the authorities and will have to serve prison sentences of 4 years each (rather than 5). we can succinctly summarize the costs incurred in these four outcomes via the following two-by-two matrix, which is called a cost matrix because it contains the cost incurred by the players for each choice of their strategies.

P_1 \ P_2	confess	silent
confess	4, 4	1, 5
silent	5, 1	2, 2

[Expressing games in this form is the standard form or the matrix form is good for small # player and # strategies; however not good for big games thus we use the implicit form]

note → (confess, confess) is the only stable solution; in each of the other three cases, at least one of the players can switch from silent to confess and improve his own payoff. on the other hand the social optimum choice is the (silent, silent) case, which is not stable.

Prisoners' Dilemma arise naturally in a lot of different situations with many players (see ISP games in chap 1 of the book of Nisan et al.)

Example ② we might have multiple outcomes which are stable

Battle of sexes: consider two players, a boy and a girl, are deciding on how to spend their evening. They both consider two possibilities: going to a baseball game or going to a softball game: the boy prefers baseball and the girl prefers softball, but they both would like to spend the evening together rather than

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③

separately. The payoffs (benefits) are as follows. whether it is softball or

Girl \ Boy	B	S
B	5, 6	1, 1
S	2, 2	6, 5

Here both attending the same game, whether it is softball or baseball, are both stable solutions.

This Battle of the Sexes is an example of coordination games arise naturally in many contexts such as the context of routing to avoid congestion (see the book)

Example ③: not all games has stable outcomes in the sense that none of players would want to individually deviate from such an outcome.

Matching Pennies Game: Two players, each having a penny, are asked to chose from among two strategies - heads (H) and tails (T). The row player (player 1) wins if the two pennies match while the column player (player 2) wins if they do not match

1 \ 2	H	T
H	1, -1	-1, 1
T	-1, 1	1, -1

1 : win
-1 : loss

It is easy to see that this game has no stable solution and the best for the players to randomize (with prob $\frac{1}{2}$) between strategies (Randomized (mixed) strategie

All examples ①, ②, and ③ are one-shot simultaneous move games, in that all players simultaneously chose an action from their set of possible strategies. **We might have repeated Games.**

☆ Formally, we have n player $\{1, 2, \ldots, n\}$. Each player i has his own set of possible strategies, say S_i. To play the game, each player i selects a strategy $s_i \in S_i$. We will use $s = (s_1, \ldots, s_n)$ to denote the vector of strategies selected by the players and $S = \times_i S_i$ denote the set of all strategies, each has a payoff for each player, i.e. $u_i(s)$ where $s \in S$ is a vector.

☆ We say a game has a dominant strategy solution if each player has a unique best strategy, independent of the strategies played by the other players. More formally for a strategy vector $s \in S$ we use s_i to denote the strategy played by player i and s_{-i} to denote the $(n-1)$-dimensional vector of the strategies played by all other player

— We say a strategy vector $s \in S$ is a dominant strategy solution if for each player i, and each alternate strategy vector $s' \in S$, we have that $u_i(s_i, s'_{-i}) \geq u_i(s'_i, s'_{-i})$. For example, Prisoners' Dilemma has a solution that always confess, but note that a dominant strategy solution may not give an optimal payoff to any of the players.

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- since games rarely possess dominant strategy solutions, we need to seek a
- less stringent and more widely applicable solution concept. (see e.g. the Battle of Sexes game)
- Nash equib. captures the notion of a stable solution:

A strategy vector $s \in S$, is said to be a Nash equib. if for all players i and each alternate strategy $s_i' \in S_i$, we have that

$$u_i(s_i, s_{-i}) \geq u_i(s_i', s_{-i})$$

Note that a solution is self-enforcing in the sense that once the players are playing such a solution, it is in every player's best interest to stick to his or her strategy.

Also clearly, a dominant strategy solution is a Nash Equib.

we may have several Nash equib. e.g. in the battle of sexes game.

★ mixed strategy Nash Equib.:

So far every thing was deterministic and thus called pure strategy equib

but matching Pennies game did not possess any pure Nash equib.

If each player picks each of his two strategies with prob. $\frac{1}{2}$, then we obtain a stable solution in a sense, since the expected payoff of each player now is 0 and neither player can improve on this by a choosing a different randomization.

To define randomized strategies formally, let us enhance the choices of players so each one can pick a probability distribution over his set of possible strategies, such a choice is called a mixed strategy

we assume players independently select strategies using the probability distribution which leads to a probability distribution of strategy vectors S.

Thm (Nash 1951): Any game with a finite set of players and finite set of strategies has a Nash equib of mixed strategies.

If we do not have finite sets, there is not a mixed Nash necessarily (see the pricing game in the book chap).

The price of anarchy (PoA) is the most popular measure of the inefficiency of equib. is defined as the ratio between the worst objective function value of an equi of the game and that of an optimal outcome (social optimum).

We are interested in a price of anarchy which is close to 1, i.e., all equib. are good approximations of an optimal solution.