A game with multiple equi. has a large price of anarchy even if only one of its equilibria is highly inefficient. The price of stability (PoS) is a measure of inefficiency designed to differentiate between games in which all equi. are inefficient and those in which some equi is inefficient. Formally the price of stability of a game is the ratio between the best objective function value of one of its equilibium and that of an optimal outcome.

In games with a unique equi. PoA = PoS

A bound on PoS is much weaker than a bound on PoA

Two Reasons for PoS: 1- in some games a nontrivial bound is possible only for PoS. (PoA is very high)

2- if we envision the outcome as being initially designed by a central authority for subsequent use by selfish players, then the best equi. is an obvious solution to the propose.

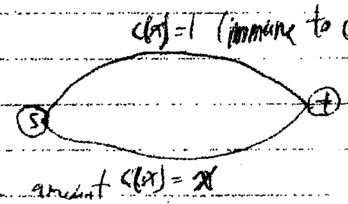
We see price of ~~Anarchy~~ Stability and more PoA in the next sessions

In our class, we consider pure Nash Equi., which are Nash Equi in which there is no randomization involved (vs. mixed Nash Equi where the players are choosing their strategies randomly)

The inefficiency of equib. cannot be bounded in general, a natural goal is to identify the classes of games in which equi. are guaranteed to be approximately optimal (which happens for lots of classes)

In this session, we consider selfish Routing.

Pigou's Example (1920)



cost function c(x) describes the cost (e.g. travel time) incurred by users at the edge, as a function of the amount of traffic routed on the edge.

Suppose there is one unit of traffic, representing a very large population of players, and that each player chooses independently between the two routes from s to t. Assuming that each player aims to minimize its cost, the lower route is a dominant strategy.

②

In the unique equib, all players follow this strateg, and all of them incur one unit of cost.

Assume that the objective function is to minimize the average cost incurred by players. In the above equal, this average cost is 1.

However splitting the traffic equally between the two links is the optimal solution with delay $\frac{1}{2}\times 1+\frac{1}{2}\times\frac{1}{2}=\frac{3}{4}$. Thus $PoS = PoA = \frac{1}{\frac{3}{4}} = \frac{4}{3}$

General selfish routing games are conceptually similar to Pigou's example, but more complex in several respects (arbitrary large directed graphs, different source-sink pairs, edge cost functions can be arbitrary nonnegative, continuous and non-decreasing functions). However even in the general case $PoA = PoS$.

★ We show that $\frac{4}{3}$ is indeed the PoA for all non-atomic selfish routings

In general, we have a directed network together with a set $(s_1,t_1),\ldots,(s_k,t_k)$ of source-target pairs, called commodities. Each player is identified with one commodity, we use P_i to denote the s_i-t_i paths of a network and $P = \bigcup_{i=1}^{k} P_i$. We allow parallel edges, and a vertex can participate in multiple source-sink pairs.

For a flow f and a path $P\in P_i$, we interpret f_P as the amount of traffic of commodity i that chooses the path P to travel from s_i to t_i.

There is demand d_i that needs to be satisfied, i.e., $\sum_{P\in P_i} f_P = d_i$

There is a cost function $c_e: R^+ \to R^+$ which is non-negative, continuous and non-decreasing

★ A non-atomic selfish routing instance is (G, d, c) → cost
network, demand

Let the cost of a path P with respect to a flow f as the sum of the costs of the constituent edges: $c_P(f) = \sum_{e\in P} c_e(f_e)$ where $f_e = \sum_{P\in P: e\in P} f_P$ (the flow on edge)

Def 1: Nonatomic equilibrium flow: Let f be a feasible flow for the nonatomic instance (G, d, c). The flow f is an equi flow, if for every commodity $i\in\{1,2,\ldots,k\}$ and every pair $P, \tilde{P} \in P_i$ of s_i-t_i paths with $f_P > 0$, $c_P(f) \le c_{\tilde{P}}(f)$.

In other words, all paths in use by an equi. flow f have minimum possible cost (see Pigou's example in which only one path carries flow)

We consider the objective of minimizing the total cost incurred by traffic. We define the cost of a flow f as $C(f) = \sum_{P \in \mathcal{P}} c_P(f) f_P = \sum_{e \in E} c_e(f_e) \cdot f(e)$ (see footnote ①)

An optimal solution is the one minimizes the cost over all feasible flows.

- (in the atomic version of the problem, each commodity represents a single player who must route a significant amount of traffic on a single path, where in the nonatomic case each commodity represents a large population of individuals, each of whom controls a negligible amount of traffic)

First we have the following theorem whose proof is omitted here but uses the potential function method (see section 18.3.1 of Algorithmic Game theory book)

- Theorem (Existence and uniqueness of equilibrium flows) See the proof in Page B

Let (G, d, c) be an nonatomic instance.

a) the instance (G, d, c) admits at least one equilibrium flow

b) If f and $\tilde{f}$ are equi flows for (G, d, c) then $c_e(f_e) = c_e(\tilde{f}_e)$ for every edge e

- Now we prove the $\frac{4}{3}$ bound. Note that since $c_e(x)$ is continuous, the optimum social exists and well-defined.

First note that if a strateg f^{EQ} is an equilibrium then

$$\langle c(f^{EQ}), f^{EQ} - f \rangle \leq 0 \quad \text{for all strateg distribution (flow) } f \qquad (1)$$

This is a direct consequence of the fact that in equi users travel on shortest paths with respect to $c(f^{EQ})$. Idea of the proof: (match the flow paths in f^{EQ} & f if it is not the case then there is a negative cycle in $f^{EQ} - f$ and we can use of EQ, we consider each f_i & f_i together) the cycle to find a shorter path. [Exercise: make it formalize]

★ Now Assume cost function $c_e(x) = c_e x + b_e$ for all edges (affine cost functions and separable) for non-negative c_e, b_e

- Thm: POA is $\frac{4}{3}$.

Let f be the optimum social flow. Then by ① then

$$\underbrace{C}_{\text{here cost}}(f^{EQ}) = \sum_e c_e(f_e^{EQ}) f_e^{EQ} \leq \sum_e c_e(f_e^{EQ}) f_e = \sum_{e} c_e(f_e) f_e + \sum_e \left(c_e(f_e^{EQ}) - c_e(f_e)\right) f_e$$

Footnote ①: $\sum_{P \in \mathcal{P}} c_P(f) f_P \overset{\text{def}}{=} \sum_{P \in \mathcal{P}} \sum_{e \in P} c_e(f(e)) f_P \overset{\text{change order}}{=} \sum_{e \in P} \sum_{P \in \mathcal{P}, e \in P} c_e(f(e)) f_P \overset{\text{trivial}}{=} \sum_{e \in P} c_e(f(e)) \sum_{P \in \mathcal{P}, e \in P} f_P \overset{\text{def}}{=} \sum_{e \in P} c_e(f(e)) f_e$

Intuition: equil. flows and optimum flows are the same things just with respect to different sets of cost functions.

Page 8

Proof of Thm: Existence and uniqueness of equilibrium flows)

let (G,r,c) be a nonatomic instance where $c_e: \mathbb{R}^+ \to \mathbb{R}^+$ are nonnegative, continuous and non-decreasing.

a) The instance (G,r,c) admits at least one equilibrium flow

b) If f and $\tilde{f}$ are equilibrium flows for (G,r,c) then $c_e(f_e) = c_e(\tilde{f}_e)$ for every edge (e).

First, see this intuitive proposition from convex optimization (more precisely, from the first order conditions of a convex opt. problem with non-negativity constraints)

Proposition 1 (characterization of optimal flows) Let (G,r,c^*) be a nonatomic instance such that for every edge e, the function $c^*_{(e)}$ is convex and continuously differentiable. Let $c^{*\prime}_e$ (derivative of c_e) denote the marginal cost function of the edge e. Then f^* is an optimal flow for (G,r,c) if and only if, for every commodity $i \in \{1,2,\dots,k\}$ and every pair $P, \tilde{P} \in \mathcal{P}_i$ of s_i–t_i path with $f^*_P > 0$, $c^{*\prime}_P(f^*) \le c^{*\prime}_{\tilde{P}}(f^*)$

Now by setting ($(c^{*\prime}_p(x) = c_p(x))$) $c^*_e(x) = \int_0^x c_e(y)\,dy$ for each edge e, we obtain the desired potential function. Moreover, since c_e is continuous and non decreasing for every edge e, every function c^*_e is both continuously differentiable and convex.

call $\Phi(f) = \sum_{e \in E} c^*_e(f_e) = \sum_{e \in E} \int_0^{f_e} c_e(x)\,dx.$ (★★)

By the definition of nonatomic equil. flow (Def A), we have the following characterization of equil. flows as the global minimizers of the potential function Φ.

Essentially follows from Prop 1

Prop 2 (potential function for equil. flows). Let (G,r,c) be a nonatomic instance. A flow feasible for (G,r,c) is an equil. flow if and only if it is a global minimum of the corresponding potential function Φ given in (★★) above.

-Now Proof of Thm: by definition the set of feasible flows of (G,r,c) can be identified with a compact (i.e., closed and bounded) subset of $|\mathcal{P}|$-dim Eucl. space (# of pairs). Since edge cost functions are continuous, the potential function is a continuous function on this set. By Weierstrass's theorem from elementary math analysis, the potential function Φ achieves a minimum value on this set. By Prop 2., every point at which Φ attains its minimum corresponds to an equi flow. Part (b) follows essentially because of convexity of Φ, which itself follows from the fact that each cost function is non-decreasing, and hence each summand on (★★) is convex. (we take two equi flow f, $\tilde{f}$ which by prob 2 minimize Φ as well and then it is not hard to see that c_e is constant between f_e and $\tilde{f}_e$. See the exact details in Page 470 of AGT book)

Since the functions c_e are non decreasing, we need to focus on the last expression for which $f_e < f_e^{EQ}$ to bound the last term in (2).

In this case $(c_e(f_e^{EQ}) - c_e(f_e))f_e$ is equal to the area of the shaded rectangle in Figure 1. Note that the area of any rectangle whose upper-left corner point is $(0, c_e(f_e^{EQ}))$ and whose lower-right corner point lies on the line corresponding $c_e(y_e) = c_e y_e + b_e$ is at most half of the triangle defined by three points $(0, c_e(f_e^{EQ}))$, $(0, b_e)$, and $(f_e^{EQ}, c_e(f_e^{EQ}))$. In turn, the area of the triangle is at most half of the rectangle by the two points $(0,0)$ and $(f_e^{EQ}, c_e(f_e^{EQ}))$. Thus

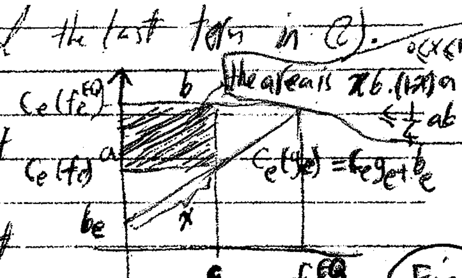
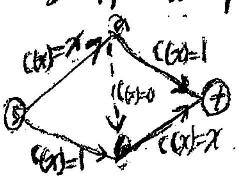


Fig 1

$$(c_e(f_e^{EQ}) - c_e(f_e))\, f_e \leq \frac{1}{4}\, c_e(f_e^{EQ})\, f_e^{EQ}$$ as desired.

Finally the result above can be generalized for different c_e functions, i.e. when $c_e(x)$ is polynomial in x, with different constant independent of the network. See the book for more information.

Finally Braess' Paradox



For one unit of flow from s to t the equil is half-half with cost $\frac{3}{2}$.
Now assume we add $v \to w$ with $c(x) = 0$.

Then the prev equal does not persist the cost of new route $s \to v \to w \to t$ is never worse than along the two original paths, and it is strictly less whenever some traffic fails to use it. Thus all flow uses the new path the cost would be 2 then. The optimum flow is the one before, i.e $\frac{3}{2}$ so the price of anarchy is $\frac{4}{3}$. But as we proved above, since the price of anarchy is always at most $\frac{4}{3}$, it says adding edges to monotonic games with affine cost functions cannot increase the cost of equi flow by more than $\frac{4}{3}$ factor and thus Braess's Paradox is the worst case indeed.