Smooth Games:
Blum, H., Ligett & Roth in STOC 2008 introduced the concept of price of total anarchy for regret-minimization algorithms (sort of learning algorithms).
Roughgarden introduced the concept of smooth games when he taught price of total anarchy and try to generalize the results as much as possible.

Generalizations of Pure Nash equilibria

Good news: every bound on the price of anarchy that is derived via a smoothness argument extends automatically with no quantitative degradation in the bound to all of the more general equilibrium concepts depicted in Figure above.

\[ \sum_i C_i(S^*_i, S_{-i}) \leq \lambda^* \text{cost}(S^*) + \mu \cdot \text{cost}(S) \quad (\lambda > \lambda^*) \]

Next we prove: If a game is \((\lambda, \mu)\)-smooth with \(\lambda > 0\) and \(\mu < 1\) then each Nash equilibrium has cost at most \(\frac{\lambda}{1 - \mu}\) times that of an optimal solution \(S^*\), i.e. \(P\text{OA} = \frac{\lambda}{1 - \mu}\).

Proof: Assuming \((\lambda, \mu)\)-smoothness, \[ \text{cost}(S) \leq \sum_i C_i(S) \leq \sum_i C_i(S_i^*, S_{-i}) \leq \lambda^* \text{cost}(S^*) + \mu \cdot \text{cost}(S) \]

Thus \(\text{cost}(S) \leq \frac{\lambda}{1 - \mu} \text{cost}(S^*)\) and thus \(P\text{OA} = \frac{\lambda}{1 - \mu}\).

Example: for non-atomic congestion games we proved \(P\text{OA} = \frac{1}{2}\) and \(P\text{OA} = \frac{1}{1 - \frac{1}{4}} = \frac{4}{3}\) as we have seen.

Why is smoothness stronger? Key point: to derive \(P\text{OA}\) bound only need to hold in special case where \(S = S^*\) Nash eq. and \(S^* = \text{optimal but smoothness requires cost for every pair } S, S' \text{ outcomes even if } S \text{ is not a Nash eq.} \)
Smoothness results so far have been used for non-atomic congestion games, atomic congestion games, weighted congestion games, coordination mechanisms and submodular maximization games.

We can relax definition of smoothness in two ways:

1) by assuming \( c(s) \leq \sum c_i(s) \) (instead of equality)
2) the equation (28) only needs to hold for some optimal solution \( s^* \) and all out comes \( s \), rather than for all pairs \( (s, s') \) of out comes.

Also note that the theorem for pure Nash only says if there is a pure Nash then PoA is bounded.

Example: Congestion Games with Affine cost functions which are generalization of non-atomic selfish routing. We have a ground set \( E \) of resources, a set of \( K \) players with strategy sets \( S_1, \ldots, S_K \subset 2^E \), and a cost function \( c_e(x) \) for each resource \( e \). We always assume that cost functions are non-negative and non-decreasing; in particular when it is affine, \( i.e., \) \( c_e(x) = ax + bx \) with \( a_e, b_e \geq 0 \) for every resource \( e \).

In selfish routing, \( E \) is the edge set and strategies are source-sink pairs \( e \in E \).

The load induced on edge \( e \) is \( x_e = \left| \{ i : i \in S_i \} \right| \), i.e., the number of players using \( e \).

The cost of player \( i \), \( c_i(s) = \sum_{e \in S_i} c_e(x_e) \), where \( x_e \) is the vector of loads induced on \( e \).

As we showed before, by reversal of sums \( c(s) = \sum_{i=1}^K c_i(s) = \sum_{e \in E} c_e(x_e) \) where \( x_e \) is the vector of loads induced on \( e \).

Thm2: Every congestion game with affine cost functions is \( \left( \frac{5}{3}, \frac{1}{3} \right) \)-smooth.

First a lemma by Christodoulou and Koutsoupias who noted:

\[ \text{Im 1: } g(2t+1) \leq \frac{5}{3} \left( y^2 + \frac{1}{3} z^2 \right) \text{ for all non-negative integers } y, z. \text{ Thus for all } a, b \geq 0 \]

\[ a g(2t+1) + b y \leq \frac{5}{3} (a g^2 + b y) + \frac{1}{3} (a z^2 + b z) \]

For large integers. Now consider a pair \( s, s' \) of outcomes.

To establish smoothness, since the number of players using resource \( e \) in the outcome \( (s_i, s'-i) \) is at most one more than in \( s \) and this resource contribution is at most \( x_e \),

\[ \sum_{i=1}^K c_i(s_i, s_i) \leq \sum_{e \in E} (a_e x_e + b_e) x_e^2 \leq \sum_{e \in E} \frac{5}{3} (a_e x_e + b_e) x_e + \sum_{e \in E} \frac{1}{3} (a_e x_e + b_e) x_e = \frac{5}{3} c(s) + \frac{1}{3} c(s') \]

Thus by Thm 1, we have upper bound \( \frac{5}{3} c(s) + \frac{1}{3} c(s') \) for PoA (and indeed it is tight). Indeed there is an extension of this for all classes of Fig 1.
The next example is payoff-maximization games, where player $i$ wants to maximize payoff function $\Pi_i(s)$. We use $W$ to denote objective function of an $\alpha$ payoff-maximization game. We call such a game $(\alpha, \mu)$-smooth if

$$\sum_{i=1}^{K} \Pi_i(s_i^*, s_{-i}) \geq \lambda (V(s^*) - W(s))$$

for every pair $s, s^*$ of outcomes. The same way of minimization, we can show a $(\alpha, \mu)$-smooth payoff-maximization game, the objective function value of every pure Nash Equil. is at least $\frac{\lambda}{1 + \mu}$ fraction of the maximum possible.

Example: Valid Utility Games: We have a ground set $E$, a non-negative submodular function $V$ defined on subsets of $E$, and a strategy set $S_i \subseteq E$ and a payoff function $\Pi_i$ for each player $i = 1, 2, \ldots, K$. A function $V : 2^E \rightarrow \mathbb{R}$ is submodular if $V(X \cup Y) + V(X \cup Y) \leq V(X) + V(Y)$ for every $X, Y \subseteq E$.

For example, the set $E$ could denote a set of locations where facility can be built, and a strategy $S_i \subseteq E$ could denote the locations at which player $i$ chooses to build facilities. For an outcome $s$, let $V(s) \subseteq E$ denote the union $\cup_i S_i$ of players' strategies $s_i$. The objective function value of an outcome $s$ is defined as $W(s) = V(U(s))$.

Also, we have two extra conditions (i) for each player $i$, $\Pi_i(s) \geq W(s) - W(s_i)$ for every outcome $s_i$ and (ii) $\Pi_i(s) \leq W(s)$ for every outcome $s_i$.

One example of these games is competitive facility location with price-taking markets and profit-maximizing firms by Goemans et al.

Theorem 3: Valid utility games with a non-decreasing objective function $V$ is $(\alpha, \mu)$-smooth and hence has (robust) PoA at least $\frac{\lambda}{1 + \mu}$.

Proof: Let $s$ and $s^*$ denote arbitrary outcomes of such a game. Let $Z_i \subseteq E$ denote the union of all of the players' strategies in $s$, together with the strategies employed by players other than $i$ in $s^*$. Then

$$\sum_{i=1}^{K} \Pi_i(s_i^*, s_{-i}) \geq \sum_{i=1}^{K} [V(U(s^*_i, s_{-i})) - V(U(s_i, s_{-i}))]$$

(because of (i) condition of valid utility game)

$$\geq \sum_{i=1}^{K} [V(Z_i) - V(Z_{-i})]$$

(since $V$ is submodular with $X = U(s^*_i, s_{-i})$ and $Y = U(s_i, s_{-i})$)

$$\geq W(s^*) - W(s)$$

(since $V$ is non-decreasing). We are done.

Extension: This implies the same result for Figl esp. regret minimizing first observed by M.C. Rost (2008).
Note on Regret Minimizing Algorithms, (first introduced in 1950 by Hannan)

In these algorithms at each time, the agent selects an action and observes the loss/gain, and the goal is to minimize loss or maximize gain, we have stochastic vs adversarial inputs and we consider the fixed optimum to compete. These are algorithms with regret approaching zero at a rate of $O(\frac{1}{\sqrt{n}})$.

These algorithms are learning algorithms.

By submodularity:
$$V(X) - V(XU) \geq V(XU) - V(Y)$$
Thus
$$V(U_i(s_i, S_{i-1})) - V(U(U_i(s_i^*, S_{i-1}))$$
$$\geq V(U(s_i, S_{i-1})) - V(U(U(s_i^*, S_{i-1}))$$
$$\geq V(U(S_i, S_{i-1})) - V(U(U(S_i^*, S_{i-1}))$$
$$\geq V(U(S_i^*, S_{i-1})) - V(U(U(S_i^*, S_{i-1}))$$