10/06

Today we talk about secretary problem, online Auctions and finally Ad Auctions.

Classic secretary problem: There are n applicants arriving online one by one. After interviewing an applicant i, we can decide its value v_i relative to all $1,\dots,i-1$ (indeed only its relative order is enough). After interviewing applicant i, we have to decide to hire i or not (decision is irrevocable). Select the best applicant.

One naive algorithm: Randomly choose some element j (give competitive ratio n in the worst case, e.g. choose the first element or not)

We can do better in Random Permutation Model in which values and thus ranks of applicant can be decided by adversary, but the order in which applicants arrive is uniformly at random.

Motivations for Random Permutation Model is when the values are coming from an unknow distribution which is practical.

Algorithm

- Interview first k applicants and don't hire anyone.
- let max_v be the (value of) best element in first k.
- Select the first element after k elements which is better than max_v.

Indeed above is a stopping rule policy. It is shown by Dynkin 1962 that the optimal policy for the problem is a stopping rule under which, we skip the first k applicants and then take the next applicant who is a candidate (i.e., who has the best relative ranking of those interviewed up that point). For an arbitrary cutoff k, the probability that the best applicant is selected is $P(k) = \sum_{j=k+1}^{n} \frac{1}{n}\left(\frac{k}{j-1}\right) = \frac{k}{n}\sum_{j=k+1}^{n}\frac{1}{j-1}$ where $\frac{1}{n}$ is the prob that jth element is the maximum and the second highst among places 1 to j appears in the first k slots. letting n tends to ∞, writing x as the limit of $\frac{k}{n}$ using t for $\frac{j}{n}$ and dt for $\frac{1}{n}$, the sum can be approximated by the integral $P(x) = x\int_x^1 \frac{1}{t}\,dt = -x\ln x$. Take the derivative of $P(x)$ with respect to x, setting it to 0 and solving for x, we find that the optimal value for $x = \frac{1}{e}$. Thus the optimal cutoff tends to $\frac{n}{e}$ as n increases and the best applicant is selected with probability $\frac{1}{e} = 0.368$.

The problem has been considered for example for unknown number of applicants as well

Secretary problem and its extensions have applications in finding a fiancee (the original motivation), hiring secretaries, dynamic auction market, online market Variation of house selling problem, the one-armed bandit problem, detecting a chang point and burglar problem and etc.

A natural extension with applications to online market:
we have to select k secretaries instead of just 1 (again in a random permutation model.

different objectives:

① Maximize the probability of selecting best k.

② minimize the expected sum of $\underbrace{\text{ranks of}}$ selected k.

OR

③ maximize the expected sum of values of selected k

we focus mainly on ③ and generalization though ① and ② can be modeled as well.

<u>submodular secretary problem</u> A function f is monotone iff $f(A) \leq f(B)$ for $A \subseteq B \subseteq S$.

A function F defined on the subsets of a universe set X is <u>submodaar</u> if for any pair of subsets of X like A and B, we have
economies of scale

$$F(A) + F(B) \geqslant F(A \cup B) + F(A \cap B)$$

$\boxed{\text{It is subadditive if } f(A)+f(B) \geq f(A \cup B)}$

An equivalent characterization is that the marginal profit of each item should be non-increasing, i.e., $f(A \cup \{a\}) - f(A) \leq f(B \cup \{a\}) - f(B)$ if $B \subseteq A \subseteq S$ and $a \in S \setminus$

we want to hire <u>k</u> secretaries such that each subset of seretaries has a value for you which not necessarily linear and this function is submodular in many practical applications.

If $F(A) = \sum_{a \in A} V(a)$, i.e. linear case, the problem has a $1 - \theta(\frac{1}{\sqrt{k}})$ competitive algorithm. Also the case that $F(A) = \max_{a \in A} V(a)$ has been considered before. (special cases)

Thm: There is a constant $(8e^2)$ competitive algorithm for the (non-monotone) submodular secretary problem. Indeed this is the most that one can hope for. subadditive secretary problem the ratio can be as bad as $\Omega(\sqrt{n})$.

For subadditive secretary problem the ratio can be as bad as $\Omega(\sqrt{n})$.

The Algorithm is simple and intuitive: we partition the input stream into k equal parts: $S_1, S_2, \dots, S_k$, and choose exactly one secretary in each part.

let T_i be the set of i secretaries we choose in the first i parts.

In part S_{i+1}, we try to choose a secretary $x \in S_{i+1}$ that maximizes $F(x \cup T_i) - F(T_i)$ (using the classic secretary problem)

We prove the expected value of $f(T_K)$ is within a constant factor of the optimum solution. let $R=\{a_{i_1}, a_{i_2}, \dots, a_{i_k}\}$ be the optimal solution. Note that the set $\{i_1, i_2, \dots, i_k\}$ is a uniformly random subset of $\{1,2,\dots,n\}$ with size k (due to random ordering.

First by submodularity we have $f(B)-f(A) \leq \sum_{a\in B\setminus A} [f(A\cup\{a\}) - f(A)]$ for any $A\subseteq B \subseteq S$ (★★)

Define $X=\{S_i : |S_i \cap R| \neq \emptyset\}$. For each $S_i \in X$, we pick one element say s_i of $S_i \cap R$ randomly. These selected items form a set called $R' = \{s_1, s_2, \dots, s_{|X|}\} \subseteq R$ of size $|X|$.

Lm 1: the expected value of the number of items in R' is at least $K(1-\frac{1}{e})$.

Pf: by standard methods (like balls and bins) we can show at most $\frac{1}{e}$ fractions of S_i have empty intersection with R. □

Another intuitive lm2: For a random subset A of R, the expected value of $f(A)$ is at least $\frac{|A|}{k} \cdot f(R)$.

Pf by randomness and submodularity. ■

The crux of the proof is that local optimizations steps indeed lead to global approximate solution. i.e;

Lm: The expected value of $f(T_K)$ is at least $O\left(\frac{|R'|}{ek} f(R)\right)$.

intuition of the proof: ~~the correct element of $R' \cap S_i$~~ Assume w.l.o.g. S_1 is the first element of R' to appear in our ordering. Define $\Delta_j = f(T_j) - f(T_{j-1})$ note that due to monotonicity $\Delta_j \geq 0$ and thus $E[\Delta_j] \geq 0$. with probability $\frac{1}{e}$, we choose the element in S_j which maximizes the value of $f(T_j)$ (given that the set T_{j-1} is fixed. we get factor $\frac{|R'|}{k} f(R)$ since by (★★) we can relate the value of $f(T_K)$ with $O(f(R'))$ which is by lm 2 is $\frac{|R'|}{k} f(R)$. ■

Finally Thm: The expected value of the output of our algorithm is at least $O\left(\frac{1-\frac{1}{e}}{e} f(R)\right)$

Pf: use lm 1 that expected value of R' is $K(1-\frac{1}{e})$. ■

The exact number is $\frac{1-\frac{1}{e}}{e} f($

For non-monotone case, this algorithm does not work, since if we select some item for the ith set, it may hurt us at ~~a~~ later step.

However slight changes works.

let divide the input stream into two equal parts and let $0 \le X \le 1$ be a uniformly random value. If $X \le \frac{1}{2}$, we run the algorithm on the first part (and thus ignore the other part), otherwise we run the algorithm on the second part and ignore the first part.

one can prove that the sum of the solutions in two parts is a constant times the opt and thus we get at least half of it. ∎

Next we consider the applications of secretary problem in online auctions.