Market equib or Market clearance price:

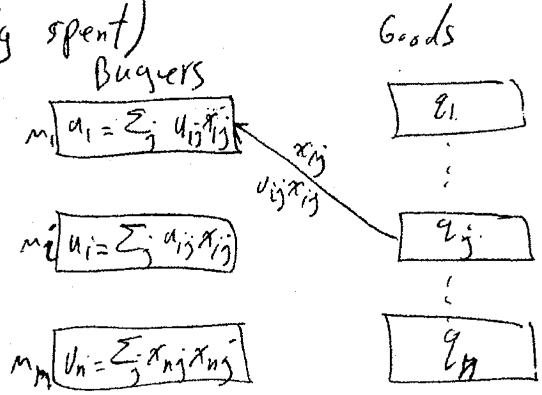
is a market price established through competition such that the amounts of goods or services sought by buyers is equal to the amount of goods or services produced by sellers. The prices will tend not to change unless demand or supply changes.

More precisely in a Fisher setting with linear Utilities, we have

- m buyers (each in B set with budget B_i) and n goods in G set for sale (each with quantity q_j)
- Each buyer has linear utility u_i, i.e. utility of i is $\text{sum}_j\, u_{ij} x_{ij}$ where $u_{ij} \geq 0$ is the utility of buyer i for good j and x_{ij} is the amount of good j bought by i.

A market equib. or market clearance is a price vector p that

- maximizes utility $\text{sum}_j\, u_{ij} x_{ij}$ of buyer i subject to his budget $\text{sum}_j\, p_j x_{ij} \leq B_i$
- The demand and supply for each good j are equal, i.e., $\text{sum}_i\, x_{ij} = q_j$ (and thus the budgets are totally spent)



Thm: Under the mild assumption that each good has a potential buyer, i.e., a buyer who derives nonzero utility from this good, market clearing prices always do exist.

Arrow–Debreu Model with linear utilities:

It is also known as the Walrasian model or the exchange model and it generalizes Fisher's model. Consider a market consisting of a set B of agents and a set G of goods, assume $|G| = n$ and $|B| = m$. Each agent i comes to the market with an initial endowment of Goods $e_i = (e_{i1}, e_{i2}, \ldots, e_{in})$. We may assume w.l.o.g. that the total amount of each good is unit (by scaling), i.e. for $1 \leq j \leq n$, $\sum_{i=1}^{m} e_{ij} = 1$. Each agent has linear utilities for these goods. The utility of agent i on deriving x_{ij} amount of good j, for $1 \leq j \leq n$ is $\sum_{j=1}^{n} u_{ij} x_{ij}$. The problem is to find prices for goods $P = (p_1, \ldots, p_n)$ so that if each agent sells

her initial endowment at these prices and buys her optimal bundle, the
market clears, i.e., there is no deficiency or surplus of any good. An agent
may have more than one optimal bundle; we will assume that we are free to give
each agent any optimal bundle to meet the market clearing condition.

Note Fisher is a special case of Arrow-Debreu, $n+1$ goods and $m+1$ agents
Assume money is the $n+1$st good and for the first m agents correspond
to the m buyers whose initial endowment is the money and the $m+1$st
agent's initial endowment is all n goods. The first m agents have
utilities for goods, as given by the Fisher market and no
utility for money whereas the $m+1$st agent has utility for money only.
(by scaling we can assume price $\$ = 1$)

Thm: there exists a market clearance prices in the Arrow-Debreu model
and indeed it is unique

See some ideas of the proof in the wireless network setting and more generally THM A

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proof of Thm A: First we observe that w.l.o.g each $q_j = 1$, by scaling
the u_{ij}'s appropriately. u_{ij} and B_i are in general rational; again by scaling
appropriately, we can assume they are integral.
Again similar to non-atomic selfish routing, we show equil. allocations
for Fisher's linear case are indeed optimal solutions to a remarkable
convex program, called the Eisenberg-Gale convex program.
The objective function in this program is not very intuitive though
there are some evidence for it. The objective is

$$\max \left(\prod_{i \in B} u_i^{B_i} \right)$$

in some sense for each buyer i is $u_i^{B_i}$ is increased
if u_i or B_i is increased. However we want buyers
with larger B_i have higher effect in the system.

(see the paper by Kamal Jain in FOCS 04 for mor intuition)

Exercise: Think more about this objective function

The log is used in Eisenberg-Gale convex prog.

$$\text{Max} \sum_{i \in B} B_i \log u_i$$

$\boxed{\text{see the page after next}}$

s.t.

$$u_i = \sum_{j \in G} u_{ij} x_{ij} \quad \forall i \in B$$

$$\sum_{i \in B} x_{ij} \leq 1 \quad \forall j \in G$$

$$x_{ij} \geq 0 \quad \forall i \in B, j \in G.$$

Note that instead max $f(x)$ where $f(x)$ is concave we can
min $-f(x)$, where $-f(x)$ is convex.

Note convex program min convex
s.t. convex body (here is linear indeed)

using Karush, Kuhn, Tucker (KKT) conditions which are essentially the generalization of duality theory for linear programming, we know that optimal solutions (and complementarity slackness conditions) to x_{ij} and p_j must satisfy the following: (here p_j's are dual variables corresponding to second set of conditions)

i) $\forall j \in G;\ p_j \geq 0$

ii) $\forall j \in G: p_j > 0 \Rightarrow \sum_{i \in B} x_{ij} = 1$

iii) $\forall i \in B, \forall j \in G: \frac{u_{ij}}{p_j} \leq \frac{\sum_{j \in G} u_{ij} x_{ij}}{B_i}$ (makes sure the good purchased by i by spending B_i has the highest utility, otherwise buy j more)

iv) $\forall i \in B, \forall j \in G: x_{ij} > 0 \Rightarrow \frac{u_{ij}}{p_j} = \frac{\sum_{j \in G} u_{ij} x_{ij}}{B_i}$

Thm: For the linear case of Fisher's Model:

a) If each Good has a potential buyer, equil. exists.

b) The set of equil allocations is convex

c) Equil utilities and prices are unique.

Proof: since for good j, there is buyer i with $u_{ij} > 0$, by (iii) $p_j \geq \frac{B_i \cdot u_{ij}}{\sum_{j \in G} u_{ij} x_{ij}} > 0$. (we set price of good j, p_j.)

Thus by (ii), $\sum_{i \in B} x_{ij} = 1$ and are goods are fully sold their prices are positive also.

iii) and (iv) imply that if buyer i gets good j (i.e. $x_{ij} > 0$) then j must be among the goods that give buyer i maximum utility per unit money spent at current prices. Hence each buyer gets only a bundle consisting of her most desired goods, i.e., an optimal bundle.

By (iv), $\frac{u_{ij}}{p_j} \times x_{ij} = \frac{\sum_{j \in G} u_{ij} x_{ij}}{B_i} \times x_{ij}$ (to handle the case of $x_{ij} = 0$) $\forall i \in B, j \in G.$

$\Rightarrow \frac{B_i u_{ij} x_{ij}}{\sum_{j \in G} u_{ij} x_{ij}} = p_j x_{ij}$

sum over all j: $\frac{B_i \sum_{j \in G} u_{ij} x_{ij}}{\sum_{j \in G} u_{ij} x_{ij}} = \sum_{j \in G} p_j x_{ij} \Longrightarrow \forall i \in B: B_i = \sum_j p_j x_{ij}$ and thus money of each buyer is fully spent. This complete the proof of (a).

- Since each equil allocation is an optimal solution to the Eisenberg-Gale
- Convex Program, the set of allocations must form a convex set. Thus we have (b).
- Since log is strictly concave, similar to non-atomic selfish routing, if there is more than one equil, the utility by each buyer should be the same and this with (iv) gives unique equil. Prices. Thus we have (c) also.