11/17: **Network Creation Games and Network Bargaining Game (cooperative game theory)**

As we discussed equilib. is a locally stable solution in which no agent can greedily improve his situation by only changing his strategy. Today we mainly focus and advocate the structure of equilib such as diameter, degree or other properties of the network. We think these are more important than PoA, e.g. in network creation games any bound on diameter gives the bound on price of anarchy.

In Network creation Games several players (the nodes) collectively attempt to build an efficient network that interconnects every one. Each player has two selfish goals: to minimize the cost spent building links (creation cost) and to minimize the average or max distance to all other nodes (usage cost). Together these goals capture the issues of both network design and network routing. In these games, they mediate the two objectives by defining the cost of each link to be a parameter α and minimizing the sum of creation cost and usage cost. The resulting behaviour of these games seems quite intericate and heavily depends on the choice of α, with most bounds and proofs applying only to specific ranges of α. Also despite much effort, the behaviour remains poorly understood for certain ranges of α, in particular when the cost of creating a link is equivalent within a logarithmic factor to decreasing the distance to all others. In the rest of this half-session, we introduce a basic network creation game which avoid parameterization by α (via slides.)

Cooperative Game Theory: → example

Now we consider **network bargaining game**: given a graph of n agents. Each agent i can participate in at most one contract (more generally in c_i contract). For each pair of agents i,j we are given a weight w_{ij} representing the surplus of a contract between i and j. Our task is to predict the set of contracts formed. The case of $c_i=1$ is the matching game that we consider in the rest of this session.

Solution Concept:
We want to predict the set of contracts $M \subseteq E$ which is a matching and the surplus $\{z_{ij}\}$ that result from bargaining among agents. A "solution" $(\{z_{ij}\}, M)$.

is such that for all $(i,j) \in M$, $z_{ij} + z_{ji} = w_{ij}$ and for all $(i,j) \notin M$, $z_{ij} = 0$. We interpret z_{ij} as the amount of money i earns from contracting with j. We also define the aggregate earnings of node i by $x_i = \sum_{j \in N} z_{ij}$ and sometimes refer to the set of earnings $\{x_i\}$ as the "out come" of our game.

The set of solutions of our game is quite large and so it is desirable to define a subset of solutions that are likely to arise as a result of the bargaining process.

We define the outside option of an agent i to be the best deal he or she could make with someone outside the contracting set M. For a fixed agent k with $(i,k) \in E \setminus M$, the best deal i can make with k is to match k's current ~~offer~~ offer (if any) and if k is not in any contract the i's outside option with k would be w_{ik}. So more formally the outside option α_i of an agent i is the best deal he or she could make with someone outside the contracting set M.

$$\rightarrow \alpha_i = \max_{k:(i,k) \in E \setminus M} \{w_{ik} - x_k\}$$ and max is zero over an empty set.

Kleinberg and Tardos define concepts of stable and balanced as follows: An out come is stable if for all i, $x_i \geq \alpha_i$ and is balanced if for all pair $(i,j) \in M$, we have $x_i - \alpha_i = x_j - \alpha_j$ (if agents have different out side options, then they will split the net surplus equally.

Cooperative Game Theory: A cooperative game is defined by a set of agents N (Grand coalition) and a value function $v: 2^N \rightarrow R^+ \cup \{0\}$ mapping subsets of agents to the non-negative real numbers. $v(\emptyset) = 0$ Intuitively, the value of a set of agents represents the maximum surplus (Payoff) they alone can achieve. Cooperative game theory suggests that the total earnings of agents in a cooperative game is fundamentally related to the values of the sets in which they are contained. To cast our network Bargaining game in the cooperative game theory terminology, we must first define the value of a subset of agents as follows:

The value $v(S)$ of a subset $S \subseteq N$ of agents is the maximum $\cancel{\sum_{(i,j)\in M} w_{ij}}$ weight matching M_S in the induced subgraph $G[S]$.

Cooperative game theory suggests that each set of agents should earn in total at least as much as they alone can achieve. In mathematical terms, we require that the sum of the earnings of a set of agents should be at least the value of the same set. We additionally require that the total surplus of all agents is fully divided among the agents. These together yield the cooperative game-theoretic notion of the core: An outcome $\{x_i\}$ is in the core if for all ① subsets $S \subseteq N$, $\sum_{i \in S} x_i \geq v(S)$ and ② for the grand coalition N, $\sum_{i \in N} x_i = v(N)$.

The core may be empty even for very simple class of games and it may be hard to test whether it is empty or not. For our games, we are able to characterize the set of matching games having a nonempty core.

Other solution concepts proposed in the cooperative game theory literature are that of kernel and prekernel. Unlike the core, the kernel and prekernel always exist. Kernel and prekernel are closely related and we only work with the prekernel in this session. The prekernel is defined by characterizing the power of agent i over agent j, and requiring that these powers are in some sense equilized. Intuitively, the power of i over j is the maximum amount i can earn without the cooperation of j.

The power of agent i with respect to agent j in the outcome $\{x_i\}$ is

$$S_{ij} = \max \{ v(S) - \sum_{k \in S} x_k : S \subseteq N, S \ni i, S \not\ni j \}$$ [maximum amount i can earn without cooperation with j]

The prekernel is then the set of outcomes x that satisfy $S_{ji}(x) = S_{ij}$ for every i, j.

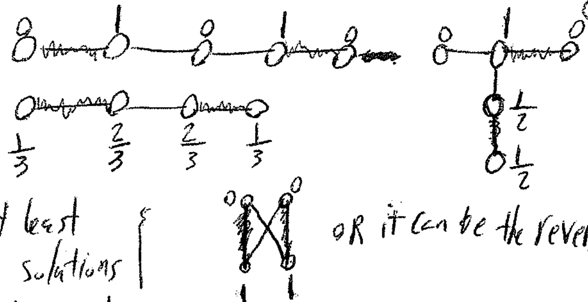
Although the definition of the prekernel is not completely intuitive, it turns out to be similar to the notion of balanced solutions in certain network settings. (The concept of kernel is even less intuitive with more conditions.)

Known economic fact: If the core is non-empty, the core intersection prekernel is non-empty. In addition we can find a point in the intersection in poly-time if certain conditions are satisfied, e.g. we can compute $S_{ij}(x)$ given any vector x in poly-time.

For example, unique balanced outcomes are as follows: (All weights $=1$)

unique solutions

at least 2 solutions



OR it can be the reverse

The goal here is to identify and characterize stable solutions and stable solutions which are also balanced.

Thm: An outcome (M, x) is stable iff payoff vector x is in the core.

Pf: we use the strong duality theorem and complementary slackness conditions. First we prove if outcome (M, x) is stable then x is in the core. First we consider the second condition of the core. since by def. $\sum_{i \in N} x_i = W_M$ where $W_M = \sum_{e \in M} W_e$, we only need to prove that M is a maximum-weight matching. consider the following LP and its dual

PRIMAL $=$ minimize $\sum_i x_i$

$$x_i + x_j \geqslant W_e \quad \forall (i,j) \in E$$

$$x_i \geqslant 0 \quad \forall i \in V$$

DUAL $= \min \sum_e W_e y_e$

$$\sum_{j: e=(i,j) \in E(G)} y_e \leq 1 \quad \forall i \in V$$

$$y_e \geqslant 0 \quad \forall e \in E$$

Note that x and M (i.e. $y_e = 1$ if $e \in M$ and $y_e = 0$ otherwise) are feasible solutions for PRIMAL and DUAL LPs. since $v(V(G)) = \sum_{k \in V} x_k = W_M$, by the strong duality theorem we can conclude that M is a maximum-weight matching. Now we consider the first condition of the core. consider an edge $e = (i,j)$. For $e \in M$, we have $x_i + x_j = W_e$ by the definition of an outcome. For $e \notin M$, by the def. of an outside option $\alpha_i \geqslant W_e - x_j$ and by the def. of stable $x_i \geqslant \alpha_i$ which results in $x_i + x_j \geqslant W_e$ as desired. so if we sum over edges of any matching in $G[S]$ we obtain the ~~first~~ condition of the core (use ✱✱)

Now we prove the reverse, there is a stable solution (M, x) with respect to any vector $x \in$ core. We prove this by showing a max-weight matching M for which for all $e \in (i,j) \in M$, $x_i + x_j = W_e$. Note that x is feasible solution for LP PRIMAL. consider any max_weight matching M and set $y_e = 1$ if $e \in M$ and $y_e = 0$ otherwise. Thus y is a feasible solution for the dual LP. since $V(V) = \sum_{k \in V} x_k = W(M)$, by the strong duality theorem x and y are optimum solutions to Primal and dual LPs. By complementary slackness conditions $y_e(x_i + x_j - W_e) = 0$ for each $e = (i,j)$. It means if $y_e > 0$, i.e., $e \in M$, then $x_i + x_j = W_e$ as desired. Now we show $x_i \geqslant \alpha_i$. Assume j is an outside option for i. Thus $x_i + x_j \geqslant W_e$ and $x_i \geqslant W_e - x_j = \alpha_i$

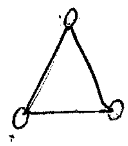
Indeed the proof gives a characterization of graphs with a non-empty set of stable/core solutions

✱✱ note that having $x_i + x_j \geqslant W_e$ though it is weaker is equivalent to $\sum_{i \in S} x_i \geqslant V(S)$ by summing over all edges $\in M_S$.

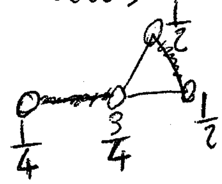
COR: A Graph has non-empty core (and thus has a stable outcome) iff LP ~~D~~ PRIMAL has integrality gap 1 for finding a max-weight matching for the graph.

For example it is well-known that the integrality gap is one for bipartite graphs and thus for them, core and stable solution are non-empty. However for general graphs we need to add the following condition $\sum_{e \in G(S)} y_e \geq \lceil \frac{|S|-1}{2} \rceil$ for all subsets $S \subseteq V(G)$ to get the integrality gap down to 1.

For example no stable solution here



but



has stable and balanced even solution (though it contains Δ)

Thm: An outcome (M,x) is stable and balanced iff the payoff vector x is in the intersection of core and prekernel.

Pf: We already showed an outcome (M,x) is stable iff vector x is in the core. In addition according to the proof ~~if (M,x) is stable~~ we can always construct a maximum matching M corresponding to a vector x in the core. We now prove if (M,x) is balanced then x is in the prekernel set. For two agents i and j we consider two cases:

1) $(i,j) \in M$ and 2) $(i,j) \notin M$.

But first let's define a simplified def. for s_{ij} (the power of agent i with respect to agent j: $s_{ij} = \max\{w_{ik} - x_i - x_k : (i,k) \in E, k \neq j\}$, we take the maximum to be $-x_i$ over an empty set. Though the condition is weaker (say by taking all $S = \{i,k\} : (i,k) \in E(G)$), indeed it is equivalent since in any (non-empty) maximum matching M_S, each edge $e = (i',j') \in M_S$ contributes w_e to $v(S)$ and at least w_e to $\sum_{k \in S} x_k$ since $x_{i'} + x_{j'} \geq w_e$ (by coreness) and thus cannot be of any help (if no matching edge, then adding more to S only makes the result more negative). $\boxed{\text{Note that } s_{ij} \leq 0 \text{ in our game.}}$

Now if $(i,j) \in M$, the same $k \neq j$ which maximizes $w_{ik} - x_i - x_k$ should maximize α_i too. Similarly the same $k' \neq i$ which maximizes $w_{jk'} - x_j - x_{k'}$ should maximize α_j too. Thus the balanced condition $x_i - \alpha_i = x_j - \alpha_j$ implies $x_i - w_{ik} + x_k = x_j - w_{jk'} + x_{k'}$ which implies $s_{ij} = s_{ji}$

Now if $(i,j) \notin M$: If there is a k such that $(i,k) \in M$ then $s_{ij} \geq w_{ik} - x_i - x_k = 0$. since x is in the core $s_{ij} \leq 0$ and thus $s_{ij} = 0$. If there is no such k, $x_i = 0$. consider an edge $(i,k) \in E(G)$, $k \neq j$. If there is no such k, then $s_{ij} = 0$ by the definition (over empty set). otherwise since $x_i = 0$, $x_k = w_{ik}$. Thus $s_{ij} \geq w_{ik} - x_i - x_k = 0$ which again implies $s_{ij} = 0$, since x is in the core. Thus in all cases $s_{ij} = 0$. Similarly $s_{ji} = 0$ and thus $s_{ij} = s_{ji}$.

Finally we prove if x is in the prekernel then x with its corresponding matching M is balanced. By the definition, $\forall i,j \in V$, $s_{ij} = s_{ji}$. For and edge $(i,j) \in M$, by the simplified definition of s_{ij} and α_i $s_{ij} = \alpha_i - x_i$. similarly $s_{ji} = \alpha_j - x_j$. Thus $s_{ij} = s_{ji}$ implies $x_i - \alpha_i = x_j - \alpha_j$ and thus (M, x) is a balanced outcome.

Kleinberg and Tardos via a combinatorial proof establish the following main theorem of their paper:

Thm: If a graph G has a stable outcome, then it has a balanced outcome, and the set of all balanced outcomes can be constructed in poly-time.

Thus the economic fact that if the core is non-empty (i.e., there is a stable outcome) the core intersection of prekernel is non-empty (it has nucleolus in it). Constructability follows the result of Faigle et al. since $s_{ij}(x)$ can be computed for any vector x is poly-time (trivial in our case).

Finally more on cooperative game: a game where groups of players ("coalition") may enforce cooperative behaviour, hence the game is a competetion between coalitions of players rather than individual players. In such a game players are assumed to choose which coalition to form, according to their estimate of the way the payment will be divided among coalition members. we can have subadditivity ($v(S \cup T) \geq v(S) + v(T)$ when $S \cap T = \emptyset$) or monotonicity ($S \subseteq T \Rightarrow v(S) \leq v(T)$) for the value function. We can have a subgame for $S \subsetneq N$, $S \neq \emptyset$ with $v_S: 2^S \to \mathbb{R}$ is $v_S(T) = v(T)$, $\forall T \subseteq S$. We can have dual game with value $v^*(S) = v(N) - v(N-S)$ $\forall S \subseteq N$. We have simple games if $v(S)$ is 1 or 0 (wining or losing). A solution concept is a vector $x \in \mathbb{R}^N$ that represents the allocation to each player. Some properties of solution concepts: Efficiency $\sum_{i \in N} x_i = v(N)$, Individual rationality $x_i \geq v(i)$, $\forall i \in N$, Existence & Uniqueness, Computational ease, Symmetry $v(S \cup \{i\}) = v(S \cup \{j\})$, $\forall S \subseteq N \setminus \{i,j\}$