Convergence of Nash Dynamics:
Equilibria and Nearly-Optimal Solutions

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1 Introduction: Games, Equilibria, and Dynamics

2 Convergence to Equilibria
   - Potential Games and PLS
   - Non-Potential Games

3 Convergence to Nearly-Optimal Solutions
   - Potential Games
   - Non-Potential Games

4 Other Dynamics
   - Equilibria
   - Nearly-optimal Solutions

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Games

Game:
- agents $\mathcal{N} = \{1, \ldots, n\}$
- $\forall i \in \mathcal{N}$: finite strategy space $\Sigma_i$
- $\forall i \in \mathcal{N}$: cost function $c_i : \Sigma_1 \times \cdots \times \Sigma_n \to \mathbb{R}$
  ($S \in \Sigma_1 \times \cdots \times \Sigma_n$ is called state.)
Games

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Example: Network Congestion Games
Game:

- agents $\mathcal{N} = \{1, \ldots, n\}$
- $\forall i \in \mathcal{N}$: finite strategy space $\Sigma_i$
- $\forall i \in \mathcal{N}$: possible paths from $s_i$ to $t_i$
- $\forall i \in \mathcal{N}$: cost function $c_i : \Sigma_1 \times \cdots \times \Sigma_n \to \mathbb{R}$

Example: Network Congestion Games

Latency function $\ell_e : \mathbb{N} \to \mathbb{R}$ for every edge $e$
Games

Game:

- agents $\mathcal{N} = \{1, \ldots, n\}$
- $\forall i \in \mathcal{N}$: finite strategy space $\Sigma_i$ possible paths from $s_i$ to $t_i$
- $\forall i \in \mathcal{N}$: cost function $c_i: \Sigma_1 \times \cdots \times \Sigma_n \to \mathbb{R}$ travel time

($S \in \Sigma_1 \times \cdots \times \Sigma_n$ is called state.)

Example: Network Congestion Games

latency function $\ell_e: \mathbb{N} \to \mathbb{R}$ for every edge $e$

\[
\begin{align*}
c_1(S) &= 8 \\
c_2(S) &= 8 \\
c_3(S) &= 4
\end{align*}
\]

We consider only games with complete information.
Nash Equilibria

\[ c_1(S) = 4 \]
\[ c_2(S) = 1 \]
\[ c_3(S) = 5 \]

**Definition**

pure Nash Equilibrium \( S \in \Sigma_1 \times \cdots \times \Sigma_n \)

\( \iff \) no player can unilaterally improve his costs in \( S \)

We do not consider mixed Nash equilibria in this tutorial.
Nash Equilibria

c_1(S) = 3
c_2(S) = 1
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**Definition**

\[ \text{pure Nash Equilibrium } S \in \Sigma_1 \times \cdots \times \Sigma_n \]
\[ \iff \text{no player can unilaterally improve his costs in } S \]

- Nash Equilibrium = stable
  - (if players are uncoordinated, rational, selfish)
- We do not consider mixed Nash equilibria in this tutorial.
A lot of research on **static properties** of equilibria:

How much does society suffer from selfish behavior?

- Let $\text{cost}$ be some measure for social cost, e.g.,
  
  $\text{cost}(S) = \sum_{i \in N} c_i(S)$ or $\text{cost}(S) = \max_{i \in N} c_i(S)$.

- \text{price of anarchy} = \max_{S \in \text{NE}} \frac{\text{cost}(S)}{\text{cost}(\text{Opt})}$
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  \]

- **price of anarchy** = \[
  \max_{S \in \text{NE}} \frac{\text{cost}(S)}{\text{cost}(\text{Opt})}
  \]

**Focus of this tutorial: Questions about dynamics**

- Do uncoordinated agents **reach an equilibrium**?
- How long does it take?
- Do they quickly reach a state with **small social cost**?
Nash Dynamics: Sequence of best responses of players.

\[ c_1(S) = 8 \]
\[ c_2(S) = 8 \]
\[ c_3(S) = 2 \]
Nash Dynamics

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Nash Dynamics

- **Nash Dynamics**: Sequence of best responses of players.

  \[ c_1(S) = 3 \]
  \[ c_2(S) = 1 \]
  \[ c_3(S) = 4 \]

- **Nash Dynamics with Liveness Property**: Each player gets a chance to play his/her best response after at most \( t \) steps.

Random Nash Dynamics: Players are chosen uniformly at random.

\( \epsilon \)-Nash Dynamics: Players change their strategy only if they can improve their own cost by a factor of at least \( 1 + \epsilon \).

Other dynamics (Noisy Nash Dynamics, Fictitious Play, Regret-minimization Dynamics) are discussed later.
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5. Conclusion
The State Graph

state graph $\mathcal{G} = (V, E)$

$V = \text{states} \quad E = \text{better/best responses}$
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state graph $G = (V, E)$

$V = \text{states} \quad E = \text{better/best responses}$

Properties of dynamics can be phrased in terms of state graph:

- pure Nash equilibrium $= \text{sink of state graph}$
state graph $\mathcal{G} = (V, E)$

$V =$ states  $E =$ better/best responses

Properties of dynamics can be phrased in terms of state graph:

- pure Nash equilibrium = sink of state graph
- potential game = acyclic state graph
  $\Rightarrow$ players eventually reach equilibrium.

Example: Congestion Games
The State Graph

**state graph** \( G = (V, E) \)

\[ (S_1, S_2, S_3) \]
\[ (S_1, T_2, S_3) \]
\[ (S_1, S_2, T_3) \]

Player 2

Player 3

\( V = \text{states} \quad E = \text{better/best responses} \)

Properties of dynamics can be phrased in terms of state graph:

- **pure Nash equilibrium** = sink of state graph
- **potential game** = acyclic state graph
  \( \Rightarrow \) players eventually reach equilibrium.
  Example: Congestion Games

- **non-potential games** = best responses may cycle.
Congestion Games

Congestion Game:

- set of players $\mathcal{N}$
- set of resources $\mathcal{R}$
  e.g., edges of a graph or set of servers

- set of strategies, $\forall i \in \mathcal{N} : \Sigma_i \subseteq 2^{\mathcal{R}}$
**Congestion Games**

**Congestion Game:**
- set of **players** $\mathcal{N}$
- set of **resources** $\mathcal{R}$
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- set of **strategies**, $\forall i \in \mathcal{N}$: $\Sigma_i \subseteq 2^\mathcal{R}$
  - $\Sigma_i = \{ P \subseteq \mathcal{R} \mid P \text{ path } s_i \rightarrow t_i \}$
  - $\Sigma_i = \{ P \subseteq \mathcal{R} \mid P \text{ path } s \rightarrow t \}$
  - $\Sigma_i = \{ T \subseteq \mathcal{R} \mid T \text{ spanning tree} \}$
  - $\Sigma_i = \{ \{ r \} \mid r \in \mathcal{R} \}$

(network congestion game)
(symmetric netw. cong. game)
(singleton congestion game)
Congestion Games

Congestion Game:
- set of players \( N \)
- set of resources \( R \)
  - e.g., edges of a graph or set of servers

\[ \frac{2}{3}, \frac{1}{8}, 4, 1 \]

\( s_1 \)
\( s_2 \)
\( s_3 \)
\( t_1 \)
\( t_2 \)
\( t_3 \)

- set of strategies, \( \forall i \in N : \Sigma_i \subseteq 2^R \)
  - \( \Sigma_i = \{ P \subseteq R \mid P \text{ path } s_i \rightarrow t_i \} \) (network congestion game)
  - \( \Sigma_i = \{ P \subseteq R \mid P \text{ path } s \rightarrow t \} \) (symmetric netw. cong. game)
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  - \( \Sigma_i = \{ \{ r \} \mid r \in R \} \) (singleton congestion game)

- latency functions \( \forall r \in R : \ell_r : \mathbb{N} \rightarrow \mathbb{N} \)
Rosenthal (Int. Journal of Game Theory 1973)

Every congestion game admits an exact potential function.

- \( \Phi : \Sigma_1 \times \cdots \times \Sigma_n \to \mathbb{N} \) with \( 0 \leq \Phi \leq n \cdot m \cdot \ell_{\text{max}} \)
- player decreases his latency by \( x \in \mathbb{N} \) \( \Rightarrow \) \( \Phi \) decreases by \( x \) as well
Rosenthal’s Potential Function

\[ \phi(S) = 2 + 2 + (1 + 8) = 13 \]

\[ \phi(S') = 2 + (2 + 3) + 1 + 1 = 9 \]

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- \( n_r = \) number of players \( i \) with \( r \in S_i \in \Sigma_i \)

\[ \phi(S) = \sum_{r \in \mathcal{R}} \sum_{i=1}^{n_r} \ell_r(i) \]
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\[ \phi(S) = \sum_{r \in R} \sum_{i=1}^{n_r} \ell_r(i) \]

⇒ Number of better response at most \( n \cdot m \cdot \ell_{\text{max}} \).
There exist network congestion games with an initial state from which all better response sequences have exponential length.
Known Results on Convergence Time

Fabrikant, Papadimitriou, Talwar (STOC 04)

There exist network congestion games with an initial state from which all better response sequences have exponential length.

Ieong, McGrew, Nudelman, Shoham, Sun (AAAI 05)

In singleton games all best response sequences have length at most \( n^2 \cdot m \).
Known Results on Convergence Time

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In singleton games all best response sequences have length at most $n^2 \cdot m$.

**Ackermann, R., Vöcking (FOCS 06)**

- In spanning tree congestion games all best response sequences have length at most $n^2 \cdot m \cdot$ number of vertices.
- In matroid congestion games all best response sequences have length at most $n^2 \cdot m \cdot$ rank.
Idea: Reduce latencies without affecting the game!

\[
\ell_r(n_r) > \ell_{r'}(n_{r'} + 1)
\]
Idea: Reduce latencies without affecting the game!

Equivalent latencies $\bar{\ell}_r(x) \leq n \cdot m$

$\forall r, r' \in \mathcal{R}, n_r, n_{r'} :$

$\ell_r(n_r) > \ell_{r'}(n_{r'} + 1)$

$\iff \bar{\ell}_r(n_r) > \bar{\ell}_{r'}(n_{r'} + 1)$

However, latency reduction works also for matroid games.
Singleton Games

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Network Congestion Games

\[ \ell_{r_1}(n_{r_1}) + \ell_{r_2}(n_{r_2}) > \ell_{r'_1}(n_{r'_1} + 1) + \ell_{r'_2}(n_{r'_2} + 1) \]
Singleton Games

- Idea: Reduce latencies without affecting the game!
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\forall r, r' \in \mathcal{R}, n_r, n_{r'} : \\
\ell_r(n_r) > \ell_{r'}(n_{r'} + 1) \\
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Network Congestion Games

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However, latency reduction works also for matroid games.
Matroid Congestion Games

Let $(\mathcal{R}, \mathcal{I})$ be any non-matroid anti-chain. Then, for every $n$, there exists an $n$-player congestion game with the following properties.

- each $\Sigma_i$ is isomorphic to $\mathcal{I}$,
- there is a best response sequence of length $2^{\Omega(n)}$. 
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- there is a best response sequence of length \(2^{\Omega(n)}\).

\[\Rightarrow\] Matroid property is the maximal property on the individual players’ strategy spaces that guarantees polynomial convergence.
Local Search Problem $\Pi$

- set of instances $I_\Pi$
- for $I \in I_\Pi$: set of feasible solutions $F(I)$
- for $I \in I_\Pi$: objective function $c: F(I) \rightarrow \mathbb{Z}$
- for $I \in I_\Pi$ and $S \in F(I)$: neighborhood $N(S, I) \subseteq F(I)$
Local Search Problem $\Pi$

- set of instances $\mathcal{I}_\Pi$
- for $I \in \mathcal{I}_\Pi$: set of feasible solutions $\mathcal{F}(I)$
- for $I \in \mathcal{I}_\Pi$: objective function $c: \mathcal{F}(I) \rightarrow \mathbb{Z}$
- for $I \in \mathcal{I}_\Pi$ and $S \in \mathcal{F}(I)$: neighborhood $\mathcal{N}(S, I) \subseteq \mathcal{F}(I)$

Johnson, Papadimitriou, Yannakakis (FOCS 85)

$\Pi$ is in PLS if polynomial time algorithms exist for

- finding initial feasible solution $S \in \mathcal{F}(I)$,
- computing the objective value $c(S)$,
- finding a better solution in the neighborhood $\mathcal{N}(S, I)$ if $S$ is not locally optimal.
PLS-reductions

**PLS-reduction**

- Polynomial-time computable function \( f: \mathcal{I}_{\Pi_1} \rightarrow \mathcal{I}_{\Pi_2} \).
- Polynomial-time computable function \( (S_2 \in \mathcal{F}(f(I))) \)

\( g: S_2 \mapsto S_1 \in \mathcal{F}(I) \)

A PLS-reduction is called tight if it does not shorten distances in the state graph.

\[ \Rightarrow \text{Exponential lower bounds are preserved.} \]
PLS-reductions

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- Polynomial-time computable function (\( S_2 \in \mathcal{F}(f(I)) \))
  \( g: S_2 \mapsto S_1 \in \mathcal{F}(I) \)
- \( S_2 \) locally optimal \( \Rightarrow \)
  \( g(S_2) \) locally optimal.

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- Polynomial-time computable function ($S_2 \in \mathcal{F}(f(I))$)
  
  $g: S_2 \mapsto S_1 \in \mathcal{F}(I)$

- $S_2$ locally optimal $\Rightarrow$ $g(S_2)$ locally optimal.

- local opt. of $\Pi_2$ easy to find $\Rightarrow$ local opt. of $\Pi_1$ easy to find
- local opt. of $\Pi_1$ hard to find $\Rightarrow$ local opt. of $\Pi_2$ hard to find
**PLS-reductions**

- Polynomial-time computable function \( f : \mathcal{I}_{\Pi_1} \rightarrow \mathcal{I}_{\Pi_2} \).
- Polynomial-time computable function \((S_2 \in \mathcal{F}(f(I)))\)
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Party Affiliation Games:

- Input: $G(V, E)$ and $w : E \rightarrow \mathbb{N}$
- agents = nodes, $\Sigma_i = \{\text{left, right}\}$
- $w(\{u, v\}) = \text{antipathy of } u \text{ and } v$
Party Affiliation Games

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Finding a locally optimal cut is PLS-complete.
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**Schäffer, Yannakakis (SIAM J. Comput. 1991)**
Finding a locally optimal cut is PLS-complete.

Very involved reduction from Circuit/Flip.

**First PLS-complete problem: Circuit/Flip**
- $C$: Boolean circuit composed of AND, OR, and NOT gates.
- Input to $C$: $x_1, \ldots, x_m \in \{0, 1\}$. Output of $C$: $y_1, \ldots, y_n \in \{0, 1\}$. 
Party Affiliation Games

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- Input to $C$: $x_1, \ldots, x_m \in \{0, 1\}$. Output of $C$: $y_1, \ldots, y_n \in \{0, 1\}$.
- Objective function: $f(x_1, \ldots, x_m) = \sum_{i=1}^{n} 2^{i-1} y_i$. 
Party Affiliation Game:

- Input: $G(V, E)$ and $w : E \rightarrow \mathbb{N}$
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Finding a **locally optimal cut** is PLS-complete.

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First PLS-complete problem: Circuit/Flip

- $C$: Boolean circuit composed of AND, OR, and NOT gates.
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- **Objective function:** $f(x_1, \ldots, x_m) = \sum_{i=1}^{n} 2^{i-1} y_i$.
- **Neighborhood** = Hamming distance 1
Congestion Games and PLS

Finding an equilibrium in a congestion game belongs to PLS:

- objective function = Rosenthal’s potential function
- \( S' \in \mathcal{N}(S) \) if \( S' \) is obtained from \( S \) by better response of one of the players.
Congestion Games and PLS

Finding an equilibrium in a congestion game **belongs to PLS**:
- objective function = Rosenthal’s potential function
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**PLS-completeness** follows by reduction from MaxCut:

\[
\begin{align*}
R_{\text{in}} & : \text{player } i \text{ on } R_{\text{in}} \iff \text{node } i \text{ on left side} \iff \text{latency of player } i \text{ on } R_{\text{in}} = \text{weight of edges from } i \text{ to the left side} \iff \text{latency on } R_{\text{in}} \leq W_i/2 \iff \text{contribution to cut when on left side} \geq W_i/2 \\
R_{\text{out}} & : \text{nodes } r_1, r_2, r_3, r_4
\end{align*}
\]
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- objective function $= \text{Rosenthal’s potential function}$
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**PLS-completeness** follows by reduction from MaxCut:

\[
R_{\text{in}} = \begin{array}{c}
0/w_{i,j} \\
r_{1,2} \\
r_{2,3} \\
r_{1,3} \\
r_{2,4} \\
r_{3,4} \\
r_{2,4}
\end{array} \\
R_{\text{out}} = \begin{array}{c}
\frac{w_{1,2} + w_{1,3}}{2} = \frac{W_1}{2} \\
\frac{w_{1,2} + w_{2,3} + w_{2,4}}{2} = \frac{W_2}{2} \\
\frac{w_{1,3} + w_{2,3} + w_{3,4}}{2} = \frac{W_3}{2} \\
\frac{w_{2,4} + w_{3,4}}{2} = \frac{W_4}{2}
\end{array}
\]

- \( g: \) player \( i \) on \( R_{\text{in}} \) \( \iff \) node \( i \) on left side
- latency of player \( i \) on \( R_{\text{in}} \) = weight of edges from \( i \) to the left side
Congestion Games and PLS

Finding an equilibrium in a congestion game belongs to PLS:
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PLS-completeness follows by reduction from MaxCut:

$g$: player $i$ on $\mathcal{R}_{\text{in}} \iff$ node $i$ on left side
- latency of player $i$ on $\mathcal{R}_{\text{in}} =$ weight of edges from $i$ to the left side
- node $i$ on left side
  $\iff$ latency on $\mathcal{R}_{\text{in}} \leq W_i/2$
  $\iff$ contribution to cut when on left side $\geq W_i/2$
Finding a pure Nash equilibrium in network congestion games is **PLS-complete**.

Reduction from Circuit/Flip that reworks MaxCut reduction.
Network Congestion Games and PLS

Fabrikant, Papadimitriou, Talwar (STOC 04)

Finding a pure Nash equilibrium in network congestion games is PLS-complete.

Reduction from Circuit/Flip that reworks MaxCut reduction.

Ackermann, R., Vöcking (FOCS 06)

Network congestion games are PLS-complete for (un)directed networks with linear latency functions.

Simple reduction from MaxCut.
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---

Ackermann, R., Vöcking (FOCS 06)

Network congestion games are PLS-complete for (un)directed networks with **linear latency functions**.

Simple reduction from MaxCut.

All these PLS-reductions are **tight**.

⇒ There exist states exponentially far from all sinks in the state graph.
Approximate Equilibria

What happens if players are lazy?

A state \( S = (S_1, \ldots, S_n) \) is called \((1 + \varepsilon)\)-approximate equilibrium if
\[
\forall i \in \mathcal{N} : \text{latency of player } i \leq (1 + \varepsilon) \cdot \text{min achievable latency of player } i
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What happens if players are lazy?

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A state $S = (S_1, \ldots, S_n)$ is called $(1 + \varepsilon)$-approximate equilibrium if

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**Positive Result:**

Chien, Sinclair (SODA 07)

In any symmetric congestion game with $\alpha$-bounded jump condition, the $(1 + \varepsilon)$-Nash dynamics converges after at most $\text{poly}(n, \alpha, \varepsilon^{-1}, \log(\ell_{\text{max}}))$ steps, assuming liveness property.
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\\]
steps, assuming liveness property.

Idea: high-cost player moves \( \Rightarrow \) significant potential drop
\( S \) not \((1 + \varepsilon)\)-equilibrium \( \Rightarrow \exists \) high-cost player that has an incentive to move. (due to \( \alpha \)-bounded jump condition and symmetry)
Approximate Equilibria

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A state $S = (S_1, \ldots, S_n)$ is called $(1 + \varepsilon)$-approximate equilibrium if $\forall i \in \mathcal{N}: \text{latency of player } i \leq (1 + \varepsilon) \cdot \text{min achievable latency of player } i$

Negative Result:

Skopalik, Vöcking (STOC 2008)

It is PLS-hard to compute an $(1 + \varepsilon)$-approximate equilibrium for any polynomial-time computable $\varepsilon$.

$\Rightarrow$ Exponentially many steps until $(1 + \varepsilon)$-approx. eq. is reached.

Very involved reduction from Circuit/Flip.
### Summary of Convergence Results

<table>
<thead>
<tr>
<th>Model</th>
<th>Nash Dynamics</th>
<th>$\varepsilon$-Nash Dynamics</th>
</tr>
</thead>
<tbody>
<tr>
<td>Matroid</td>
<td>poly</td>
<td>poly</td>
</tr>
<tr>
<td>Symmetric Network</td>
<td>exp</td>
<td>poly</td>
</tr>
<tr>
<td>Asymmetric Network</td>
<td>exp, PLS-complete</td>
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<tr>
<td>Symmetric General</td>
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<td>Cut Games</td>
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   - Non-Potential Games

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   - Nearly-optimal Solutions

5. Conclusion
Non-potential Games

State Graph

Sink equilibrium: [Goemans, M., Vetta (FOCS 2005)]
strongly connected comp. of state graph without outgoing edges

⇒ random Nash dynamics eventually reaches sink equilibrium
Two agents: \((r_1 = 1, r_2 = 2)\).

\(l_1(x) = x + 33, \ l_2(x) = 13x, \ l_3(x) = 3x^2, \ l_4(x) = 6x^2, \ l_5(x) = x^2 + 44, \) and \(l_6(x) = 47x.\)
Two agents: \((r_1 = 1, r_2 = 2)\).

Only Sink equilibrium: \(\{(P_1, P_2), (P_3, P_2), (P_3, P_4), (P_1, P_4)\}\).

**No Pure Nash equilibrium.**
Non-potential Games

Sink equilibrium: [Goemans, M., Vetta (FOCS 2005)]
strongly connected comp. of state graph without outgoing edges
Non-potential Games

**Sink equilibrium:** [Goemans, M., Vetta (FOCS 2005)]
strongly connected comp. of state graph without outgoing edges

Complexity Questions about Nash Dynamics and sink equilibria:

1. Given a state in a game, is it in a sink equilibrium?
2. Given a game, determine if it has a pure Nash equilibrium?
3. Given a game, determine if it has any non-singleton sink equilibrium?
Non-potential Games

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**Theorem (M., Skopalik, EC 2009)**

*For many classes of games with succinet representation, it is PSPACE-hard to answer questions 1 and 3, and it is NP-hard to answer question 2.*

- Player-specific and weighted congestion games.
- Anonymous Games and Graphical Games.
- Many-to-one two-sided market games.
Interesting subclass: Games with only singleton sink equilibria

Milchtaich (Games and Economics Behaviour, 1996)

Player-specific singleton congestion games:
- Pure Nash equilibria exist, but best resp. dyn. can cycle.
- From every state there is a sequence of best responses to an equilibrium.
How to find a stable marriage?

Let’s get to the really important problems...
The Stable Marriage Problem

Set of women $\mathcal{X}$

Set of men $\mathcal{Y}$
The Stable Marriage Problem

Set of women $\mathcal{X}$

Every person has a preference list.
The Stable Marriage Problem

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Set of women \( \mathcal{X} \)

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Every person has a preference list.
Stable Matching

A matching is stable if there does not exist a blocking pair.
Formal Definition

Stable Matching

A matching is stable if there does not exist a blocking pair.

$w, m$ is a blocking pair $\iff$

1) $w$ prefers $m'$ to $m$
2) $m'$ prefers $w$ to $w'$
A matching is stable if there does not exist a blocking pair.

\[(w, m') \text{ is blocking pair} \iff \]

1) \(w\) prefers \(m'\) to \(m\)
2) \(m'\) prefers \(w\) to \(w'\)
A matching is stable if there does not exist a blocking pair. A stable matching can be computed efficiently.
Applications and Previous Work

- **Many Applications**: Interns/Hospitals, College Admission, Labor market.

Main Question

What happens without central authority?

Do players reach a stable matching?

How long does it take?

Consecutive resolving of blocking pairs:

Knuth observed a cycle.

Roth and Vande Vate showed that there is no non-trivial sink equilibrium (Econometrica 1990).
Many Applications: Interns/Hospitals, College Admission, Labor market.

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**Main Question**

What happens **without central authority**?

- Do players **reach a stable matching**?
- How **long** does it take?

Consecutive resolving of blocking pairs:

- Knuth observed a **cycle**.
- Roth and Vande Vate showed that there is **no non-trivial sink equilibrium** (Econometrica 1990).
Best Response Dynamics

Matching not stable $\Rightarrow$ Choose woman, let her play best response.
Best Response Dynamics

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Matching not stable ⇒ Choose woman, let her play best response.
Best Response Dynamics

Matching not stable ⇒ Choose woman, let her play best response.
The best response dynamics can cycle.

Was shown for better response dynamics by Knuth.
The best response dynamics can cycle. Was shown for better response dynamics by Knuth.

From every matching there exists a sequence of $2n^2$ best responses to a stable matching. Was shown for better response dynamics by Roth and Vande Vate. $\Rightarrow$ Random best response dynamics reaches a stable matching with probability 1.
The best response dynamics can cycle.

Was shown for better response dynamics by Knuth.

From every matching there exists a sequence of $2n^2$ best responses to a stable matching.

Was shown for better response dynamics by Roth and Vande Vate.

⇒ Random best response dynamics reaches a stable matching with probability 1.

There exist instances such that the expected number of best responses is $\Omega(c^n)$ for some constant $c > 1$.

Similar exponential bound holds for better response dynamics.
Theorem
From every matching there exists a sequence of $2n^2$ best responses to a stable matching.

Claim 1
If only married women play best responses, after at most $n^2$ steps every married woman is happy.

Claim 2
If every married woman is happy, every sequence of best responses terminates after at most $n^2$ steps.
Claim 1

If only married women play best responses, after at most $n^2$ steps every married woman is happy.

Proof.

Use the following potential function:

$$\Phi = \sum_{\text{married woman } w} \text{rank of } w's \text{ current partner}$$

$0 \leq \Phi \leq n^2$ and $\Phi$ decreases with every best response.
Claim 1

If only married women play best responses, after at most \( n^2 \) steps every married woman is happy.

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$\Rightarrow$ Men are never dumped.
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Use the following potential function:

$$\Psi = \sum_{\text{married man } m} n + 1 - \text{rank of } m\text{'s current partner}$$

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\[
\begin{array}{c}
\bigcirc & \bigcirc & 1 \\
\bigcirc & \bigcirc & 2 \\
\bigcirc & \bigcirc & \Psi = 5
\end{array}
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$$\Psi = 5 \quad \Psi = 6 \quad \Psi = 5$$
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Proof.
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Use the following potential function:
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\]

$0 \leq \Psi \leq n^2$ and $\Psi$ increases with every best response.

\[
\begin{array}{c}
\circ - \circ & 1 & \circ - \circ & 1 \\
\circ - \circ & 2 & \circ - \circ & 1 \\
\circ & \circ & \circ & \circ \\
\Psi = 5 & \Psi = 6 & \Psi = 5 & \Psi = 8 \\
\end{array}
\]
Further Results – Correlated Instances

Good news: Correlation helps!

Monotone Instances
Input: complete, weighted bipartite graph $G = (V, E, w)$. Every player tries to maximize the weight of her/his relationship.
Good news: Correlation helps!

**Monotone Instances**

Input: complete, weighted bipartite graph $G = (V, E, w)$. Every player tries to maximize the weight of her/his relationship.

![Graph](https://via.placeholder.com/150)

**Theorem**

Random best/better responses converge in polynomial time whp.
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Price of Anarchy and Convergence

- Price of anarchy = \( \frac{\text{Social Value of the worst equilibrium}}{\text{Optimal Social Value}} \).
Price of Anarchy and Convergence

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- **Large** Price of Anarchy: Need for Central Regulation.
- **Small** Price of Anarchy: Does not indicate good performance.
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- Players may not converge to those equilibria.
- Convergence to equilibria may take exponential time.
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**Question 1:** Potential Games: How fast do players converge to approximate solutions? (and not to equilibria).

**Question 2:** Non-Potential Games: What is the quality of solutions that players converge to?
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Price of anarchy: 2.5 (Koutsoupias, Christoudolou, 05 and Awerbuch, Azar, Epstein, 05).
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How about convergence time to constant-factor approximate solutions?
Convergence to Nearly-optimal Solutions

Theorem (Awerbuch, Azar, Epstein, M., Skopalik, EC 2008)

Convergence time of Nash dynamics with liveness property to constant-factor optimal solutions in linear congestion games might be exponential.

Proof Idea: Three lemmas:
- In any bad state, there exists a player who improves the average by a large margin, thus there is a state.
- In any bad state, the expected value of the change incurred by players is not too bad.
- Use induction on the above lemmas.

⇒ The price of anarchy for sink equilibrium is a constant.
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This is in contrast to:

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For a large class of potential games that are $\beta$-nice, and satisfy bounded-jump condition, after polynomial steps of $\epsilon$-Nash dynamics with a liveness property, players converge to a solution with approximation factor of price of anarchy.
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Bounded-jump condition (informal): After a player $i$ plays a best response, the change in the payoff (cost) of other players is bounded by the new payoff (cost) of player $i$. 

- Congestion games with constant-degree polynomial delay functions,
- Weighted congestion games with linear delay functions,
- Party affiliation games,
- Market sharing games.
Convergence to Nearly-optimal Solutions

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- **Bounded-jump condition** (informal): After a player $i$ plays a best response, the change in the payoff (cost) of other players is bounded by the new payoff (cost) of player $i$.

- For example:
  - Congestion games with constant-degree polynomial delay functions,
  - Weighted congestion games with linear delay functions,
  - Party affiliation games,
  - Market sharing games.
Convergence to Nash equilibria: exponential
Convergence to nearly-optimal solutions:

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<thead>
<tr>
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<tr>
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<td>poly, 70</td>
<td>poly, 2.5 + ϵ</td>
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<tr>
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<td>poly, $O(2^{2d})$</td>
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<td>Wei. Lin. Cong.</td>
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<td>poly, $\frac{1}{6}$</td>
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<tr>
<td>Market Sharing</td>
<td>$\frac{1}{2}$</td>
<td>poly, $\frac{1}{\log n}$</td>
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For other games, check the $β$-nice and bounded jump condition.
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Question 2 (Non-Potential Games): What is the quality of solutions that players converge to?
Sink Equilibria and Convergence

• **Question 2 (Non-Potential Games):** What is the quality of solutions that players converge to?

• Price of anarchy for mixed NE might be good, but how about convergence to good-quality solutions in non-potential games?
Question 2 (Non-Potential Games): What is the quality of solutions that players converge to?

Price of anarchy for mixed NE might be good, but how about convergence to good-quality solutions in non-potential games?

In other words, what is the price of anarchy of sink equilibria?
A sink equilibrium is a set of states.
Each state has a social value.
Price of Anarchy for Sink equilibria

- A sink equilibrium is a set of states.
- Each state has a social value.

**Social Value of a Sink equilibrium?**

Social Value of a Sink equilibrium = Average Social value of states on a random best-response walk.

- *Random Best-response Walk*: Choose a player uniformly at random at each step.
A sink equilibrium is a set of states.
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Social Value of a Sink equilibrium?
Social Value of a Sink equilibrium = Average Social value of states on a random best-response walk.

Random Best-response Walk: Choose a player uniformly at random at each step.

Price of anarchy for sink equilibrium = value of the worst sink equilibrium \[ \text{Opt} \].
### Theorem (Goemans, M., Vetta, FOCS 2005)

For weighted congestion games with constant-degree polynomial delay functions, the price of anarchy for sink equilibria is constant.

Related to convergence of random Nash dynamics to constant-factor approximate solutions.
Sink Equilibria and Convergence

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Theorem (Goemans, M., Vetta, FOCS 2005)

For a general class of market sharing games (aka valid-utility games), even though the price of anarchy for mixed NE is constant ($1/2$), the price of anarchy for sink equilibria is very poor ($1/n$).

⇒ Players may converge to a bad-quality solution and they may get stuck there.
## Sink Equilibria and Convergence

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<tbody>
<tr>
<td>For a general class of market sharing games (aka valid-utility games), even though the price of anarchy for mixed NE is constant ($1/2$), the price of anarchy for sink equilibria is very poor ($\frac{1}{n}$).</td>
</tr>
</tbody>
</table>

⇒ Players may converge to a bad-quality solution and they may get stuck there.
What if players follow other dynamics?
Outline

1 Introduction: Games, Equilibria, and Dynamics

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4 Other Dynamics
   - Equilibria
   - Nearly-optimal Solutions

5 Conclusion
Natural Distributed/Synchronous Dynamics

- Fictitious Play
- Replicator dynamics
- Noisy Nash dynamics
- No-regret dynamics
Natural Distributed/Synchronous Dynamics

- **Fictitious Play**
  - Best response to the empirical distribution of the opponents.
  - Nash equilibrium is an “absorbing state”
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Fictitious Play
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Replicator dynamics
- Each strategy survives according to its excess payoff
- Most reasonable variants converge in potential games
  - [Sandholm JET 2001]
  - Convergence rate [Fischer, Räcke, Vöcking STOC06]

Noisy Nash dynamics
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- Convergence properties in Congestion Games [Asadpour, Saberi 2009].

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Noisy Nash dynamics.
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No-regret dynamics.
- Known to converge in specific games to Nash equilibrium.
- There exist games on which uncoupled dynamics do not converge [Hart and Mas-Collel].
No-External-Regret

Is there a strategy that guarantees that the total routing time will take almost as time as the best fixed path in hindsight?
No regret in Congestion Games

No-External-Regret

Is there a strategy that guarantees that the total routing time will take almost as time as the best fixed path in hindsight?

No-Internal-Regret

Is there a strategy that guarantees that the total routing time when it took path P will take almost as time as the best fixed path in hindsight for that time steps?
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Is there a strategy that guarantees that the total routing time when it took path P will take almost as time as the best fixed path in hindsight for that time steps?

We say that algorithm is No X-Regret if its regret to best static decision, $R(T)$ is sublinear.
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Correlated Equilibria [Aumann 1974]

- Distribution over $N$-tuples.
Correlated Equilibria [Aumann 1974]

- Distribution over $N$-tuples.
- Nash Equilibrium with a shared signal

Properties:
- Contains the convex hull of Nash equilibrium.
- Can be computed efficiently.
Correlated Equilibria [Aumann 1974]

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Equilibria Types

- Mixed Nash Equilibrium
- Pure Nash Equilibrium
- Correlated Equilibrium
- No Regret
No Regret convergence

- **No-internal-regret convergence to Correlated equilibria**
  - [Hart and Mas-Collel, Foster and Vohra] If every player plays a no internal regret algorithm, then the empirical distributions of play converge almost surely as \( t \to \infty \) to the set of correlated equilibrium distributions of the game.
  - The convergence is of the empirical distributions and not at a specific time.

- **No-external-regret and zero sum games**

- **No-external-regret and Routing games**

- **Atomic games specific update rule** [Kleinberg, Piliouras and Tardos STOC 09]

- **Parallel links** [Blum, Even-dar and Ligett PODC 06]

- **Splittable traffic** [Even-dar, Mansour and Nadav STOC 09]

- **Infinitesimal users (Wardrop model)** [Blum, Even-dar and Ligett PODC 06]
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Quality of playing no-regret

- In congestion games same bounds hold through similar arguments [Roughgarden STOC 09]
- Valid utility games and Hotelling games [Blum et al. STOC 08]
Recall

No Regret

Correlated Equilibrium

Mixed Nash Equilibrium

Pure Nash Equilibrium

Quality of playing no regret

\[ \text{price of No regret} \geq \text{price of Correlated} \geq \text{price of Mixed N.E} \geq \text{price of Pure N.E} \]
Recall

price of No regret $\geq$ price of Correlated $\geq$ price of Mixed N.E $\geq$ price of Pure N.E
Load balancing example

Consider $n$ parallel links and $n$ identical users and Makespan metric then:

- Pure N.E and sink: $P_{ofA} = 1$
- Mixed N.E: $P_{ofA} = \frac{\log n}{\log \log n}$
- Correlated Eq. and No regret: $P_{ofA} = \sqrt{n}$
Consider $n$ parallel links and $n$ identical users and Makespan metric then:

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Pure N.E and sink : $PofA = 1$
Consider valid-utility games then:
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Sink Eq.: $\text{PofA} \geq n$

Pure N.E to No Regret: $\text{PofA} = 2$
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Learning Algorithms

- In many realistic games learning algorithms can lead to Nash equilibrium or high quality state.
  - Can be used for computing Nash equilibria.
- What can we say about games where nice behavior is not guaranteed?
- Effect of using machine learning algorithms and game dynamics in (ad) auctions (or everywhere...)
Conclusions and Future Directions

Questions about Dynamics

1. What do players converge to?
   Find potential functions? Characterize sink equilibria?

2. How long does it take?
   PLS-complete?

3. Do they quickly reach a state with small social cost?
   Performance of equilibria? Random or $\varepsilon$-dynamics.

4. Take your favorite game and answer these questions.
   Ad auctions, scheduling games, distributed caching games, …
Thank You

Special thanks to Eyal Even Dar for sharing his slides with us from another joint tutorial.