Auction: we have a set of buyers and sellers in the market. The more the situation becomes close to the perfect market scenario, the more the situation becomes like the one of an auction, where there is only one seller—i.e., the auctioneer. The auction rules define the social choice, i.e., the identity of the winner. We have a set of alternatives (situations) \( A \) and a set \( n \) of players \( I \).

We have a valuation function \( v_i : A \to \mathbb{R} \), where \( v_i(a) \) denotes the "value" that \( i \) assigns to alternative \( a \) being chosen. The value is defined as the utility that \( i \) gets from receiving \( a \).

In terms of some currency, we assume that if \( a \) is chosen, then player \( i \)'s utility is \( v_i(a) \). This utility being the abstraction of what the player desires, and \( i \) is additionally given some quantity \( m \) of money, then its utility is denoted as \( u_i = v_i(a) + m \), this utility being the abstraction of what the player desires, and \( i \)'s utility is maximized. Utilities of this form are called quasi-linear preferences.

**Vineyard's Second Price Auction**

- **Vineyard's Second Price Auction**

  - We have a single item to sell and we have \( n \) players.
  - Each player \( i \) has a valuation \( v_i \) that he is "willing to pay" for this item. If \( i \) wins the item, \( i \) has to pay \( v_i \) for it, and the utility is \( v_i - p \), where \( p \) is the second-highest bid. If someone else wins the item, then \( i \)'s utility is \( 0 \), thus the set of alternatives here is the set of possible winners, \( A = \{ i \text{-wins} \} \cup \{ j \text{-wins} \} \) and the valuation of each bidder \( i \) is
    
    \[ v_i( i \text{-wins}) = v_i \quad \text{and} \quad v_i( j \text{-wins}) = 0 \quad \text{for all } j \neq i. \]

  - A social choice, i.e., an aggregation of the preferences of the different participants toward a single joint decision, is to allocate the item to the player who values it highest: choose \( i \text{-wins} \), where \( i = \arg \max_j v_j \).

  - However, we do not know all values, which are private, and we want to make sure that our mechanism decides on the allocation in a way that cannot be strategically manipulated.

  - Our degree of freedom is the definition of the payment by the winner.
Consider some options:
- No payment: we give the item for free to the highest bid. It can be easily manipulated by exaggerating.
- Pay your bid. Again there is a problem.
  A player with value $w_j$ who wins and pays $w_i$ gets a total utility of 0. Thus he should declare a somewhat lower value $w_i'$ that still wins. He pays less and his utility $u_i = w_i - w_i' > 0$.
  He essentially can bid the second price, if he knows that.

Here is the solution:
Vickrey's second price auction: let the winner be the player with the highest declared value of $w_i$ and let $i$ pay the second highest declared bid $p^* = \max_{j \neq i} w_j$.

**Proposition (Vickery)** For every $w_1, \ldots, w_n$ and every $w_i$ let $u_i$ be its utility if he bids $w_i$ and $u_i'$ his utility if he bids $w_i'$. Then $u_i > u_i'$.

**Proof:** Assume that the valuation of $u_i$ and the second highest value is $p^*$, then $u_i = w_i - p^* > 0$. Now for an attempted manipulation of $w_i$ to $p^*$, if I would still win if I bids $w_i$ and would still pay $p^*$, thus $u_i = u_i$.

On the other hand, for $w_i < p^*$, I would lose so $u_i = 0 < u_i'$.

Now, if I loses by bidding $w_i$ then $u_i = 0$. Let $j$ be the winner in this case, and thus $w_j > w_i$. For $w_i < w_j$, I would still lose and so $u_i = 0 < u_i$.

For $w_i > w_j$, I would win but would pay $w_j$, thus his utility would be $u_j = w_i - w_j < 0 = u_i'$.

Thus this mechanism reliably computes a function (any $\max$) of $n$ numbers (the $w_i$'s) that are each held secretly by a different self-interested player. We need lots of these case analyses for the proof of truthfulness in general.
Def: A (direct revelation) function (A.K.A. mechanism) is a social choice function \( f : V_1 \times \ldots \times V_n \to A \) and a vector of payment functions \( p_1, \ldots, p_n \) where \( p_i : V_1 \times \ldots \times V_n \to \mathbb{R} \) is the amount that player \( i \) pays.

Def: A mechanism \( (f, p_1, p_2, \ldots, p_n) \) is called incentive compatible if for every player \( i \), every \( v_i \in V_i \), \( v_j \in V_j \) and \( v = (v_i, v_j) \) if we denote \( a = f(v_i, v_j) \) and \( a' = f(v_i', v_i) \) then

\[ v_i(a) - p_i(v_i, v_j) \geq v_i(a') - p_i(v_i', v_i). \]

It means player \( i \) prefers telling the truth \( v_i \) to the mechanism rather than any possible "lie" \( v_i' \), since this gives him higher (in the weak sense) utility.

When there is money, there is an incentive compatible mechanism for the most natural social choice function: optimizing the (oftentimes) social welfare, which is for an alternative \( a \in A \), is the sum of valuations of all players for this alternative, \( \sum v_i(a) \).

Def: A mechanism \( (f, p_1, p_2, \ldots, p_n) \) is called a Vickrey-Clarke-Grof (VCG) mechanism if

\[ f(v_i, \ldots, v_n) \in \text{argmax}_{a \in A} \sum v_i(a); \text{ that is } f \text{ maximizes the social welfare.} \]

For some functions \( h_1, \ldots, h_n \), where \( h_i : V_i \to \mathbb{R} \) (i.e., \( h_i \) does not depend on \( v_i \)), we have that for all \( v_i \in V_i \), \( v_j \in V_j \),

\[ p_i(v_1, \ldots, v_n) = h_i(v_i) - \sum_{j 
eq i} v_j(f(v_i, \ldots, v_n)) \]

each player is paid an amount equal to the sum of the values of all other players. When we add this to \( v_i(h(v_1, \ldots, v_n)) \) the sum becomes exactly the total social welfare of \( (v_1, \ldots, v_n) \).
The (VCG): every VCG mechanism is incentive

Truthful: \( a = f(v_i, v_{-i}) \)

\[
V_i(a) + \sum_{j \neq i} V_j(a^j) - h_i(v_{-i}) \]

lies: \( a' = f(v_i, v_{-i}) \)

But since \( a = f(v_i, v_{-i}) \) maximizes social welfare over all alternatives,

\[
V_i(a) + \sum_{j \neq i} V_j(a^j) \geq V_i(a') + \sum_{j \neq i} V_j(a^j) \]

and thus the same inequality holds when we subtract term \( h_i(v_{-i}) \) from both sides.

E.g., in auction of a single item, finding the player with higher value is exactly equivalent to maximizing \( \sum V_i(i) \) since only a single player gets non-zero value. The payment is Clarke pivot payment.

If \( h_i = 0 \), then though the mechanism is simple but the mechanism pays money.

Def: The choice \( h_i(v_{-i}) = \max b \geq \sum V_i(b) \)

is called the Clarke pivot payment. Under this rule the payment of player \( i \) is \( p_i(v_1, \ldots, v_n) = \max b \sum V_i(b) \)

where \( a = f(v_1, \ldots, v_n) \).

Intuitively, \( p_i \) pays in amount equal to the total damage that he causes the other players. The difference between the social welfare of the others with and without his participation.

Lim: A VCG with Clarke pivot payments makes the case that no player's always gets non-negative utility.

Pf: Trivial

Clarke pivot rules does not fit many situations where valuations are but it is fairly general rule payment.
Combinatorial Auctions: (A very general auction setting)

Def: A valuation \( v \) is a real-valued function that for each subset \( S \) of items, \( v(S) \) is the value that bidder \( i \) obtains if he receives this bundle of items. A valuation must have two "disposability", i.e., be normalized: for \( S \subseteq T \) we have that \( v(S) \leq v(T) \) and it should be monotone: 

\[
v(\emptyset) = 0.
\]

Usually, we have subadditivity, i.e., 

\[
v(S \cup U) \leq v(S) + v(U)
\]

Again, the utilities of bidders are "quasi-linear" in the money, i.e., if the utilities of bidders are "quasi-linear" in the money, i.e., if bidder \( i \)'s bundle \( S \) and pays a price of \( p \) for it then his utility is 

\[
v_i(S) - p.
\]

Also we assume that there are no externalities, i.e., a bidder only values effects to or from other players. (We have positive or negative externalities)

Def: An allocation of the items among the bidders is \( S_1, \ldots, S_n \) where \( S_i \cap S_j = \emptyset \) for every \( i \neq j \). The social welfare obtained by an allocation is \( \sum v_i(S_i) \). A socially efficient allocation (among bidders with valuations \( v_1, \ldots, v_n \)) is an allocation with maximum social welfare among all allocations.

Note that if we use VCG payments, then these payments essentially charge each bidder his "externality": the amount by which his allocated bundle reduced the total reported value of the bundles allocated to the others. So, this is incentive compatible.

- Computational complexity: The allocation problem is \( \mathrm{NP} \)-hard even for simple cases.
- Representation and communication: The valuation functions are exponential sizes, how can we even represent them?
The second issue itself forces us to look for languages that allow succinct representations of valuations. We will call them bidding languages.

Here we face expressiveness vs. simplicity issues.

Express succinctly as many naturally occurring

Eg. We can use combination of atomic bids with OR & XOR

atomic bid offer of p for bundle S of items or any T ⊇ S and

Then we have \((\{a,b\}, 3) \lor (\{c,d\}, 5)\) then \(v(\{a,b\}, 3) = 0, v(\{c,d\}, 5) = 5\)

\\(e.g. \{\{a\}, 2\}\) OR \((\{c,d\}, 5)\) then \(v(\{a\}, 2) = 0, v(\{c,d\}, 5) = 8\)

\((\{a,b\}, 3) \cup (\{c\}, 5)\) \(v(\{a,b\}, 3) = 5\) since we can satisfy only one.

This is a natural approach see more details of bidding language in the book AGT, Sec 11.4. This is an active research e.g. for google.

Sometimes simple bidders are considered, i.e., single-minded bidders that is a bidder \(i\) for which there is a set \(A_i \subseteq S\) of goods and a value \(v_i \geq 0\) such that

(a) \(v_i(T_i) = v_i\) whenever \(T_i \supseteq A_i\); and \(v_i(T_i) = 0\) otherwise.

Even for the special case of single-minded bidders, VCG mechanism can not be implemented in polynomial-time since the optimization problem of winner determination, i.e., given single-minded bidders \((A_1, v_1) \ldots (A_n, v_n)\), grant a set of disjoint bids (i.e. a subset of players such that the corresponding \(A_i's\) are pairwise disjoint) to maximize the sum \(\sum_{i} v_i\) of the values of the granted bids, is \(NP\)-hard (indeed \(SL(n容)\)-hard to approximate).

Proof: by a reduction from \(NP\)-hard Weighted Independent Set (input a graph \(G = (V, E)\) and a weight \(w_v\) for each vertex \(V\) \(E\)). The set of goods is the set \(E\) of edges; the set of players is the set of vertices \(V\); for a vertex/player \(v \in V\), set \(w_v = w_v\) and \(\{v\} = \{v\}\) equal to the set of edges of \(G\) incident to \(v\). A set of player is independent iff the subset can be granted simultaneously.