Auction: we have a set of buyers and sellers in the market, the more people the more the situation becomes close to the Perfect Market scenario

An exterme opposite case is where there is only a single seller_an auctioneer

The auction rules define the social choice, i.e. the identity of the winner

We have a set of alternatives (situations) A and a set n of players I.

We have a valuation function $v_i : A \to R$, where $v_i(a)$ denotes the "value" (or valuation) that i assigns to alternative a being chosen. The value is in terms of some currency. We assume that if a is chosen and then player i is additionally given some quantity m of money, then its utility is $u_i = v_i(a) + m$, this utility being the abstraction of what the player desires and aims to maximize. Utilities of this form are called quasilinear preferences denoting the separable and linear dependence on money.

A famous Auction: English (ascending) auction e.g. e-Bay. The auctioneer raises the current price by small amount until there is only one bidder remains at the current price. Highest bidder wins.

Vickery's Second Price Auction

We have a single item to sell and we have n players. Each player i has valuation v_i that he is "willing to pay" for this item. If he wins the item, he has to pay some price p for it then his utility is $w_i - p$, while if someone else wins the item then its utility is 0. Thus the set of alternatives here is the set of possible winners, $A = \{i\text{-wins} \mid i \in I\}$ and the valuation of each bidder i is $v_i(i\text{-wins}) = w_i$ and $v_i(j\text{-wins}) = 0$ for all $j \neq i$.

A social choice, i.e., an aggregation of the preferences of the different participants toward a single joint decision, is to allocate the item to the player who values it highest: choose i-wins, where $i = \arg\max_j w_j$.

However we do not know all values, which are private, and we want to make sure that our mechanism decides on the allocation_ the social choice _ in a way that cannot be strategically manipulated (be truthful or strategy proof or Incentive Compatible). Our degree of freedom is the definition of the payment by the winner.

Consider some options:

- No payment: we give the item for free to the highest bid. It can be easily manipulated by exaggerating.
- Pay your bid. Again there is a problem. A player with value w_i who wins and pays w_i gets a total utility of 0. Thus he should declare a somewhat lower value $w_i' < w_i$ that still wins. He pays less w_i' and and his utility $u_i = w_i - w_i' > 0$. He essentially can bid the second price, if he knows that.

Here is the solution:
Vickery's second Price Auction: let the winner be the player i with the highest declared value of w_i and let i pay the second highest declared bid $p^* = \max_{j \neq i} w_j$.

Proposition (Vickery) For every $w_1, \dots, w_n$ and every w_i', let u_i be i's utility if he bids w_i and u_i' his utility if he bids w_i'. Then $u_i \geq u_i'$

Proof: Assume that the valuation of i is w_i and the second highest value is p^*, then $u_i = w_i - p^* \geq 0$. Now for an attempted manipulation (reporting) $w_i' > p^*$, i would still win if he bids w_i' and would still pay p^*, thus $u_i' = u_i$ on the other hand, for $w_i' \leq p^*$, i would lose so $u_i' = 0 \leq u_i$
Now, if i loses by bidding w_i, then $u_i = 0$. let j be the winner in this case, and thus $w_j \geq w_i$. For $w_i' < w_j$, i would still lose and so $u_i' = 0 = u_i$
For $w_i' \geq w_j$, i would win but would pay w_j, thus his utility would be $u_i' = w_i - w_j \leq 0 = u_i$ □

Thus this mechanism reliably computes a function (arg max) of n number (the w_i's) that are each held secretly by a different self-interested player. We need lots of these case analysis for the proof of truthfulness in general.

Def: A (direct revelation) Auction (A.K.A mechanism) is a social choice function $f: V_1 \times \cdots \times V_n \to A$ and a vector of payment functions $p_1, \ldots, p_n$
where $p_i: \underbrace{V_1 \times \cdots \times V_n}_{\text{or } A \text{ (in several cases)}} \to R$ is the Amount that player i pays.
(A: action or alternative)

Def: A mechanism $(f, p_1, p_2, \ldots, p_n)$ is called incentive compatible (strategy-proof or truthful) if for every player i, every $v_1 \in V_1, \ldots, v_n \in V_n$ and every $v_i' \in V_i$, if we denote $a = f(v_i, v_{-i})$ and $a' = f(v_i', v_{-i})$ then
$v_i(a) - p_i(v_i, v_{-i}) \geq v_i(a') - p_i(v_i', v_{-i})$.
It means player i prefer "telling the truth" v_i to the mechanism rather than any possible "lie" v_i', since this gives him higher (in the weak sense) utility.

when there is money, there is an incentive compatible mechanism for the most natural social choice function: optimizing the social welfare, (or efficiency) which is for an alternative $a \in A$ is the sum of valuations of all players for this alternative, $\sum_i v_i(a)$.

Def: A mechanism $(f, p_1, \ldots, p_n)$ is called a Vickrey-Clarke-Groves (VCG) mechanism if

- $f(v_1, \ldots v_n) \in \arg\max_{a \in A} \sum v_i(a)$; that is f maximizes the social welfare

and

- for some functions $h_1, \ldots h_n$, where $h_i: V_{-i} \to R$ (i.e., h_i does not depend on v_i), we have that for all $v_1 \in V_1, \ldots v_n \in V_n$,

$$p_i(v_1, \ldots, v_n) = h_i(v_{-i}) - \underbrace{\sum_{j \neq i} v_j(f(v_1, \ldots v_n))}_{\substack{\text{each player is paid} \\ \text{an amount equal to the sum} \\ \text{of the values of all other players.}}}$$

when we add this to $v_i(f(v_1, \ldots v_n))$ the sum becomes exactly the total social welfare of $f(v_1, \ldots v_n)$.

Thm (VCG): every VCG mechanism is incentive

Truthful: $a = f(v_i, v_{-i})$

$v_i(a) + \sum_{j \neq i} v_j(a) - h_i(v_{-i})$

lies: $a' = f(v_i', v_{-i})$

$v_i(a') + \sum_{j \neq i} v_j(a') - h_i(v_{-i})$.

But since $a = f(v_i, v_{-i})$ maximizes social welfare over all alternatives

$v_i(a) + \sum_{j \neq i} v_j(a) \geq v_i(a') + \sum_{j \neq i} v_j(a')$ and thus the same inequality holds when we subtract term $h_i(v_{-i})$ from both sides.

E.g. in Auction of a single Item, finding the player with highest value is exactly equivalent to maximizing $\sum_i v_i(i)$ since only a single player gets non-zero value. The payment is clarke pivot payment below

If all $h_i = 0$, then though the mechanism is simple but the mechanism pays money.

Def: The choice $h_i(v_{-i}) = \max_{b \in A} \sum_{j \neq i} v_i(b)$ is called the clarke pivot payment. Under this rule the payment of player i is $P_i(v_1, \dots, v_n) = \max_b \sum_{j \neq i} v_i(b) - \sum_{j \neq i} v_i(a)$ where $a = f(v_1, \dots, v_n)$.

Intuitively, i pays an amount equal to the total damage that he causes the other players. The difference between the social welfare of the others with and without its participation.

Lm: A VGG with clarke pivot payments makes the case that no players ~~always get non-negative utility~~ is ever paid.

Pf: Trivial

clarke pivot rules does not fit many situations where valuations are negative but it is fairly general rule payment.

Combinatorial Auctions: (A very General Auction Setting)

Def: A valuation v is a real-valued function that for each subset S of items, $v(S)$ is the value that bidder i obtains if he receives this bundle of items. A valuation must have free "disposal", i.e. be monotone: for $S \subseteq T$ we have that $v(S) \le v(T)$ and it should be normalized: $v(\emptyset) = 0$.

usually we have subaddivity, i.e., $v(S \cup T) \le v(S) + v(T)$

Again the utilities of bidders are "quasi-linear" in the money, i.e. if bidder i wins bundle S and pays a price of p for it then his utility is $v_i(S) - p$.

Also we assume that there are "no externalities" (effects to or from other players. (We have positive or negative externalities)), i.e., a bidder only cares about the item that he receives and not about how the other items are allocated among the other bidders.

Def: An allocation of the items among the bidders is $S_1, \ldots, S_n$ where $S_i \cap S_j = \emptyset$ for every $i \ne j$. The social welfare obtained by an allocation is $\sum_i v_i(S_i)$. A socially efficient allocation (among bidders with valuations $(v_1, \ldots, v_n)$) is an allocation with maximum social welfare among all allocations.

Note that if we use VCG payments, then these payments essentially charge each bidder his "externality": the amount by which his allocated bundle reduced the ~~total~~ reported value of the bundles allocated to the others. So this is incentive compatible. but of course there are <u>some issues</u>:

- Computational complexity: The allocation problem is NP-hard even for simple cases
- Representation & Communication: The valuation functions are exponential sizes. how can we even represent them?

The second issue itself ~~s~~ forces us to look for languages that allow succinct representations of valuations (to be used in practice). We will call them bidding languages.

Here we face <u>expressiveness</u> v.s. <u>simplicity</u> issues.

expressiveness: express succinctly as many naturally occurring

simplicity: simple both for humans to express and for programs to work

E.g. we can use combination of atomic bids with OR & XOR

atomic bid (S,p) is an offer of p for bundle S of items or any $T \supseteq S$ and zero for other

Then we have e.g. $(\{a,b\},3)$ XOR $(\{c,d\},5)$ then $V(\{a,c\})=0$, $V(\{a,b,c,d\})=5$

$(\{a,b\},3)$ OR $(\{c,d\},5)$ $V(\{a,c\})=0$, $V(\{a,b,c,d\})=8$

$(\{a,b\},3)$ OR $(\{a,c\},5)$ $V(\{a,b,c\})=5$ since we can satisfy only one.

This is a natural approach see more details of bidding language in the book AGT, sec 11.4. This is an active research e.g. for google

Sometimes very simple bidders are considered, i.e., single-minded bidders that is a bidder i for which there is a set $A_i \subseteq S$ of goods and a value $\alpha_i \geq 0$ such that (a) $v_i(T_i)=\alpha_i$ whenever $T_i \supseteq A_i$; and $v_i(T_i)=0$ otherwise.

Even for the special case of single-minded bidders, VCG mechanism can not be implemented in polynomial-time since the optimization problem of winner determination, i.e., given single-minded bidders $(A_1,\alpha_1) \cdots (A_n,\alpha_n)$, grant a set of disjoint bids (i.e. a subset of players such that the corresponding A_i's are pairwise disjoint) to maximize the sum $\sum \alpha_i$ of the values of the granted bids, is NP-hard (indeed $\Omega(n^{1-\varepsilon})$-hard to approximate).

<u>Proof</u>: by a reduction from NP-hard Weighted Independent Set (input a graph $G=(V,E)$ and a weight w_v for each vertex $v \in V$). The set of goods is the set E of edges; the set of players is the set of vertices V; for a vertex/player $v \in V$, set $\alpha_v = w_v$ and A_v equal to the set of edges of G incident to v. A set of player is independent iff the subset can be granted simultaneously. □